On the 'Size' of Einstein's Spherically Symmetric Universe

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It is alleged by the Standard Cosmological Model that Einstein's Universe is finite but unbounded. Although this is a longstanding and widespread allegation, it is nonetheless incorrect. It is also alleged by this Model that the Universe is expanding and that it began with a Big Bang. These are also longstanding and widespread claims that are demonstrably false. The FRW models for an expanding, finite, unbounded Universe are inconsistent with General Relativity and are therefore invalid.

1 Historical basis

Non-static homogeneous models were first investigated theoretically by Friedmann in 1922. The concept of the Big Bang began with Lemaître, in 1927, who subsequently asserted that the Universe, according to General Relativity, came into existence from a "primaeval atom".

Following Friedmann, the work of Robertson and Walker resulted in the FRW line-element,

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right],$$

from which is obtained the so-called "Friedmann equation",

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2,$$

where ρ is the macroscopic proper density of the Universe and k a constant. Applying the continuity condition $T^{\mu\nu}{}_{;\mu} = 0$, to the stress tensor $T_{\mu\nu}$ of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu},$$

where p is the pressure and u_μ the covariant world velocity of the fluid particles, the equation of continuity becomes

$$R\dot{\rho} + 3\dot{R}(\rho + p) = 0.$$

With the ad hoc assumption that $R(0) = 0$, the Friedmann equation is routinely written as

$$\dot{R}^2 + k = \frac{A^2}{R},$$

where A is a constant. The so-called "Friedmann models" are:

1. k = 0 — the flat model
2. k = 1 — the closed model
3. k = -1 — the open model,

wherein $t = 0$ is claimed to mark the beginning of the Universe and $R(0) = 0$ the cosmological singularity.

Big Bang and expansion now dominate thinking in contemporary cosmology. However, it is nonetheless easily proved that such cosmological models, insofar as they relate to the FRW line-element, with or without embellishments such as "inflation", are in fact inconsistent with the mathematical structure of the line-elements from which they are alleged, and are therefore false.

2 Spherically symmetric metric manifolds

A 3-D spherically symmetric metric manifold has, in the spherical-polar coordinates, the following form ([1, 2]),

$$ds^2 = B(R_c)dR_c^2 + R_c^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (1)$$

where $B(R_c)$ and $R_c = R_c(r)$ are a priori unknown analytic functions of the variable r of the simple line element

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (2)$$

$$0 \le r \le \infty.$$

Line elements (1) and (2) have precisely the same fundamental geometric form and so the geometric relations between the components of the metric tensor are exactly the same in each line element. The quantity R_c appearing in (1) is not the geodesic radial distance associated with the manifold it describes. It is in fact the radius of curvature, in that it determines the Gaussian curvature $G = 1/R_c^2$ (see [1, 2]). The geodesic radial distance distance, R_p, from an arbitrary point in the manifold described by (1) is an intrinsic geometric property of the line element, and is given by

$$R_p = \int \sqrt{B(R_c)}\, dR_c + C = \int \sqrt{B(R_c)}\, \frac{dR_c}{dr} dr + C,$$

where C is a constant of integration to be determined ([2]). Therefore, (1) can be written as

$$ds^2 = dR_p^2 + R_c^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where

$$dR_p = \sqrt{B(R_c)}dR_c,$$

and

$$0 \le R_p < \infty,$$

with the possibility of the line element being singular (undefined) at $R_p = 0$, since $B(R_c)$ and $R_c = R_c(r)$ are a priori unknown analytic functions of the variable r. In the case of (2),

$$R_c(r) \equiv r, \quad dR_p \equiv dr, \quad B(R_c(r)) \equiv 1,$$

from which it follows that $R_c \equiv R_p \equiv r$ in the case of (2). Thus $R_c \equiv R_p$ is not general, and only occurs in the special case of (2), which describes an Efcleethean* space.

