EXPERIMENTAL VIOLATION OF HEISENBERG'S UNCERTAINTY RELATIONS BY THE SCANNING NEAR-FIELD OPTICAL MICROSCOPE

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Abstract. The recent development of the scanning near-field apertureless optical microscope, allowing spatial resolutions of the $\lambda/50$ order, or even better (thus breaking the $\lambda/2$ limit derived by Abbe) seems to have deep implications in the very foundations on the quantum physics. This looks like the first known experimental proof that the usual Heisenberg uncertainty relations are, in fact, a first approximation to describe certain aspects of the quantum world and are, consequently, in need of revision.

Key words: Violation of Heisenberg's uncertainty relations, scanning near-field optical microscope, apertureless microscope, Fourier non local analysis, `wavelet local analysis, foundations of quantum physics, non-linear quantum mechanics.

1 - Introduction

The first demonstration of the scanning near-field apertureless optical microscope was given by Pohl et al.[1], with a spatial resolution of $\lambda/20$. Today[2] it is possible to obtain resolutions of $\lambda/50$ or better, the theoretical limit not known. Since these experimental high spatial resolutions break the limit imposed by Abbe's rule for the common aperture microscopes: $\delta x \geq \lambda/2$, this implies that the usual Heisenberg's uncertainty relations, are in practice violated, contrary to more than a half-century of quantum mechanics claims that this was not possible. The aim of this letter is to show that these microscope observations in fact violate the usual uncertainty relations.

2 - The Uncertainty Relations

Contrary to certain common knowledge, that often text books on quantum mechanics sustain, the uncertainty relations do not forbid an actual measurement violating the limits imposed by the inequalities, as stressed early by Heisenberg[3], Popper[4] and many others. What Heisenberg's relations in fact forbid is that after the measurement, after the interaction, one can predict position and momentum with uncertainties that violate these inequalities. But, in certain conditions, the operation of the scanning near-field optical microscope violates the uncertainty relations, since after the interaction the product of the future uncertainty in position and momentum are less than predicted.

In light of the above considerations let us consider the Heisenberg microscope, as has been done in almost every textbook of quantum mechanics[5,6].

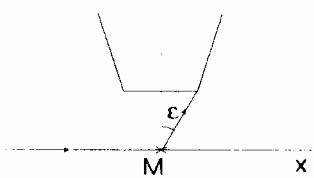

Fig.1. Scanning near-field optical microscope.

In Fig1, there is a schematic representation of the scanning near-field optical microscope where an horizontal incident photon is diffused by a very small object M. For this kind of apertureless microscope, experiments have shown that it is possible to attain spatial resolutions of $\delta x \leq \lambda/50$ or better.

The principles underlying the working of this apertureless scanning optical microscope are, essentially, the following: The photon after striking the corpuscle M is diffused and, after that, is caught by a tiny photon sensing device. This photon detector can be made of an optical fiber that conduct the light to a large detector. In the limit it is possible to make the photon detector so small that there is no need for the conducing optical fiber, the tip being the very detector. The electric pulse, produced in the light detector, feeds an electronic system connected to a computer. After the necessary scannig lines the image of the corpuscle M is seen in the screen.

In order to determine the uncertainty in momentum of the object M after the interaction with the photon it is necessary to consider the global conservation of momentum, photon plus diffusing corpuscle. Before the interaction the momentum of the diffusing photon is $|\vec{p}'| = p' = h/\lambda$. After the interaction it shall be $\vec{p}''$ making an angle ε with the axis of the point probe. Let p_x be the x-component of the particle M after the interaction. The conservation of momentum for the system object plus incident photon allows us to write

$$p_x = p' - p'' \sin\varepsilon \approx p'(1 - \sin\varepsilon) = \tfrac{h}{\lambda}(1 - \sin\varepsilon), \tag{1}$$

where it is assumed that, after the interaction, the absolute value of the momentum of the diffusing photon remains practically unchanged, $p'' \approx p'$. This means that the value of p_x lies between

$$\tfrac{h}{\lambda}(1 - \sin\varepsilon) \leq p_x \leq \tfrac{h}{\lambda}(1 + \sin\varepsilon). \tag{2}$$

In such conditions the uncertainty in the x-component of the momentum is given by

$$\delta p_x = 2\tfrac{h}{\lambda} \sin\varepsilon, \tag{3}$$

the maximum corresponding to a diffusion angle of $\pi/2$ and the above expression turns into

$$\delta p_x = 2\tfrac{h}{\lambda}, \tag{3'}$$

The product of the two uncertainties $\delta x = \lambda/50$ and $\delta p_x = 2h/\lambda$ is

$$\delta x \delta p_x = h/25. \tag{4}$$

If, instead of the experimental resolution of the scanning near-field microscope, the usual Abbe's criterion $\delta x = \lambda/2\sin\varepsilon$ were used, the product of the uncertainties would give the usual Heisenberg's relation

$$\delta x \delta p_x = h. \tag{5}$$

Conclusion

I was shown, in a similar manner to the one presented initially by Heisenberg, and since then used in many textbooks on quantum mechanics, that the uncertainty principle is violated experimentally by a factor of 1/25 in the above conditions. This means that, in certain very special experimental settings, it is possible to predict, before actual measurement (interaction) takes place, the future uncertainty of position and momentum in a way that their product is less than h.

