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An improved scheme for solving the ECE equations for electric and magnetic fields

In note 1 a solution scheme was given for the equations

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \underline{j}^a \qquad (1)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2}\frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a \qquad (2)$$

The scheme showed up some convergence problems. In this note we derive an alternative scheme wich is similar to the method used in Finite Element Analysis for solving Maxwell's equations. The basic idea was pointed out by Bob Flower.

In equation (1) we have subsumed the factor μ_0 of the homogeneous current into $\underline{j}^a$. Thus the quantities $\underline{j}^a$ and $\underline{J}^a$ have equivalent physical units (V/m^2 and A/m^2). Both are considered to be predefined in this note. Assuming a harmonic time dependence

$$e^{-i\omega t}$$

of all quantities as in note 1 leads to the equations

$$\underline{\nabla} \times \underline{E}^a - i\omega \underline{B}^a = \underline{j}^a \qquad (3)$$

$$\underline{\nabla} \times \underline{B}^a + i\frac{\omega}{c^2} \underline{E}^a = \mu_0 \underline{J}^a \qquad (4)$$

The complex factors give rise to a phase shift between E and H fields. The currents are the driving terms and can be described by real-valued amplitudes. Imaginary parts would only change the initial phase.

Applying the curl operator to both equations (1), (2) gives

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}^a) + \frac{\partial}{\partial t} \underline{\nabla} \times \underline{B}^a = \underline{\nabla} \times \underline{j}^a \qquad (5)$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}^a) - \frac{1}{c^2}\frac{\partial}{\partial t} \underline{\nabla} \times \underline{E}^a = \mu_0 \underline{\nabla} \times \underline{J}^a \qquad (6)$$

Inserting (1) into (5) and (2) into (6) leads to

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}^a) + \frac{1}{c^2}\frac{\partial^2 \underline{E}^a}{\partial t^2} + \mu_0 \frac{\partial \underline{J}^a}{\partial t} = \underline{\nabla} \times \underline{j}^a \tag{7}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}^a) + \frac{1}{c^2}\left(\frac{\partial^2 \underline{B}^a}{\partial t^2} - \frac{\partial \underline{j}^a}{\partial t}\right) = \mu_0 \underline{\nabla} \times \underline{J}^a \tag{8}$$

Assuming the harmonic time dependence again leads to the simplified equations

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}^a) - \frac{\omega^2}{c^2}\underline{E}^a - i\omega\mu_0 \underline{J}^a = \underline{\nabla} \times \underline{j}^a \tag{9}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}^a) - \frac{\omega^2}{c^2}\underline{B}^a + i\frac{\omega}{c^2}\underline{j}^a = \mu_0 \underline{\nabla} \times \underline{J}^a \tag{10}$$

We replace the doubled curl operator by the Laplace and the grad div operator:

$$-\nabla^2 \underline{E}^a + \underline{\nabla}(\underline{\nabla}\cdot\underline{E}^a) - \frac{\omega^2}{c^2}\underline{E}^a = i\omega\mu_0 \underline{J}^a + \underline{\nabla} \times \underline{j}^a \tag{11}$$

$$-\nabla^2 \underline{H}^a + \underline{\nabla}(\underline{\nabla}\cdot\underline{H}^a) - \frac{\omega^2}{c^2}\underline{H}^a = -i\omega\varepsilon_0 \underline{j}^a + \underline{\nabla} \times \underline{J}^a \tag{12}$$

Now we have arrived at a form of the equations ready for numerical solution. It is an extended form of the Poisson equation, similar to the resonance equations derived in the numerical paper.

The discretization of the Laplacian for a function $f(x,y,z) = f_{i,j,k}$ is

$$\nabla^2 f \approx \left(f_{i+1,j,k} + f_{i-1,j,k} + f_{i,j+1,k} + f_{i,j-1,k} + f_{i,j,k-1} + f_{i,j,k+1} - 6 f_{ijk}\right)\frac{1}{h^2} \tag{13}$$

with grid spacing h in all directions. This lets us write eqs. (11), (12) as

$$6\underline{E}^a_{i,j,k} = \underline{E}^a_{i+1,j,k} + \ldots + h^2\left(-\underline{\nabla}(\underline{\nabla}\cdot\underline{E}^a) + \frac{\omega^2}{c^2}\underline{E}^a + i\omega\mu_0 \underline{J}^a + \underline{\nabla}\times\underline{j}^a\right) \tag{14}$$

$$6\underline{H}^a_{i,j,k} = \underline{H}^a_{i+1,j,k} + \ldots + h^2\left(-\underline{\nabla}(\underline{\nabla}\cdot\underline{H}^a) + \frac{\omega^2}{c^2}\underline{H}^a - i\omega\varepsilon_0 \underline{j}^a + \underline{\nabla}\times\underline{J}^a\right) \tag{15}$$

As the last step we bring the terms with ω^2/c^2 to the left-hand side and divide by the scalar factor:

$$\underline{E}^a_{ijk} = \frac{1}{6 - h^2 \frac{\omega^2}{c^2}} \left(\underline{E}^a_{i+1,j,k} + \ldots + h^2 \left(-\underline{\nabla}(\underline{\nabla} \cdot \underline{E}^a) + i \omega \mu_0 \underline{J}^a + \underline{\nabla} \times \underline{j}^a \right) \right) \quad (16)$$

$$\underline{H}^a_{ijk} = \frac{1}{6 - h^2 \frac{\omega^2}{c^2}} \left(\underline{H}^a_{i+1,j,k} + \ldots + h^2 \left(-\underline{\nabla}(\underline{\nabla} \cdot \underline{H}^a) - i \omega \varepsilon_0 \underline{j}^a + \underline{\nabla} \times \underline{J}^a \right) \right) \quad (17)$$

The grad div operator of $\underline{E}^a$ and $\underline{H}^a$ has also to be discretized as described in the numerical paper. As shown there, the discretization of the first derivative in the symmetric form does not contain the grid point (i,j,k) of the fields. We have arrived at an iteration scheme where all grid points (i,j,k) are located at the left. The right-hand side depends only on neighbouring points and the current terms at (i,j,k). The curls of the currents need only be computed once before the iteration starts.