The volume V of (1), and therefore of (2), is

$$V = \int_0^{R_p} R_c^2\, dR_p \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= 4\pi \int_{R_c(0)}^{R_c(r)} R_c^2(r)\sqrt{B(R_c(r))}\, dR_c(r),$$

$$= 4\pi \int_0^{r} R_c^2(r)\sqrt{B(R_c(r))}\, \frac{dR_c(r)}{dr}\, dr,$$

although, in the general case (1), owing to the a priori unknown functions $B(R_c(r))$ and $R_c(r)$, the line element (1) may be undefined at $R_p(R_c(0)) = R_p(r=0) = 0$, which is the location of the centre of spherical symmetry of the manifold of (1) at an arbitrary point in the manifold. Also, since $R_c(r)$ is a priori unknown, the value of $R_c(0)$ is unknown and so it cannot be assumed that $R_c(0) = 0$. In the special case of (2), both $B(R_c(r))$ and $R_c(r)$ are known.

Similarly, the surface area S of (1), and hence of (2), is given by the general expression,

$$S = R_c^2(r)\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = 4\pi R_c^2(r).$$

This might not ever be zero, since, once again, $R_c(r)$ is an a priori unknown function and so $R_c(0)$ might not be zero. It all depends an the explicit form for $R_c(r)$, if it can be determined in a given situation, and on associated boundary conditions. References [1, 2] herein describe the mathematics in more detail.

*For the geometry due to Efcleethees, usually and abominably rendered as Euclid.

3 The 'radius' of Einstein's universe

Since a geometry is entirely determined by the form of its line element, everything must be determined from it. One cannot, as is usually done, merely foist assumptions upon it. The intrinsic geometry of the line element and the consequent geometrical relations between the components of the metric tensor determine all.

Consider the usual non-static cosmological line element

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1+\frac{k}{4}\bar{r}^2\right)^2}\left[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{3}$$

wherein it is usually simply assumed that $0 \le \bar{r} < \infty$ ([3–6]). However, the range on $\bar{r}$ must be determined, not assumed. It is easily proved that the foregoing usual assumption is patently false.

Once again note that in (3) the quantity $\bar{r}$ is not a radial geodesic distance. In fact, it is not even a radius of curvature on (3). It is merely a parameter for the radius of curvature and the proper radius, both of which are well-defined by the form of the line element (describing a spherically symmetric metric manifold). The radius of curvature, R_c, for (3), is

$$R_c = e^{\frac{1}{2}g(t)}\frac{\bar{r}}{1+\frac{k}{4}\bar{r}^2}. \tag{4}$$

The proper radius for (3) is given by

$$R_p = e^{\frac{1}{2}g(t)}\int \frac{d\bar{r}}{1+\frac{k}{4}\bar{r}^2}$$

$$= \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}}\left(\arctan\frac{\sqrt{k}}{2}\bar{r} + n\pi\right), \quad n = 0, 1, 2, ... \tag{5}$$

Since $R_p \ge 0$ by definition, $R_p = 0$ is satisfied when $\bar{r} = 0 = n$. So $\bar{r} = 0$ is the lower bound on $\bar{r}$. The upper bound on $\bar{r}$ must now be ascertained from the line element and boundary conditions.

It is noted that the spatial component of (4) has a maximum of $\frac{1}{\sqrt{k}}$ for any time t, when $\bar{r} = \frac{2}{\sqrt{k}}$. Thus, as $\bar{r} \to \infty$, the spatial component of R_c runs from 0 (at $\bar{r} = 0$) to the maximum $\frac{1}{\sqrt{k}}$ (at $\bar{r} = \frac{2}{\sqrt{k}}$), then back to zero, since

$$\lim_{\bar{r}\to\infty} \frac{\bar{r}}{1+\frac{k}{4}\bar{r}^2} = 0. \tag{6}$$

Transform (3) by setting

$$R = R(\bar{r}) = \frac{\bar{r}}{1+\frac{k}{4}\bar{r}^2}, \tag{7}$$

which carries (3) into

$$ds^2 = dt^2 - e^{g(t)}\left[\frac{dR^2}{1-kR^2} + R^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{8}$$

