Einstein-Cartan-Evans Unified Field Theory

The Geometrical Basis of Physics

Volume 1: Classical Physics

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In memoriam amici mei
Myron W. Evans
(1950 - 2019)

Dedicated to my heart sister
Momo Bähr
who gave me mystical insights into the universe

Contents

1. Introduction

1.1 Geometry throughout history

Geometry is visible everywhere in daily life. It appears both in natural objects and in those that have been engineered. Geometry has been used for centuries in art and architecture (Fig. 1.1), and it also plays an important role in pure sciences like mathematics and physics. The mathematical description of geometry consists of the logic elements of geometry itself, for example, the geometrical construction of triangles (Fig. 1.2). This type of logical treatment dates back to the beginning of recorded time, which is assumed be around 3500 B.C., when the first written documents appeared in Mesopotamia. For earlier times, we have to rely on documents of stone, like the pyramids in Egypt, which are probably much older than commonly assumed. The Pyramid of Cheops has been charted in detail, and correlations have been found to the circumference of the Earth, hinting that geometry had played an important role even in the Stone Age. At that time, Europe had a flourishing Celtic culture, from which numerous stone relics exist, and even the runes used by the druids were geometrical signs.

Ancient philosophy, in particular natural philosophy, culminated in Greece. Pythagoras is said to have been the first founder of mathematics, and the Pythagorean theorem is one of the first geometrical concepts that we learn. In Athens, where democracy was born, the "triumvirate" of Socrates, Plato and Aristotle founded classical philosophy, starting at around 400 B.C. Their schools were valid for about a thousand years. Euclid, who wrote the pivotal treatise on geometrical reasoning, *Elements*, was a member of the Platonic school.

During medieval times, knowledge from the Roman Empire was preserved by monasteries of the ecclesia and by Arabian philosophers and mathematicians. The Renaissance, which began in Italy in the 14th century and spread to the rest of Europe in the 15th and 16th centuries, was both a rebirth of ancient knowledge, and the beginning of modern empirical natural philosophy. This philosophy is connected with Galileo Galilei, who developed the method of experimental proofs, and Johann Kepler, who established our modern heliocentric model of the Solar System, which was first presented as a hypothesis by Nicolaus Copernicus.

Since the 17th century, there has been great progress in the mathematical description of physics. Isaac Newton published the law of gravitation, which actually goes back to his mentor Robert

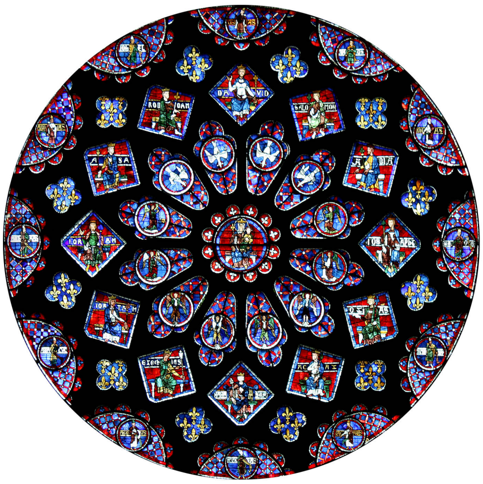


Figure 1.1: Example of geometry - rosette window in the Chartres Cathedral.

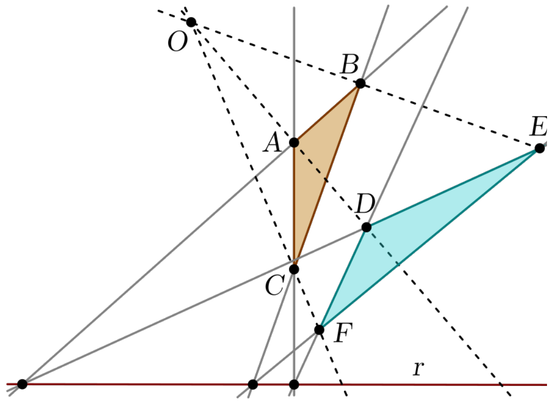


Figure 1.2: Example of geometry - construction of triangles.

Hooke. This represented a significant advancement in natural philosophy, because celestial events could now be predicted mathematically, although this has become completely feasible only since the advent of computers.

The 18th century was the golden age of mechanics. Newton's laws, and their generalizations by Lagrange and Hamilton, paved the way for mathematical physics for the next 300 years. Initially, geometry was used mainly for describing the motion of bodies and particles. The emergence of quantum mechanics in the 20th century extended geometry to the atomic and subatomic realm. For example, the atoms that comprise solids and molecules exhibit a geometrical structure (Fig. 1.3) that is essential for their macroscopic properties. Similar arguments hold for electrodynamics (see, e.g., Fig. 1.4). Faraday's lines of force describe a close-range effect, which was a basis for the geometrical description of electrodynamics, culminating in Maxwell's equations.

The use of geometry changed again at the beginning of the 20th century, when Einstein introduced his theory of general relativity, in which he based physics on non-Euclidean geometry. Gravitation was no longer described by a field imposed externally on space and time, but instead the "spacetime" itself was considered to be an object of description, and altered so that force-free bodies move on a virtually straight line (geodetic line) through space. Spacetime was considered

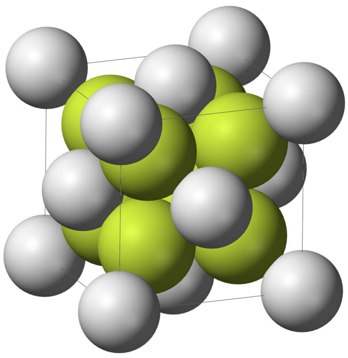


Figure 1.3: Example of geometry - unit cell of Fluorite crystal. White: Calcium; green: Fluorine.