In order to accommodate the experimental results it looks like the usual meaning of Heisenberg uncertainty relations need to be revised. We would like to point out that the relevant aspects mentioned here for the experiment with the Scanning Near-Field Optical Microscope do not involve, contrary to the usual reasoning, the wave-like nature of the light: we are thus making a measure where Bohr's[7] Complementary principle does not apply. It is therefore only natural that the Fourier analysis for the formation of the particle image does not apply to this new type of microscope.

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ON THE UNCERTAINTY RELATIONS

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Abstract: An overall view of some possible interpretations and forms for the uncertainty relations and certain aspects of quantum non-linear theories correlated with the local analysis by wavelets are presented. Also two possible experiments that may test the general validity of the usual Eisenberg uncertainty relations are discussed.

Key words: uncertainty relations, meaning of Fourier spectral analysis, local wavelet analysis, X-ray interferometry, foundations of quantum mechanics.

1. INTRODUCTION

Since Bohr[1] interpretation of the Eisenberg uncertainty relations in terms of the Fourier analysis, there was a general acceptation of them, at least on its mathematical form. It is true that were and still are lots of arguing on the actual meaning[2] of those relations, nevertheless almost[3] everybody seems to agree with its actual mathematical formulation.

Some, like de Broglie and many others, have always felt uneasy with some physical implications[4] of those relations, like for instance the infinite spreading of the matter wave packets, given by the known relation

$$\Delta x(t) \approx \frac{\hbar}{2m} t . \qquad (1)$$

If this relation holds true for all cases, then an electron ejected by the Sun,

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Those, and other reasons induced some to believe that the master equation for full describing the quantum phenomena must be a nonlinear equation. These kind of equations have nondispersive localized solutions in space and time, which are very well suited for describing the evolution of nondispersive causal particles. Recently some authors, like Mackinnon[5], Gueret[6], Vigier[7], and many others have proposed finite nondispersive, soliton-like , forms for the wave function solutions of a general master wave equation.

If quantum phenomena really need for full explanation a more general theory formulated in non linear terms, then the usual nonlocal Fourier transform will most likely need to be replaced by a different set of transforms. The recent development of the theory of wavelets[8], or finite waves, seems to indicate that a local analysis, both in time and space, of the functions is possible. These facts open the way for a more general set of uncertainty relations, derived, not as Bohr has done from the Fourier nonlocal transforms, but from a local analysis. Of course, these new uncertainty relations must in the most cases coincide with the usual ones.

2. A NEW FORM FOR THE UNCERTAINTY RELATIONS

In order to derive mathematically a different set of uncertainty relations[9], it is convenient to assume:

- 1 Quantum phenomena need for full explanation a nonlinear theory.
- 2 The nonlocal Fourier analysis, in a more complete theory, need to be replaced by a local wavelet type analysis.
- 3 Free quantum particles, are in general small, and can be localized, at any time, in a finite nondispersive volume.

Let us consider an unidimensional finite source of approximate size σ_s, each point of which emits monochromatic pulses of width σ_p described by a finite wave of the form

$$g(x,t;\varepsilon,k) = A\exp\left[-(x-\varepsilon-vt)^2/2\sigma_p^2\right]\exp[i(kx-\omega t)] \qquad (1)$$

It is also assumed that each point of the unidimensional source consists of many oscillators emitting pulses at wave numbers k that are Gaussian, distributed with average k_0,

$$f(x) = \alpha\exp\left[-(k-k_0)^2/2\sigma_k^2\right] \qquad (2)$$

and that the spacial distribution of these emitters at each instant is also approximately Gaussian, with a width σ_s of the line source

$$s(\varepsilon) = \beta \exp\left[-\varepsilon^2 / 2\sigma_s^2\right]. \tag{3}$$

The total pulse from this source will then be

$$\psi(x) = A\alpha\beta \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s(\varepsilon) f(k) g(x,t;\varepsilon,k) d\varepsilon\, dk, \tag{4}$$

which upon integration on ε gives

$$\psi(x,t) = A\alpha\beta\sqrt{2\pi} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-vt)^2}{2(\sigma_p^2+\sigma_s^2)} - \frac{(k-k_0)^2}{2\sigma_k^2} + i(kx-\omega t)\right] dk\,. \tag{5}$$