The quantity R appearing in (8) is not a radial geodesic distance. It is only a radius of curvature in that it determines the Gaussian curvature $G = \frac{1}{e^{g(t)}R^2}$. The radius of curvature of (8) is

$$R_c = e^{\frac{1}{2}g(t)}R, \tag{9}$$

and the proper radius of Einstein's universe is, according to (8),

$$R_p = e^{\frac{1}{2}g(t)} \int \frac{dR}{\sqrt{1-kR^2}}$$

$$= \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left(\arcsin \sqrt{k}R + 2m\pi \right), \quad m = 0, 1, 2, ... \tag{10}$$

Now according to (7), the minimum value of R is $R(\bar{r} = 0) = 0$. Also, according to (7), the maximum value of R is $R(\bar{r} = \frac{2}{\sqrt{k}}) = \frac{1}{\sqrt{k}}$. $R = \frac{1}{\sqrt{k}}$ makes (8) singular, although (3) is not singular at $\bar{r} = \frac{2}{\sqrt{k}}$. Since by (7), $\bar{r} \to \infty \Rightarrow R(\bar{r}) \to 0$, then if $0 \le \bar{r} < \infty$ on (3) it follows that the proper radius of Einstein's universe is, according to (8),

$$R_p = e^{\frac{1}{2}g(t)} \int_0^0 \frac{dR}{\sqrt{1-kR^2}} \equiv 0. \tag{11}$$

Therefore, $0 \le \bar{r} < \infty$ on (3) is false. Furthermore, since the proper radius of Einstein's universe cannot be zero and cannot depend upon a set of coordinates (it must be an invariant), expressions (5) and (10) must agree. Similarly, the radius of curvature of Einstein's universe must be an invariant (independent of a set of coordinates), so expressions (4) and (9) must also agree, in which case $0 \le R < \frac{1}{\sqrt{k}}$ and $0 \le \bar{r} < \frac{2}{\sqrt{k}}$. Then by (5), the proper radius of Einstein's universe is

$$R_p = \lim_{\alpha \to \frac{2}{\sqrt{k}}} e^{\frac{1}{2}g(t)} \int_0^\alpha \frac{d\bar{r}}{1 + \frac{k}{4}\bar{r}^2}$$

$$= \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left[\left(\frac{\pi}{4} + n\pi \right) - m\pi \right], \quad n, m = 0, 1, 2, ...$$

$$n \ge m.$$

Setting $p = n - m$ gives for the proper radius of Einstein's universe,

$$R_p = \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left(\frac{\pi}{4} + p\pi \right), \quad p = 0, 1, 2, ... \tag{12}$$

Now by (10), the proper radius of Einstein's universe is

$$R_p = \lim_{\alpha \to \frac{1}{\sqrt{k}}} e^{\frac{1}{2}g(t)} \int_0^\alpha \frac{dR}{\sqrt{1-kR^2}}$$

$$= \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left[\left(\frac{\pi}{2} + 2n\pi \right) - m\pi \right], \quad n, m = 0, 1, 2, ...$$

$$2n \ge m.$$

Setting $q = 2n - m$ gives the proper radius of Einstein's universe as,

$$R_p = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left(\frac{\pi}{2} + q\pi \right), \quad q = 0, 1, 2, ... \tag{13}$$

Expressions (12) and (13) must be equal for all values of p and q. This can only occur if $g(t)$ is infinite for all values of t. Thus, the proper radius of Einstein's universe is infinite.

By (4), (7) and (9), the invariant radius of curvature of Einstein's universe is,

$$R_c \left(\frac{2}{\sqrt{k}} \right) = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}}, \tag{14}$$

which is infinite by virtue of $g(t) = \infty \ \forall \ t$.

4 The volume of Einstein's universe

The volume of Einstein's universe is, according to (3),

$$V = e^{\frac{3}{2}g(t)} \int_0^{\frac{2}{\sqrt{k}}} \frac{\bar{r}^2 d\bar{r}}{\left(1 + \frac{k}{4}\bar{r}^2\right)^3} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{4\pi e^{\frac{3}{2}g(t)}}{k^{\frac{3}{2}}} \left(\frac{\pi}{4} + p\pi \right), \quad p = 0, 1, 2, ... \tag{15}$$