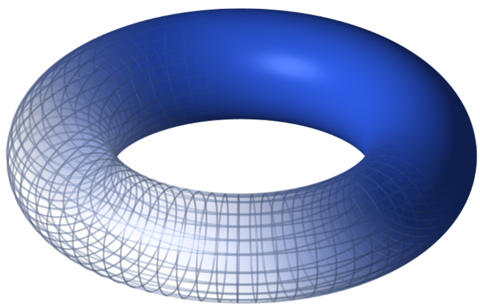


Figure 1.4: Example of geometry - torus.

to be curved, and the curving described the laws of gravitation. In this interpretation, geometry is considered to be an abstract concept described by numbers and mathematical functions. This approach is known as analytical geometry, and its simplest form uses coordinate systems and vectors.

Einstein's geometrical concept was the first paradigm shift in physics since Newton introduced his laws of motion, more than two hundred years before. Attempts at experimental validation of Einstein's general relativity have been rare, and have mainly concerned the Solar System, like the deflection of light by the Sun and the precession of the orbit of Mercury. In spite of this limitation, the theory was taken as a basis for cosmology, from which the Big Bang and dark matter were later extrapolated.

Unfortunately, this approach to cosmology has introduced self-contradictory inconsistencies. For example, the concept that the speed of light is an absolute upper limit is incompatible with the fact that, to explain the first expansive phase of the universe, one has to assume that this happened with an expansion velocity faster than the speed of light. This example is only one of the criticisms of Einstein that have yet to be answered properly and scientifically.

Later, after Einstein's death, the velocity curve of galaxies was observed by astronomers. They observed that stars in the outer arms of galaxies do not move according to Newton's law of gravitation, but have a constant velocity. However, Einstein's theory of general relativity is not able to explain this behavior. Both of these theories break down in cosmic dimensions, and when a theory does not match experimental data, the scientific method requires that the theory be improved or replaced with a better concept. In the case of galactic velocity curves, however, it was "decided" that Einstein was right and that there had to be another reason why stars behave in this way. Dark matter that interacts through gravity and is distributed in a way that accounts for observed orbits was then postulated. Despite an intensive search for dark matter, even on the sub-atomic level, nothing has been found that could interact with ordinary matter through gravity, but not interact with observable electromagnetic radiation, such as light. Although Einstein's theory has become increasingly problematic, the scientific community has been reluctant to abandon it.

Starting around 2001, the members of the AIAS institute, with Myron Evans as the guiding researcher, developed a new theory of physics, Einstein-Cartan-Evans (ECE) theory [1–5], which overcomes the problems in Einstein's general relativity. They were even able to unify this new theory with electrodynamics and quantum mechanics. This enabled significant progress in several fields of physics, and the most important aspects are described in this textbook.

ECE theory is based entirely on geometry, as was Einstein's general theory of relativity. Therefore, Einstein is included in the name of this new theoretical approach. Both theories take the geometry of spacetime (three space dimensions, plus one time dimension) as their basis. While Einstein thought that matter curves spacetime and assumed matter to be a "source" of fields, we will see that ECE theory is based entirely on the field concept and does not need to introduce external sources. This idea of sources created a number of difficulties in Einstein's theory.

Another reason for these difficulties is that Einstein made an unavoidable but significant mathematical error in his original theory (1905 to 1915), because all of the necessary information was not yet available. Riemann inferred the metric around 1850, and Christoffel inferred the connection around the 1860s. The idea of curvature was introduced at the beginning of the twentieth century, by Levi-Civita, Ricci, Bianchi and colleagues in Pisa. However, the concept of torsion was not fully developed until the early 1920s, when Cartan and Maurer introduced the first structure equation.

Therefore, in 1915, when Einstein published his field equation, Riemann geometry did not include torsion, and there was no way of determining that the Christoffel connection must be antisymmetric. The connection was assumed to be symmetric (probably for ease of calculation), and the inferences of Einstein's theory ended up being based on incorrect geometry. Setting torsion to zero and using a symmetric connection leads to a contradiction with significant consequences, as has been shown by the AIAS Institute, in great detail [6].

Torsion (which is a twisting of space) turns out to be essential and inextricably linked to curvature, because if the torsion is zero then the curvature vanishes [6]. In fact, torsion is even more important than curvature, because the unified laws of gravitation and electrodynamics are basically physical interpretations of twisting, which is formally described by the torsion tensor.

ECE theory unifies physics by deriving all of it directly and deterministically from Cartan geometry, and does so without using adjustable parameters in the foundational axioms. The parameters that combine geometry with physics are derived from experimental data and are thus not arbitrarily adjustable. Spacetime is completely specified by curvature and torsion, and ECE theory uses these underlying fundamental qualities to derive all of physics from differential geometry, and to predict quantum effects without assuming them (as postulates) from the beginning. It is the first (and only) generally covariant, objective and causal unified field theory.