Now assuming that for the chosen range of k the velocity of each pulse is practically constant, one is allowed to write

$$\omega = vk, \quad v \approx c^{te}, \tag{6}$$

and by substitution in (5) it yields

$$\psi(x,t) = \gamma \frac{2\pi\sigma_p\sigma_s\sigma_k}{\sqrt{\sigma_p^2+\sigma_s^2}} \exp\left[-(x-vt)^2 / 2\sigma_x^2\right] \exp\left[i(kx-\omega t)\right] \tag{7}$$

where

$$\sigma_x^2 = \left[\sigma_k^2 + 1/(\sigma_p^2+\sigma_s^2)\right]^{-1}. \tag{8}$$

This expression (8) represents a generalized dispersion relation. Although σ_s has been used to describe the approximate size of the one.dimension source, it could as well be regarded as representing the coherence length of a point-like source with a coherence time τ, such that $\sigma_s = v\tau$.

Some remarks can be made on the new dispersive relation (8), considering the following cases:

(a) sources with a relative "large" coherence length σ_s,
(b) wide-band sources, i.e. sources with "large" σ_k,
(c) truly monochromatic sources, $\sigma_k \approx 0$,
(d) very sharp pulses, σ_p "small."

In cases (a) and (b), sources with large σ_s $(1/(\sigma_p^2 + \sigma_s^s) \approx 0)$ and/or relatively wide-band $(1/(\sigma_p^2 + \sigma_s^2) \ll \sigma_k^2)$, the general dispersion relation (8) becomes the usual one

$$\sigma_x \sigma_k = 1. \tag{9}$$

In cases (c) and (d), since in these two cases $(1/(\sigma_p^2 + \sigma_s^2) \gg \sigma_k^2)$, one gets

$$\sigma_x = \sqrt{\sigma_p^2 + \sigma_s^2}. \tag{10}$$

As can be seen from Fig.1, in most practical instances relation (8) and (9) do give the same results.

The essential difference between the new dispersion relation (8) and the usual (9) lies then in the fact that, with the former, one can have a source with practically no dispersion $(\sigma_k \approx 0)$ giving a finite pulse, whereas with (9) the size of the pulse grows indefinitely.

It was seen that starting from very general assumptions for quantum "particle" represented by finite, nondispersive waves, it was possible, by pure formal calculations, to obtain a more general set of uncertainty relations that avoid the shortcomings of the usual ones. The question of the physical meaning of such relations remains, of course, an open question. In situations of this kind only experiment can decide which is the better formula for the uncertainty relations

Fig.1. Usual Heisenberg dispersion relations (dotted line) and generalized dispersion relations (full) for different values of $(1/(\sigma_p^2 + \sigma_s^2))$.

3. EXPERIMENTS TO TEST THE GENERAL VALIDITY OF HEISENBERG UNCERTAINTY RELATIONS

There seems to be many ways to investigate the general validity of the usual uncertainty relations, however in the present work I shall discuss only two possible experiments[10,11].

In quantum mechanics, as it is well known, the mean life of an excited state τ_0 is identified with the coherence time of the emitted particle. When taken to its ultimate consequences, together with the uncertainty relations this leads to some strange conclusions. For instance, it is known that the 5+ state of the nucleus $^{126}_{51}Sb$ has a mean life of 1140 s and decays to the 8- state emitting gamma photons of 17.7 kev[12]. Consequently, this leads to a photon wave packet of the same coherence time $\tau_0 = \Delta t_0 \approx 1140s$. From this coherence time it is possible to calculate the approximate coherence length, which gives a longitudinal size for the wave packet very large, in fact it is greater than the diameter of Earth's orbit.

Since the above inference seems to defeat the physical feeling, which assumes the single gamma photon as a very small entity, it would be interesting to submit the conclusion to an experimental validation.

An experiment to test the general validity of the Heisenberg uncertainty relations, based on those ideas, seems possible with the modern methods of X-ray interferometry. Essentially the experiment consists in using an X-ray in which the source is a suitable gamma emitter of reasonable energy and mean life. X-ray sources, even if the same energy of the gamma emitters, are not suitable because the mean life of the excited states are too short. The interferometer, shown in Fig.2, has a shutter placed in one arm, that chops the passing wave packet, with an initial coherence time Δt_0, in minute pieces with a coherent time $\Delta t \lll \Delta t_0$. Now the question is: What are the predicted results for the interference pattern at the detector?

The answer to the question depends on the assumptions made on the nature of the quantum particle.