The volume of Einstein's universe is, according to (8),

$$V = e^{\frac{3}{2}g(t)} \int_0^{\frac{1}{\sqrt{k}}} \frac{R^2 dR}{\sqrt{1-kR^2}} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi$$

$$= e^{\frac{3}{2}g(t)} \frac{2\pi}{k^{\frac{3}{2}}} \left[\frac{\pi}{2} + (2n - m)\pi \right], \quad n, m = 0, 1, 2, ...$$

$$2n \ge m,$$

and setting $q = 2n - m$ this becomes,

$$V = \frac{2\pi e^{\frac{3}{2}g(t)}}{k^{\frac{3}{2}}} \left(\frac{\pi}{2} + q\pi \right), \quad q = 0, 1, 2, ... \tag{16}$$

Since the volume of Einstein's universe must be an invariant, expressions (15) and (16) must be equal for all values of p and q. Equality can only occur if $g(t)$ is infinite for all values of the time t. Thus the volume of Einstein's universe is infinite.

In the usual treatment (8) is transformed by setting

$$R = \frac{1}{\sqrt{k}} \sin\chi, \tag{17}$$

to get

$$ds^2 = dt^2 - \frac{e^{g(t)}}{k} \left[d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2) \right], \tag{18}$$

where it is usually asserted, without any proof (see e.g. [3–6]), that

$$0 \le \chi \le \pi \quad (\text{or } 0 \le \chi \le 2\pi), \tag{19}$$

and whereby (18) is not singular. However, according to (7), (11), (12), and (13), χ can only take the values

$$2n\pi \le \chi < \frac{\pi}{2} + 2n\pi, \quad n = 0, 1, 2, ...$$

so that the radius of curvature of Einstein's universe is, by (18),

$$R_c = \frac{e^{\frac{1}{2}g(t)} \sin\chi}{\sqrt{k}}$$

which must be evaluated for $\chi = \frac{\pi}{2} + 2n\pi$, $n = 0, 1, 2, ...$, giving

$$R_c = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}}$$

as the radius of curvature of Einstein's universe, in concordance with (4), (7), and (9). The proper radius of Einstein's universe is given by

$$R_p = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \int_{2n\pi}^{\frac{\pi}{2}+2n\pi} d\chi = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \frac{\pi}{2}, \tag{20}$$

and since the proper radius of Einstein's universe is an invariant, (20) must equal (12) and (13). Expression (20) is consistent with (12) and (13) only if $g(t)$ is infinite for all values of the time t, and so Einstein's universe is infinite.

According to (18), the volume of Einstein's universe is,

$$V = \frac{e^{\frac{3}{2}g(t)}}{k^{\frac{3}{2}}} \int_{2n\pi}^{\frac{\pi}{2}+2n\pi} \sin^2\chi \, d\chi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{\pi^2 e^{\frac{3}{2}g(t)}}{k^{\frac{3}{2}}} \frac{\pi}{2}. \tag{21}$$

Since this volume must be an invariant, expression (21) must give the same value as expressions (15) and (16). This can only occur for (21) if $g(t)$ is infinite for all values of the time t, and so Einstein's universe has an infinite volume.

5 The 'area' of Einstein's universe

Using (3), the invariant surface area of Einstein's universe is

$$S = R_c^2 \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi R_c^2$$

which must be evaluated for $R_c(\bar{r} = \frac{2}{\sqrt{k}})$, according to (4), and so

$$S = \frac{4\pi e^{g(t)}}{k}.$$

By (8) the invariant surface area is

$$S = e^{g(t)} R^2 \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi = 4\pi R^2 e^{g(t)},$$

which must, according to (7), be evaluated for $R(\bar{r} = \frac{2}{\sqrt{k}}) = \frac{1}{\sqrt{k}}$, to give

$$S = \frac{4\pi e^{g(t)}}{k}.$$

By (18) the invariant surface area is

$$S = \frac{e^{g(t)}}{k} \sin^2\chi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{4\pi e^{g(t)}}{k} \sin^2\chi,$$

and this, according to (17), must be evaluated for $\chi = \left(\frac{\pi}{2} + 2n\pi\right)$, $n = 0, 1, 2, ...$, which gives

$$S = \frac{4\pi e^{g(t)}}{k}.$$

Thus the invariant surface area of Einstein's universe is infinite for all values of the time t, since $g(t)$ is infinite for all values of t.