Instead of using Einstein's field equation, ECE theory uses the equations of Cartan geometry, which can be written in a form equivalent to Maxwell's equations, for electrodynamics as well as for gravitation. By comparing this form of the Cartan equations with Maxwell's original equations (which include charges and currents), one can define charge and current terms that consist of field terms combined with curvature and torsion terms. The same can be done for gravitation, and this allows unification to happen via geometry. If a charge is present, we have electromagnetism, if not, we have only gravitation. In ECE theory, there are no sources a priori, only fields. Matter is considered to be a "condensed field" of general relativity, and spacetime itself may be interpreted as a vacuum or aether field that exists everywhere. Matter as condensed fields leads directly to quantum mechanics, and avoids extra concepts like quantum electrodynamics.

This book first introduces the mathematics on which ECE theory is based, so that the foundations of the theory can be explained systematically. Mathematical details are kept to a minimum, and explained only as far as is necessary to ensure understanding of the underlying Cartan geometry. This allows the fundamental ECE axioms and theorems to be introduced in a simple and direct manner. The same equations are shown to hold for electrodynamics, gravitation, mechanics and fluid dynamics, which places all of classical physics on common ground. In the second volume of this textbook, physics is extended to the microscopic level by introducing canonical quantization and quantum geometry. The quantum statistics that is used is classically deterministic, thus there is no need for renormalization and quantum electrodynamics. All known effects, up to and including the structure of the vacuum, can be explained within the ECE axioms, which are based on Cartan geometry. This is the great advance that this textbook will explain and clarify.

1.2 Table of ECE development

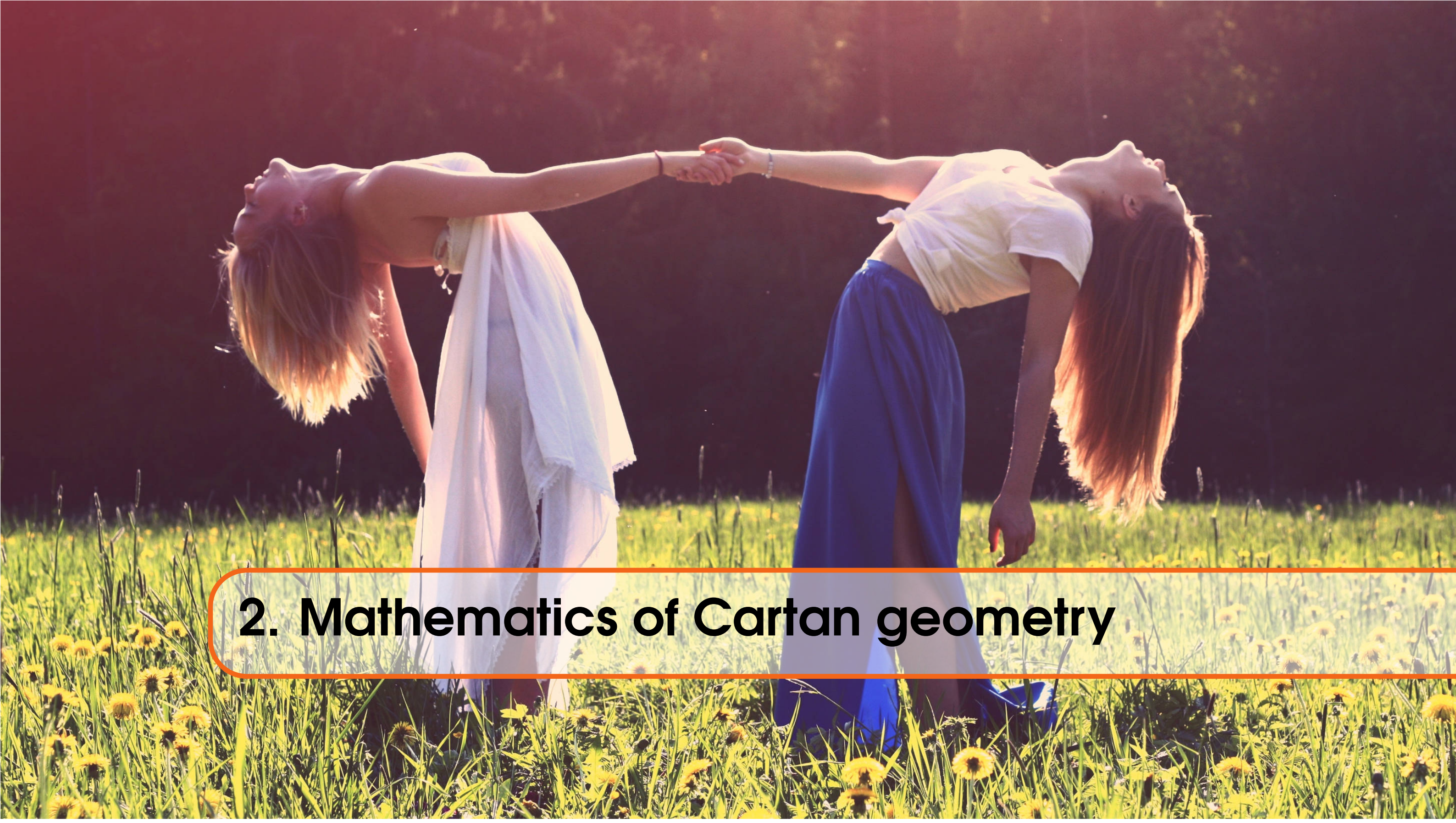
Myron Evans developed the B(3) field and O(3) electrodynamics in the 1980s and 90s. He started developing the formal foundations of ECE theory around 2001, using Cartan geometry as the basis. A complete presentation of early research, starting with 1973, can be found in the Omnia Opera Section of the AIAS website [28]. Significant milestones in ECE theory, starting with the first publications in 2003 and continuing to the present day, are listed in Table 1.1.