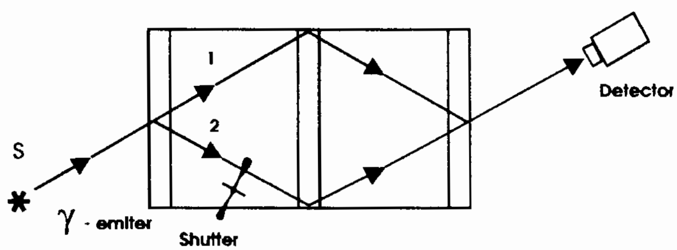

Fig.2. Proposed experiment. X-ray interferometer with a shutter in one arm.

3.1. Quantum Particles are Small Entities

In this case the action of the chopper on the beam two, when the time interval between two cuts is much greater than the "true" coherence time of the particle, is to absorb more or less wavelets. This situation is depicted in Fig.3.

Fig.3. Overlapping of small waves.

It is possible to show[10] that the visibility for this case is given by

$$\gamma_s = \frac{2\alpha}{1+\alpha}, \tag{11}$$

where α is the absorption factor, such that $I_2 = \alpha I_1$, $(0 \le \alpha \le 1)$. When the rate of incident particles is small one may have $\alpha \approx 1$ which means a visibility practically one. That is for small particle emission rate the action of the chopper on the arm two of the interferometer does not greatly change the visibility of the interference pattern.

3.2. Heisenberg Uncertainty Relations $\Delta E \Delta t \geq h$ Holds True in All Cases

In such circumstances the action of the chopper on the beam two of the interferometer is to change the initial minimum energy dispersion of the packet from ΔE_0 to ΔE so that $\Delta E \ggg \Delta E_0$, since the coherence time of the chopped pieces is much smaller than the initial coherence time ($\Delta t \lll \Delta t_0$). Therefore at the overlapping region one wave practically monochromatic superimposes to many small wave packets, see Fig. 4, producing a specific interference pattern.

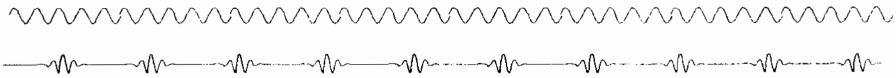

Fig.4. Overlapping of two waves: one practically monochromatic, the other composed of many small wave packets with large energy dispersion.

In this situation two cases may be considered:

(a) The action of the chopper on the beam produces small wave packets with a random relative phase, the most likely case. Then no interference pattern is to be expected at the detector.

(b) It is assumed, with Gähler and Golub[13] in what they call *diffraction in time*, that chopper is made in such a way that it maintain the relative phase among the small wave packets nearly constant. It is possible to show, under reasonable approximations, that, even in this very optimistic situation, the visibility γ_u is smaller than γ_s, obtained under the assumption that the quantum particles are small entities.

The other possible experiment to test the usual uncertainty relations is based on the spreading of the matter wave packets, given by formula (1), $\Delta x \approx [\hbar / m(\Delta x_0)]t$. According to this formula, for instance, Auger electrons produced in a monolayer of vacuum-deposited gold atoms on a perfect crystal surface, as proposed by Scheer et all.[14], the uncertainty of the initial position Δx_0 can be of the order of the atom layer, say $\Delta x_0 = 10$ Å, which by substitution in the spreading formula (1) gives after one second $\Delta x \approx 100 km$. Meaning that, after one second, the initial wave packet of about 10 Å increases to a size of 100 km. This conclusion is very striking. Even so the usual interpretation is perfectly able to deal with it. In this contest it means that after one second one can detect an electron somewhere within a distance of about 100 km, with a velocity distribution equal to the minimum initial one $\Delta v \geq \Delta v_0$. Since

$$\Delta x_0 \, \Delta v_0 \geq \frac{\hbar}{2m}, \tag{12}$$

which, after substituting Δx_0, gives $\Delta v \geq \Delta v_0 \approx v_M - v_m \approx 10^7$ cm s^{-1}.

Other explanation for the spreading of the matter wave packets is possible in context of a non linear theory where each single quantum free particle is described by a finite wave of constant shape.

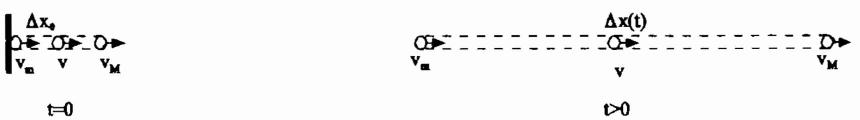

Fig.5. At time t=0, particles with different velocities occupy an initial length Δx_0; as the time increases, they will be spread over a much larger length $\Delta x(t)$.

In this natural model, the spreading of the "wave train" is only a way of saying that an initial burst of many particles, of constant small size, increase the distance between each other to a value that depends only of the elapsed time and of their respective range of velocities Δv_0.