In similar fashion the invariant great 'circumference', $C = 2\pi R_c$, of Einstein's universe is infinite at any particular time, given by

$$C = \frac{2\pi e^{\frac{1}{2}g(t)}}{\sqrt{k}}.$$

6 Generalisation of the line element

Line elements (3), (8) and (18) can be generalised in the following way. In (3), replace $\bar{r}$ by $|\bar{r} - \bar{r}_0|$ to get

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left(1 + \frac{k}{4}|\bar{r} - \bar{r}_0|^2\right)^2} \left[d\bar{r}^2 + |\bar{r} - \bar{r}_0|^2 (d\theta^2 + \sin^2\theta d\varphi^2)\right], \tag{22}$$

where $\bar{r}_0 \in \Re$ is entirely arbitrary. Line element (22) is defined on

$$0 \le |\bar{r} - \bar{r}_0| < \frac{2}{\sqrt{k}} \quad \forall\, \bar{r}_0,$$

i.e. on

$$\bar{r}_0 - \frac{2}{\sqrt{k}} < \bar{r} < \frac{2}{\sqrt{k}} + \bar{r}_0 \quad \forall\, \bar{r}_0. \tag{23}$$

This corresponds to $0 \le R_c < \frac{1}{\sqrt{k}}$ irrespective of the value of $\bar{r}_0$, and amplifies the fact that $\bar{r}$ is merely a parameter. Indeed, (4) is generalised to

$$R_c = R_c(\bar{r}) = \frac{|\bar{r} - \bar{r}_0|}{1 + \frac{k}{4}|\bar{r} - \bar{r}_0|^2},$$

where (23) applies. Note that $\bar{r}$ can approach $\bar{r}_0$ from above or below. Thus, there is nothing special about $\bar{r}_0 = 0$. If $\bar{r}_0 = 0$ and $\bar{r} \ge 0$, then (3) is recovered as a special case, still subject of course to the range $0 \le \bar{r} < \frac{2}{\sqrt{k}}$.

Expression (7) is generalised thus,

$$|R - R_0| = \frac{|\bar{r} - \bar{r}_0|}{1 + \frac{k}{4}|\bar{r} - \bar{r}_0|^2},$$

where R_0 is an entirely arbitrary real number, and so (8) becomes

$$ds^2 = dt^2 - e^{g(t)} \left[\frac{dR^2}{1 - k|R - R_0|^2} + |R - R_0|^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right], \tag{24}$$

where

$$R_0 - \frac{1}{\sqrt{k}} < R < \frac{1}{\sqrt{k}} + R_0 \quad \forall \ R_0. \tag{25}$$

Note that R can approach R_0 from above or below. There is nothing special about $R_0 = 0$. If $R_0 = 0$ and $R \ge 0$, then (8) is recovered as a special case, subject of course to the range $0 \le R < \frac{1}{\sqrt{k}}$.

Similarly, (18) is generalised, according to (24), by setting

$$|R - R_0| = \frac{1}{\sqrt{k}} \sin|\chi - \chi_0|,$$

where χ_0 is an entirely arbitrary real number, and

$$2n\pi \le |\chi - \chi_0| < \frac{\pi}{2} + 2n\pi, \quad n = 0, 1, 2, ...$$

$$\forall \ \chi_0 \ \in \ \Re.$$

Note that χ can approach χ_0 from above or below. There is nothing special about $\chi_0 = 0$. If $\chi_0 = 0$ and $\chi \ge 0$, then (18) is recovered as a special case, subject of course to the range $2n\pi \le \chi < \frac{\pi}{2} + 2n\pi$, $n = 0, 1, 2, ...$

The corresponding expressions for the great circumference, the surface area, and the volume are easily obtained in like fashion.

7 Conclusions

Einstein's universe has an infinite proper radius, an infinite radius of curvature, an infinite surface area and an infinite volume at any time. Thus, in relation to the Friedmann-Robertson-Walker line-element and its variations considered herein, the concept of the Big Bang cosmology is invalid.

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