In addition, a summary that shows how all referenced areas of physics are based on the same equations of geometry is available at [7].

Year	Development	First Reference
2003	First ECE version	Published in [1]
2006	Spin connection resonance	UFT Paper 61
2007	Explanation of whirlpool galaxy structures	UFT Paper 76
2007	Criticism of the Einstein field equation	UFT Paper 96
2008	Explanation of the cosmological red shift	UFT Paper 118
2009	Theory of the fermion	UFT Paper 129
2009	Antisymmetry law of potentials	UFT Paper 131
2009	Classical ECE vacuum	UFT Paper 292
2010	Light deflection and photon mass	UFT Paper 150
2011	Quantum-Hamilton equations	UFT Paper 175
2011	x theory of precession	UFT Paper 192
2012	Proof of antisymmetry of Christoffel connections	UFT Paper 209
2012	Refutation of standard electroweak theory	UFT Paper 225
2012	Explanation of Low Energy Nuclear Reaction (LENR)	UFT Paper 226
2013	New Resonance Spectroscopies from the ECE Fermion Equation	UFT Paper 251
2014	Beltrami fields in ECE theory	UFT Paper 257
2015	ECE2 theory	UFT Paper 315
2015	ECE2 orbital precession from the Lagrangian of special relativity	UFT Paper 325
2016	ECE2 vacuum	UFT Paper 337
2016	General theory of precessions	UFT Paper 346
2016	ECE2 fluid electrodynamics	UFT Paper 349
2017	Analytical mechanics of the gyroscope	UFT Paper 367
2017	ECE2 vacuum fluctuations of E and B fields	UFT Paper 392
2018	Generally relativistic m theory of gravitation	UFT Paper 415
2019	Relativistic quantum m theory	UFT Paper 428
2020	Light deflection in m theory	UFT Paper 442
2022	Intrinsic structure of fields	UFT Paper 447

Table 1.1: Historical development of ECE theory.

Part One: Geometry



2. Mathematics of Cartan geometry

2.1 Coordinate transformations

Before we can discuss the foundations of non-Cartesian and Cartan geometry on a mathematical level, we need to review the basics of analytical geometry.

2.1.1 Coordinate transformations in linear algebra

To start our discussion of analytical geometry, we will first review a few basic concepts from linear algebra. Cartan geometry is a generalization of these concepts, in a sense. Points in space are described by coordinates that are n-tuples for an n-dimensional vector space. The tuple components are numbers that describe how a point in space is reached by combining specific motions in different directions. The directions are called basis vectors. For a three-dimensional Euclidian space, we have the basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1}$$

A point with coordinates (X,Y,Z) is specified by the vector

$$\mathbf{X} = X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3. \tag{2.2}$$

We have the freedom to choose any basis in a vector space, rectangular or not, but when vector analysis is applied to the vector space, it is beneficial to have a rectangular basis. The basis vectors have to be normalized so that the basis will be orthonormal (consist of an orthogonal set of unit vectors).

A question arises as to what happens when the basis vectors are changed. The position of points in the vector space should be independent of the basis, and we will encounter this fundamental requirement often in Cartan geometry. The coordinates will change when the basis changes. An important part of linear algebra deals with describing this mathematically. Taking the above basis vectors $\mathbf{e}_i$, a new basis $\mathbf{e}'_i$ in an n-dimensional vector space will be a linear combination of the original basis:

$$\mathbf{e'}_i = \sum_{j=1}^{n} q_{ij}\mathbf{e}_j, \tag{2.3}$$

where the coefficients q_{ij} represent a matrix that is commonly called the *transformation matrix*. Equation (2.3) can therefore be written as a matrix equation:

$$\begin{pmatrix} \mathbf{e'}_1 \\ \vdots \\ \mathbf{e'}_n \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_n \end{pmatrix} \tag{2.4}$$

with

$$\mathbf{Q} = (q_{ij}), \tag{2.5}$$

and the unit vectors formally arranged as a column vector. $\mathbf{Q}$ must be of rank n and invertible. In Eq. (2.4) the unit vectors can be written with their components as row vectors, and if we denote the j$^{\text{th}}$ component of the unit vector $\mathbf{e}_i$ by $(e_i)_j = e_{ij}$, we can set up a matrix from the unit vectors and write (2.4) in the following form:

$$\begin{pmatrix} e'_{11} & \dots & e'_{1n} \\ \vdots & & \vdots \\ e'_{n1} & \dots & e'_{nn} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} e_{11} & \dots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \dots & e_{nn} \end{pmatrix}. \tag{2.6}$$

The basis transformation then becomes a matrix multiplication by $\mathbf{Q}$. The matrix for the inverse transformation is obtained by multiplying (2.4) or (2.6) by the inverse matrix $\mathbf{Q}^{-1}$:

$$\begin{pmatrix} e_{11} & \dots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \dots & e_{nn} \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} e'_{11} & \dots & e'_{1n} \\ \vdots & & \vdots \\ e'_{n1} & \dots & e'_{nn} \end{pmatrix}. \tag{2.7}$$

Multiplying $\mathbf{Q}$ by $\mathbf{Q}^{-1}$ gives the unit matrix, which can be expressed by the Kronecker symbol:

$$\mathbf{Q}\,\mathbf{Q}^{-1} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix} = (\delta_{ij}). \tag{2.8}$$