Therefore, in this model, an electron "wave train" of 100 km means that the possible distance between the faster and slower electron after one second is of that magnitude as depicted in Fig.5.

In order to see which is the right model consider the following experiment: An electron source emits, at fixed known time intervals t_0, electrons with an uncertainty in the position $\Delta x_0 \approx 100\,\text{Å}$, which according to the Heisenberg relations correspond to a minimum energy dispersion $\Delta v_0 \approx 10^{-6}\, cm.s^{-1}$. A millisecond later the initial wave packet will have spread to a size of 10 m. Suppose now, that from each electron wave packet, and always for the same elapsed time $t=1$ ms, one "slices " a piece of, say, $\delta x = 100$ Å. This slice from the wave packet can in principle be made with an electromagnetic field, triggered by the emission of the electron from the source, as seen in Fig.6.

Now, if one measures the velocity dispersion δv of these small slices of length δx from the expanded electron packet, what is to be expected?

According to the Heisenberg uncertainty relations, one must always have, $\delta x \delta v \geq \hbar / 2m$; therefore as $\delta x \approx \Delta x_0$, and $\Delta x_0\, \Delta v_0 \cong \hbar / 2m$, this implies that $\delta v \geq \Delta v_0$. That is, the velocity dispersion of the arriving electron, from the small slice, will be equal or greater than the initial minimum dispersion Δv_0. One may conclude that when one "slices" the expanded electron wave train, one interacts with all the Fourier components of the packet, modifying them in such a way, that at the end the predicted velocity distribution must be at least equal to the initial one. If it could be otherwise, it would be contra the Heisenberg relations and therefore against the usual quantum mechanics.

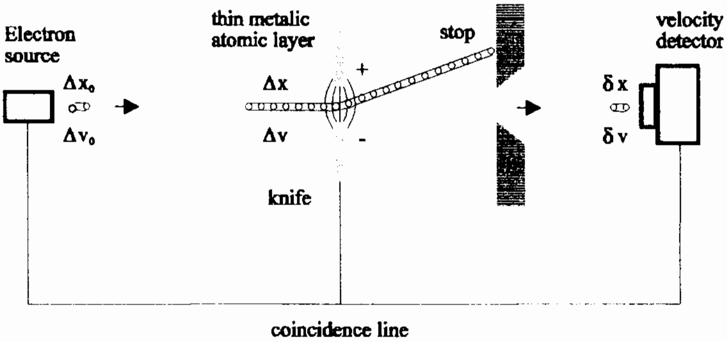

Fig.6. Schematic representation of the proposed experiment.

Assuming that the quantum phenomena need for full explanation a non linear theory and therefore that the quantum particle must be described by a finite local wave, the spreading of the "wave packet" has no real meaning for the individual electrons. It is only a mathematical description for the average separation between particles of different velocities. One is really slicing nothing. What happens is a selection from the expanded "packet" of particles that have a smaller range of velocities. Only those particles that fall in the selected smaller velocity range have the chance to be detected. Therefore if the nonlocal Fourier spectral decomposition is only a mathematical device for the average particle dispersion, a smaller velocity dispersion is to be predicted $\delta v \ll \Delta v_0$, as shown in Fig.7.

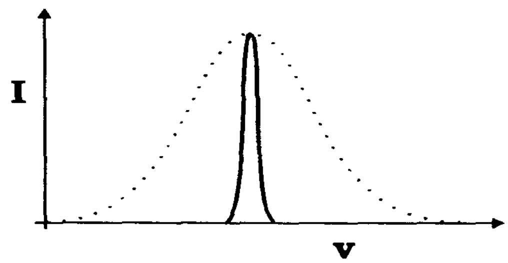

Fig.7. Predicted results for the velocity distribution of the electrons from the slice δx of the expanded wave packet. Usual theory (dotted line) vs nonlinear theory (solid line).

Because $\delta x \approx \Delta x_0$ and $\delta v \ll \Delta v_0$ $(\Delta x_0 \, \Delta v_0 \cong \hbar / 2m)$, one concludes that $\delta x \delta v \ll \hbar / 2m$. the experiment violating the Heisenberg uncertainty relations.

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Dispersion Relations for the Superposition of Monochromatic Finite Pulses

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Abstract. We obtain the general dispersion relations for the superposition offinite, monochromatic pulses emited by a pontual or unidimensional source intoboth dispersive and non-dispersive mediae. It is shown under what conditionsthose relations revert into the form of Heisenberg's dispersion relations.

Résumé. On obtient les relations générales de dispersion lorsqu'il y a la superposition de pulsations monochromatiques finies émises par une source ponctuelle ou unidimensionelledans des milieux dispersifs ou non-dispersifs. On discute dans quelles conditions ces relations se réduisent aux relations de dispersion de Heisenberg.

Quantum Mechanics is, in its traditional formulation, a linear theory which uses Fourier duality as main mathematical tool: from the transformation theory and the equivalence between **q**- and **p**-spaces one can derive as a direct consequence the Heisenberg uncertainty relations.

This has urged many authors, starting with de Broglie[1], to propose, from the very beginning of quantum theory, more general, *non-linear* theories for the interpretation of quantum phenomena. De Broglie formulated some non-linear aspects of the theory but never gave an explicit form for the non-linear equation that would sit as the cornerstone of his non-linear theory. Recently, many autjors following de Broglie's ideas, like Mackinnon[2], Gueret[3], Vigier[3 4] and many others, have proposed finite, non-dispersive, soliton-like forms for the wave-function solution of a non-linear general wave equation.

In the present letter we whish to draw attention to some possible implications of the assumption of non-dispersive solution for the master non-linear equation for quantum mechanics. If quantum phenomena really needs for its full explanation a theory formulated in non-linear terms, then the usual Fourier-transform tool will most likely need to be replaced by a different set of transforms. In fact, the recent developments of the mathematical theory of wavelets[5], or finite waves, seems to indicate that from finite waves of constant shape it is possible to analize both discrete and continuous objects. These facts may have implications on the usual Heisenberg uncertainty relations: by assuming that the particles are to be described by finite wavelets of constant shape, it may happen that the common uncertainty relations will change.

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Some strange physical situation like the prediction of infinite spreading of matter wave-packets[6] in free space and others[7] that can (in principle) be tested experimentally, seem to indicate the need to obtain a more general set of uncertainty relations, containing the usual ones as particular cases, in order that we may avoid these problems.

We show, with a very simple localized non-dispersive model for particle waves, that when one considers the sum of many particles emmited by a source in pulse form, one gets a more general set of dispersion relations from the product of the dispersion of conjugate variables, and study the conditions under which they revert to the usual ones.

If the wave-functions Ψ_1 and Ψ_2 are solutions to some master equation which is non-linear, then $\Psi = \Psi_1 + \Psi_2$ need not be in general another solution: nevertheless, there must be a certain $\Psi = \Psi(\Psi_1, \Psi_2)$ which is a solution of that non-linear equation, otherwise such equation would not even describe the emission of two or more particles. It is also assumed that a deeper non-linear quantum theory must, at the large-ensemble statistical/thermodynamic limit, approach the usual one, at least at the formal and predictive level.

In another work we show that, when the number of emmiting oscillators is very large with a gaussian-distributed amplitude, the solution $\Psi = \Psi(\Psi_1, \Psi_2, \ldots, \Psi_N)$ approaches the sum $\sum_{i=1}^{\infty} \Psi_i$. In the case of mono-chromatic, gaussian pulses in non-dispersive media it is possible to show exactly that the integral sum is a solution to a certain non-linear Schrodinger equation. In the more complex case of propagation in dispersive medium, we can say as much in the asymptotic limit of very small t or fixed position and very large t.

General Setup

Consider a one-dimensional, finite source $\mathcal{S}$ of approximate size σ_s, each point of which emits limited (half-width $= \sigma_p$), monochromatic (uncorrelated) pulses of the form

(1) $$\phi(x, t\colon x_0, k) = A e^{-\frac{(x - x_0 - vt)^2}{2\sigma_p{}^2}} e^{i(kx - \omega t)}$$

We will assume that each point of the source $\mathcal{S}$ consists of many oscillators, emmiting at wave-numbers k that appear gaussian-distributed with average k_0

(2) $$f(k) = \alpha e^{-\frac{(k - k_0)^2}{2\sigma_k^2}}$$

and that the spatial distribution of the emiters at each instant is also approximately gaussian, with the width σ_s of the line-source $\mathcal{S}$ itself,

(3) $$g(x_0) = \beta e^{-\frac{x_0^2}{2\sigma_s^2}}$$

(β could eventually be a function of time t).

The total pulse from the source $\mathcal{S}$ will thus be

(4) $$\Psi(x, t) = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dk\, f(k) g(x_0) \phi(x, t\colon x_0, k)$$

and after some arrangement

(5)
$$\Psi(x,t) = A\alpha\beta \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dx_0 e^{\frac{\sigma_p^2+\sigma_s^2}{2\sigma_p^2\sigma_s^2}\left(x_0 - \frac{\sigma_s^2}{\sigma_p^2+\sigma_s^2}(x-vt)\right)^2} e^{-\frac{(k-k_0)^2}{2\sigma_k^2} - \frac{(x-vt)^2}{2\left(\sigma_p^2+\sigma_s^2\right)} + i(kx-\omega t)}$$

which upon integration on x_0 gives

(6)
$$\Psi(x,t) = A\,\alpha\,\beta \sqrt{\frac{2\pi\,\sigma_p^2\,\sigma_s^2}{\sigma_p^2+\sigma_s^2}} \int_{-\infty}^{+\infty} dk e^{-\frac{(k-k_0)^2}{2\sigma_k^2} - \frac{(x-vt)^2}{2\left(\sigma_p^2+\sigma_s^2\right)} + i(kx-\omega t)}$$