■ **Example 2.1** The rotation of bases by an angle ϕ in a two-dimensional vector space can be described by the rotation matrix

$$\mathbf{Q} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{2.9}$$

The basis of unit vectors $(1,0)$, $(0,1)$ is then transformed into the new basis vectors

$$\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \tag{2.10}$$

which means that

$$\mathbf{e'}_1 = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \mathbf{e'}_2 = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}. \tag{2.11}$$

Both basis sets are depicted in Fig. 2.1. ■

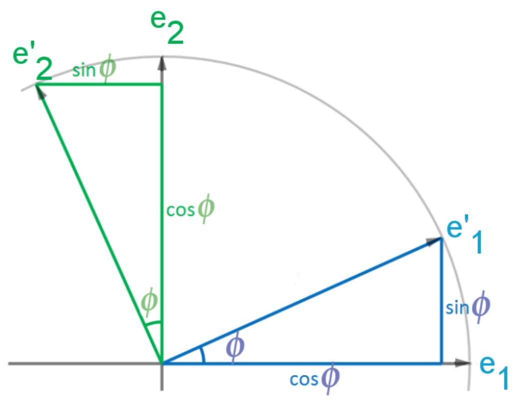


Figure 2.1: Basis vector rotation by an angle ϕ.

Now that we understand the basis transformation, we want to find the transformation law for vectors that transforms the components (the coordinates) of vectors in one basis into the components in another basis. From the definition, (2.2), a vector with coordinates x_i can be written as

$$\mathbf{X} = \sum_i x_i \mathbf{e}_i, \tag{2.12}$$

and may be transformed into a representation in a second basis with coordinates x_i':

$$\mathbf{X}' = \sum_i x_i' \mathbf{e}'_i. \tag{2.13}$$

Since the vector should remain the same in both bases, we can set $\mathbf{X} = \mathbf{X}'$. Inserting the basis transformations into this relation shows us that the transformation law of coordinates is

$$\mathbf{X}' = \mathbf{Q}^{-1}\,\mathbf{X} \tag{2.14}$$

or, in coordinates:

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{2.15}$$

Please notice that in this important result the coordinates transform with the *inverse* matrix (in contrast to the basis vectors, see (2.4)). and vice versa.

■ **Example 2.2** The transformation matrix of coordinates for the rotation in two dimensions is

$$\mathbf{Q}^{-1} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{2.16}$$

This can be seen easily, because the reverse rotation is by an angle $-\phi$. The sine function reverses sign but the cosine function does not, and the vectors on the basis axes are transformed in the following way:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix}. \tag{2.17}$$

■

By comparing this with (2.9), we see that the columns of the transformation matrix **Q** represent the coordinates of the transformed unit *vectors*, not the *basis*. In general:

$$\begin{pmatrix} x_{11} & \dots & x_{n1} \\ \vdots & & \vdots \\ x_{1n} & \dots & x_{nn} \end{pmatrix} = \mathbf{Q}^{-1} \begin{pmatrix} 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \tag{2.18}$$

where $x_{ij} = (x_i)_j$ denotes the j$^{\text{th}}$ component of the transformed unit vector $\mathbf{e}_i$. Please notice that the index scheme for x_{ij} is transposed, compared to the usual matrix definition.

2.1.2 General coordinate transformations and coordinate differentials

In the framework of general relativity, coordinate transformations are mappings from one vector space to another, and these mappings are multidimensional functions. In the preceding section we restricted ourselves to linear transformations (or mappings), while in general relativity we operate with nonlinear transformations.

Space is described by a four-dimensional manifold, using advanced mathematics. However, in this book we do not develop these concepts to any great extent, but only explain the parts that are required for a basic understanding. The mathematical details can be found in textbooks on general relativity, for example, see [8–12].

We will use one time-coordinate plus three space-coordinates for general relativity, with indices numbered from 0 to 3. Such vectors are also called *4-vectors*. The functions and maps (and later, tensors) defined in this space are functions of the coordinates: $f(x_i)$, i=0...3, and this form is particularly convenient for describing coordinate transformations. Consider two coordinate systems A and B that describe the same space and are related by a nonlinear transformation. Let X_i be the components of a 4-vector **X** in system A, and Y_i be the components of a 4-vector **Y** in system B. The coordinate transformation function $f : \mathbf{X} \to \mathbf{Y}$ can then be expressed as a functional dependence of the components:

$$Y_i = f_i(X_j) = Y_i(X_j), \tag{2.19}$$

for all components i of f and all pairs i, j. We will now consider the transformations between a rectangular, orthonormal coordinate system, defined by basis vectors $(1,0,...), (0,1,...)$, etc., and coordinates

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}, \tag{2.20}$$

and a curvilinear coordinate system with coordinates

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{2.21}$$

The transformation functions from the curvilinear to the cartesian coordinate system is defined by

$$X_i = X_i(u_j), \tag{2.22}$$

as discussed above. The inverse transformations define the *coordinate functions* of u:

$$u_i = u_i(X_j). \tag{2.23}$$

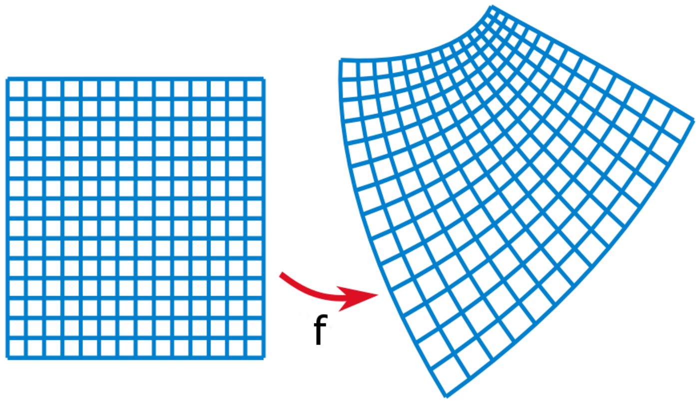


Figure 2.2: Transformation to curvilinear coordinates.