Propagation in Non-Dispersive Medium

In non-dispersive media, where pulses of any wavelength propagate with velocity c, the usual dispersion relation

(7)
$$\omega = ck$$

can be used to carry out the integral in (4), yielding

(8)
$$\Psi(x,t) = \gamma \frac{\sigma_p}{\sqrt{{\sigma_p}^2 + \sigma_s^2}} e^{-\frac{(x-ct)^2}{2\sigma_x^2}} e^{i(k_0 x - \omega_0 t)}$$

where

(9)
$$\sigma_x^2 = \frac{1}{\sigma_k^2 + \frac{1}{{\sigma_p}^2+\sigma_s^2}}.$$

Expression (9) is a *generalized dispersion relation.*

Although σ_s has been used to describe the approximate size of a one-dimensional source, it could as well be regarded as representing the *coherence-length* of a point-like source with a*coherence time* τ, i.e. such that $\sigma_s = c\tau$.

We will look at formula (9) in the following cases:

(a) Source with relatively large coherence length σ_s;
(b) Wide-band sources, i.e. sources with "large" σ_k;
(c) Truly monochromatic sources, i.e. $\sigma_k \approx 0$;
(d) Very sharp pulses.

In cases (a)-(b), sources with large coherence-length $((\frac{1}{{\sigma_p}^2+\sigma_s^2}) \approx 0)$ and/or relatively wide-band $((\frac{1}{{\sigma_p}^2+\sigma_s^2}) \ll \sigma_k^2)$, the general dispersion relation (7) becomes the usual one[8]

(10)
$$\sigma_x \sigma_k = 1$$

In cases (c)-(d) on the other hand, since $(\frac{1}{{\sigma_p}^2+\sigma_s^2}) \gg \sigma_k^2$ we have

(11)
$$\sigma_x = \sqrt{{\sigma_p}^2 + \sigma_s^2}$$

As can be seen from *Fig. 1*, in most pratical instances relations (9) and (10) do give the same result. The essential difference between the dispersion relation (9) and the usual one (10) lies in the fact that in the former one can have a source with pratically no dispersion ($\sigma_k \approx 0$) giving a finite pulse, whereas in the latter the size of the total pulse has to grow indefinitely in order to achieve the same low-dispersion regimes.

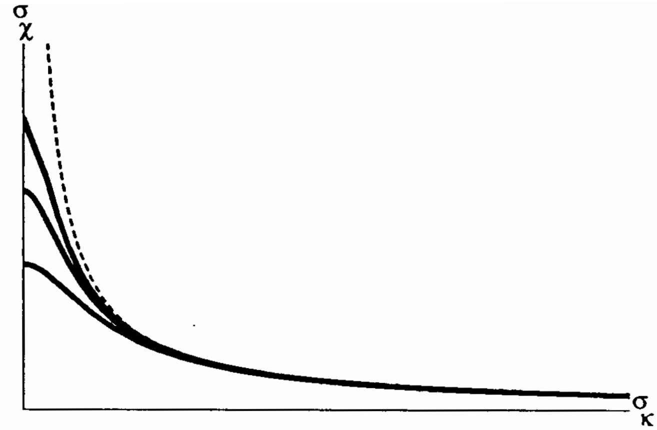

Fig. 1
Usual dispersion relation (dotted line) and generalized dispersion relation (full) for different values of $\frac{1}{\sigma_p^2+\sigma_s^2}$.

Propagation in Dispersive Medium

In the case of dispersive media, waves of different frequencies travel with different 'velocities', and therefore there will be some functional dependence $v = v(k)$ as well. Taking this into account, the remaining integral in (6) can be written as

(12) $$e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{+\infty} dk e^{-\frac{(x-vt)^2}{2(\sigma_p{}^2+\sigma_s{}^2)} - \frac{(k-k_0)^2}{2\sigma_k{}^2} + i((k-k_0)x - t(w-\omega_0))}$$