The functions $u_i =$ constant define *coordinate surfaces* (see Fig. 2.2, for example).

The degree of change in each direction is given by the change of arc length and is expressed by the *scale factors*

$$h_i = \left| \frac{\partial \mathbf{X}}{\partial u_i} \right|. \tag{2.24}$$

The unit vectors in the curvilinear space are computed using

$$\mathbf{e}_i = \frac{1}{h_i} \frac{\partial \mathbf{X}}{\partial u_i}. \tag{2.25}$$

The *tangent vector* of the coordinate curves at each point of space is defined by

$$\nabla u_i = \sum_j \frac{\partial u_i}{\partial X_j} \mathbf{e}_j. \tag{2.26}$$

We want the curvilinear coordinate system to be orthonormal at each point of space, and we can guarantee this by requiring that the tangent vectors of the coordinate curves at each point satisfy the condition

$$\nabla u_i \cdot \nabla u_j = \delta_{ij}. \tag{2.27}$$

Also, as an alternative to (2.24), the scale factors can be expressed by the modulus of the tangent vector:

$$h_i = \frac{1}{|\nabla u_i|}. \tag{2.28}$$

■ **Example 2.3** We consider the transformation from cartesian coordinates to spherical coordinates in Euclidean space. The curvilinear coordinates of a point in space are (r, θ, ϕ), where r is the radius, θ the polar angle and ϕ the azimuthal angle (see Fig. 2.3), and the cartesian coordinates are (X, Y, Z). The transformation equations from the curvilinear to the rectangular coordinate system are

$$\begin{aligned} X &= r \sin\theta \cos\phi \\ Y &= r \sin\theta \sin\phi \\ Z &= r \cos\theta \end{aligned} \tag{2.29}$$

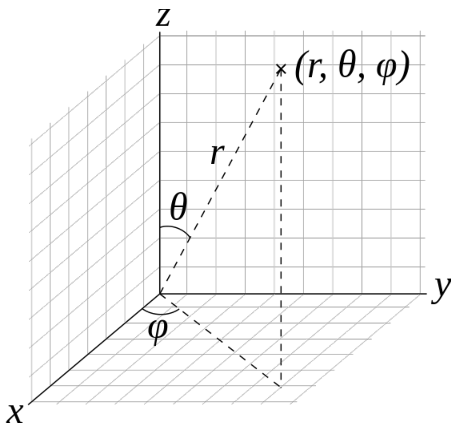


Figure 2.3: Spherical polar coordinates [179].

and the inverse transformations are

$$\begin{aligned} u_r &= r = \sqrt{X^2+Y^2+Z^2} \\ u_\theta &= \theta = \arccos\frac{Z}{\sqrt{X^2+Y^2+Z^2}} \\ u_\phi &= \phi = \arctan\frac{Y}{X}. \end{aligned} \tag{2.30}$$

The vector of scale factors (2.24) is

$$\mathbf{h} = \begin{pmatrix} 1 \\ r \\ r\sin\theta \end{pmatrix}, \tag{2.31}$$

and the matrix of column unit vectors (2.25) is

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \begin{pmatrix} \cos\phi\,\sin\theta & \sin\phi\,\sin\theta & \cos\theta \\ \cos\phi\,\cos\theta & \sin\phi\,\cos\theta & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}. \tag{2.32}$$

The components of **h** have to be positive. The sine function may have positive and negative values, but in spherical coordinates the range of θ is between 0 and π, and therefore this function is always positive. ■

In orthonormal coordinate systems, the length of vectors is conserved. This must also hold for time-dependent processes in 4-space. For example, a distance vector changing over time is

$$\Delta\mathbf{X} = \mathbf{v}\Delta t - \mathbf{X}_0, \tag{2.33}$$

where **v** is the velocity vector of a mass point and $\mathbf{X}_0$ is an offset. The squared distance is

$$s^2 = v^2\Delta t^2 - (\mathbf{X}_0)^2. \tag{2.34}$$

Notice that a minus sign appears in front of the space part of s^2. This is different from pure "static" Euclidean 3-space, where we have

$$s_E^2 = X^2 + Y^2 + Z^2. \tag{2.35}$$

We will now generalize Eq. (2.34). When the differences in time as well as in space between two points are infinitesimally small, we can write the distance between these points using coordinate differentials:

$$ds^2 = c^2\, dt^2 - dX^2 - dY^2 - dZ^2, \tag{2.36}$$

where ds is the differential line element. We have added a factor c to the time coordinate t so that all coordinates have the physical dimension of length. The signs of the time and space differentials are different. This is a consequence of the coordinate-independence of the line element as required by special relativity. Therefore, we can express the same line element in another coordinate system, for example, one with coordinates u:

$$ds^2 = (du_0)^2 - (du_1)^2 - (du_2)^2 - (du_3)^2. \tag{2.37}$$

So far, we have dealt with a Euclidian 3-space, augmented by a time component. More generally, the above equations can be written in the following form:

$$ds^2 = \sum_{ij} \eta_{ij}\, dx_i\, dx_j, \tag{2.38}$$

where η_{ij} represents a matrix of constant coefficients that lead directly to (2.36) or (2.37):

$$(\eta_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.39}$$

Formally, we can write the coordinates as a 4-column vector

$$(x^\mu) = \begin{pmatrix} ct \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}, \tag{2.40}$$

where μ runs from 0 to 3 and is written as an upper index. We can do the same for the coordinate differentials:

$$(dx^\mu) = \begin{pmatrix} c\,dt \\ dX^1 \\ dX^2 \\ dX^3 \end{pmatrix}. \tag{2.41}$$