Introducing now the relations

(13) $$\omega = \frac{\hbar}{2m} k^2 \qquad ; \qquad v = \frac{\hbar}{m} k$$

that we assume valid for dispersive media, and making the following changes

(14) $$\begin{aligned} \kappa &= k - k_0 \\ \tau &= \frac{\hbar}{m} t \\ \sigma_0^2 &= \sigma_1^2 + \sigma_2^2 \\ \chi(x,\tau) &= x - k_0 \tau \end{aligned}$$

in the integrand for convenience, we get

(15) $$\int_{-\infty}^{+\infty} d\kappa e^{-\frac{\chi^2}{2\sigma_0^2} + \chi(i + \frac{\tau}{\sigma_0^2})\kappa - \frac{1}{2}(i\tau + \frac{1}{\sigma_k^2} + \frac{\tau^2}{\sigma_0^2})\kappa^2} =$$

$$e^{-\frac{1}{2}(\frac{1}{\sigma_0^2} + \frac{\beta(\tau)^2}{\alpha(\tau)})\chi^2} \int_{-\infty}^{+\infty} d\kappa e^{-\frac{1}{2}\alpha(\tau)(\kappa - \frac{\beta(\tau)}{\alpha(\tau)})^2} = \frac{\sqrt{2}\,\Gamma(\frac{1}{2})}{\alpha(\tau)^{1/2}} \; e^{-\frac{1}{2}(\frac{1}{\sigma_0^2} + \frac{\beta(\tau)^2}{\alpha(\tau)})\chi^2}$$

where

$$\beta(\tau) = i + \frac{\tau}{\sigma_o^2}$$

(16)

$$\alpha(\tau) = i\tau + \frac{1}{\sigma_k^2} + \frac{\tau^2}{\sigma_o^2}$$

After some work, the development of this expression leads to

(17) $$\Psi(x,t) = \frac{A}{\sqrt{d_3}} e^{-\frac{1}{2}(d_1 + i d_2)(x - \frac{m}{\hbar} v_0 \tau)^2} e^{i(k_0 x - \frac{m}{\hbar}\omega_0 \tau)}$$

where we have made

$$d_1 = \frac{\tau^2 \sigma_k^2 + \sigma_o^2 \left(1 + \sigma_k^2 \sigma_o^2\right)}{(\tau \sigma_k^2 \sigma_o^2)^2 + \left(\tau^2 \sigma_k^2 + \sigma_o^2\right)^2}$$

(18) $$d_2 = -\tau \sigma_k^2 \cdot \frac{\tau^2 \sigma_k^2 + \sigma_o^2 \left(2 + \sigma_k^2 \sigma_o^2\right)}{(\tau \sigma_k^2 \sigma_o^2)^2 + \left(\tau^2 \sigma_k^2 + \sigma_o^2\right)^2}$$

$$d_3 = \frac{\sqrt{(\tau \sigma_k^2 \sigma_o^2)^2 + \left(\tau^2 \sigma_k^2 + \sigma_o^2\right)^2}}{\sigma_k^2 \sigma_o^2}$$

Setting $\Delta x_0^2 = \frac{1}{\sigma_k^2 + \frac{1}{\sigma_o^2}}$, these formulas indicate that

(19) $$\sigma_x^2(\tau) = \Delta x_0^2 \cdot \frac{1 + \tau^4 (\frac{\sigma_k^2}{\sigma_o^2})^2 + \tau^2 \left(\frac{\sigma_k^2}{\sigma_o^2} + \frac{\sigma_k^2}{\Delta x_0^2}\right)}{1 + \tau^2 \frac{\sigma_k^2 \Delta x_0^2}{\sigma_0^4}}$$

gives the time-dependence of the total pulse's size. At $\tau = 0$ we obtain the value $\sigma_x^2(0)$ of the initial size of the pulse from

(20) $$\sigma_x^2(0) = \Delta x_0{}^2 = \frac{1}{\sigma_k^2 + \frac{1}{\sigma_0^2}}$$

Thus, the size of a pulse of superposing, finite waves travelling with different speeds starts out to be equal to (9), i.e. that of a pulse of waves propagating with the same velocity.

On the other hand, in the domain of large τ, expression (19) becomes

(21) $$\sigma_x(t) \approx \frac{\hbar}{m} t \sqrt{\frac{1}{\Delta x_0^2} - \frac{1}{\sigma_0^2}}$$

For reasonable coherence length sources, the previous expression reduces to

$$\sigma_x(t) \approx \frac{1}{\Delta x_0}\frac{\hbar}{m}t \tag{22}$$

which is the approximate spreading for the matter wave packet as presented inmost textbooks[9].

Conclusion

In conclusion, we wish to stress that, starting from very general assumptions of "particle" representation by finite, non-dispersive waves, it is possible to obtain a more general set of uncertainty/dispersion relations that avoid the shortcomings of the usual ones, which are valid for the standard Quantum Mechanics and follow from those derived in this paper. The question of the physical meaning of such relations remains, of course, open. In a situation of this kind only experiment can decide which is a better formula for the uncertainty relations.

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