At this point, we should notice that the determinant of the matrix (2.39) is -1. The η matrix is called the *metric* of the space, here the time-extended flat Euclidean space, which is also called *Minkowski space*. We see that the metric is negative definite. Sometimes η is defined with reverse signs, but the result is the same. At this point, we enter the realm of special relativity, but we need not deal with Lorentz transformations in this book. Since the spacetime metric is an essential physical quantity in ECE theory as well as in general relativity, we will introduce special relativity, but only from the viewpoint that the line element ds is independent of the coordinate system. Later we will see that this leads to the gamma factor of special relativity. This is the only formalism that Einstein's relativity and ECE theory have in common. We will come back to this when physical situations are considered where very high velocities occur, because they require a relativistic treatment (in the sense of special relativity).

2.1.3 Transformations in curved spaces

To this point, our linear algebra discussion has been limited to Euclidean spaces, but we will now extend the concepts of the preceding section to curved spaces. In these spaces, equidistant coordinate values do not describe line elements equal in length, but please note that this is not evidence that space is "curved" in any way. As we have seen in Example 2.1 in the preceding section, a curvilinear coordinate system can describe a Euclidean "flat" space perfectly.

We now consider two coordinate systems existing in the same space, denoted by primed and un-primed differentials dx and dx'. According to Eqs. (2.22, 2.23), we have a functional dependence between both coordinates:

$$x^\mu = x^\mu(x'^\nu) \tag{2.42}$$

and

$$x'^\nu = x'^\nu(x^\mu). \tag{2.43}$$

Differentiating these equations gives

$$dx'^\mu = \sum_\nu \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \tag{2.44}$$

$$dx^\mu = \sum_\nu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu. \tag{2.45}$$

To make these equations similar to the transformations in linear algebra (see Section 2.2.2), we define transformation matrices

$$\alpha^\mu_\nu = \frac{\partial x^\mu}{\partial x'^\nu}, \tag{2.46}$$

$$\overline{\alpha}^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}, \tag{2.47}$$

so that any vector V with components V^μ in one coordinate system can be transformed into a vector V' in the other coordinate system in the following way:

$$V'^\mu = \alpha^\mu_\nu\, V^\nu, \tag{2.48}$$

$$V^\mu = \overline{\alpha}^\mu_\nu\, V'^\nu. \tag{2.49}$$

These matrices, however, are matrix functions and not elements of linear algebra, because we are not working with linear transformations. α is the inverse matrix function of $\overline{\alpha}$ and vice versa. This means that

$$\sum_\rho \alpha^\mu_\rho\, \overline{\alpha}^\rho_\nu = \delta^\mu_\nu \tag{2.50}$$

with the Kronecker delta

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu, \\ 0 & \text{if } \mu \neq \nu. \end{cases} \tag{2.51}$$

Here we have written α with an upper and lower index intentionally. This allows us to introduce the *Einstein summation convention*: if the same index appears as an upper and a lower index on one side of an equation, this index is summed over. Such an index is also called a *dummy index*. We will use this feature extensively, when tensors are introduced later. With this convention, which we will use without notice in the future, we can write

$$\alpha^\mu_\rho \overline{\alpha}^\rho_\nu = \delta^\mu_\nu . \tag{2.52}$$

Since space is not necessarily flat, the metric coefficients of (2.39) are not constant, and non-diagonal terms may appear. This general metric is conventionally called $g_{\mu\nu}$ and defined by the line element as before:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu . \tag{2.53}$$

For flat spaces with cartesian coordinates, we have:

$$g_{\mu\nu} = \eta_{\mu\nu} . \tag{2.54}$$

■ **Example 2.4** In this example, we compute a representative transformation matrix. From Example 2.3 (transformation between cartesian coordinates and spherical polar coordinates), using Eqs. (2.29) and (2.30), we obtain

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta , \end{aligned} \tag{2.55}$$

and the inverse transformations

$$\begin{aligned} x'^1 &= r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2} \\ x'^2 &= \theta = \arccos\frac{x^3}{\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}} \\ x'^3 &= \phi = \arctan\frac{x^2}{x^1} . \end{aligned} \tag{2.56}$$

According to (2.46), the transformation matrix is

$$\begin{aligned} \alpha^1_1 &= \frac{\partial x^1}{\partial x'^1} = \frac{\partial}{\partial r}(r\sin\theta\cos\phi) = \sin\theta\cos\phi \\ \alpha^1_2 &= \frac{\partial x^1}{\partial x'^2} = \frac{\partial}{\partial \theta}(r\sin\theta\cos\phi) = r\cos\theta\cos\phi \\ &\dots , \end{aligned} \tag{2.57}$$

which gives us the 3x3 matrix

$$\boldsymbol{\alpha} = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} . \tag{2.58}$$

Obviously, this matrix is not symmetric and even has a zero on the main diagonal. Nonetheless, it is of rank 3 and invertible, as can be checked. We omit the details here, since the inverse matrix is a bit complicated. The determinant of α is $r^2\sin\theta$, and the determinant of the inverse matrix $\overline{\alpha}$ is $1/(r^2\sin\theta)$. By insertion, one can check that

$$\boldsymbol{\alpha}\cdot\overline{\boldsymbol{\alpha}} = \mathbf{1} . \tag{2.59}$$

This example is available as code for the computer algebra system Maxima [117]. ■

■ **Example 2.5** Continuing with the same theme, we will compute the metric of the coordinate transformation from the previous example (also see computer algebra code [118]). So far, we have not identified a formal method for doing this. For Euclidean spaces, the simplest method is the one going back to Gauss. If the metric **g** (a matrix) is known for one coordinate system x^μ, the invariant line element of a surface (which is hypothetical in our case) is given by

$$ds^2 = \begin{bmatrix} dx^1 dx^2 dx^3 \end{bmatrix} \mathbf{g} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}. \tag{2.60}$$

The metrical matrix belonging to another coordinate system x'^μ can then be computed by

$$\mathbf{g}' = \mathbf{J}^T \, \mathbf{g} \, \mathbf{J}, \tag{2.61}$$

where **J** is the Jacobian of the coordinate transformation:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^1}{\partial x'^3} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} & \frac{\partial x^2}{\partial x'^3} \\ \frac{\partial x^3}{\partial x'^1} & \frac{\partial x^3}{\partial x'^2} & \frac{\partial x^3}{\partial x'^3} \end{bmatrix}. \tag{2.62}$$

By comparing this with Eq. (2.57), we see that the transformation matrix α is identical to the Jacobian, so we can also write

$$\mathbf{g}' = \boldsymbol{\alpha}^T \, \mathbf{g} \, \boldsymbol{\alpha}. \tag{2.63}$$

The metric of the cartesian coordinates is simply

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.64}$$

and can be inserted into (2.63), together with $\boldsymbol{\alpha}$ from the preceding example. The result is

$$\mathbf{g}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix}, \tag{2.65}$$

which is the metric of the spherical coordinates. When written as the line element, this becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{2.66}$$

The metric is symmetric in general, and diagonal in most relevant cases. We will learn other methods of determining the metric in curved spaces as we proceed through this book. ■

2.2 Tensors

Now that we have explored coordinate transformations and their matrix representations, including the metric, we will extend this formalism from vectors to tensors. First, we have to define what tensors are, and then we can see how they are transformed.

In Section 1.1.3 we introduced the formalism of writing matrices and vectors by indexed quantities, with an upper or lower index, where this position was chosen more or less arbitrarily, for example, to satisfy the Einstein summation convention. We now introduce k-dimensional objects (k ranging from 0 to any integer number) with upper and lower indices of the form

$$T^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m}. \tag{2.67}$$

T has n upper indices μ_i and m lower indices ν_i with $n+m=k$. It is not required that all upper indices appear first, for example,

$$T^{3\ \ 30}_{\ 1} \tag{2.68}$$

is a valid object. The indices μ_i,ν_i represent the coordinate indices for each dimension, ranging from 0 to k-1 by definition. In the above example we have $k=4$, so

$$T^{3\ \ 40}_{\ 5} \tag{2.69}$$

would not be a valid object. For $k=2$ such an object represents a matrix, for $k=1$ a vector, and for $k=0$ (without index) a scalar value. A tensor is defined by objects of type (2.67), which follow certain transformation rules for the upper and lower indices. Given a coordinate transformation α^μ_ρ between two coordinate systems, this transformation has to be applied to each index of a tensor separately. For example, a 2-dimensional tensor T may be transformed into T' by

$$T'^{\mu\nu} = \alpha^\mu_\rho\ \alpha^\nu_\lambda\ T^{\rho\lambda}. \tag{2.70}$$

Furthermore, for lower indices, we require the use of inverse transformation matrices:

$$T'_{\mu\nu} = \overline{\alpha}^\rho_\mu\ \overline{\alpha}^\lambda_\nu\ T_{\rho\lambda} \tag{2.71}$$

and, consequently, for mixed cases:

$$T'^{\mu}{}_{\nu} = \alpha^\mu_\rho\ \overline{\alpha}^\lambda_\nu\ T^{\rho}{}_{\lambda}\ . \tag{2.72}$$

Please notice that the α matrices are defined by the differentials of the transformation, see Eqs. (2.44, 2.45).

In Section 2.1.1 we have seen that, if α^μ_ρ transforms the basis vectors, then the inverted matrix $\overline{\alpha}^\nu_\lambda$ transforms the coordinates of vectors. Therefore, the upper indices of tensors transform like coordinates, while the lower indices transform like the basis. Upper indices are also called *contravariant indices*, while lower indices are called *covariant indices*. A tensor containing both types of indices is called a *mixed index tensor*.

We conclude this section with the hint that the metric introduced in the previous section is also a tensor. It would be mathematically more precise to then restrict all tensors to live in metric spaces, but we will include that level of mathematical detail only when it is directly relevant. The metric $g_{\mu\nu}$ in curved spaces is a symmetric matrix and a tensor of dimension 2. The inner product of two vectors v, w can be written with the aid of the metric:

$$s = g_{\mu\nu}\ v^\mu w^\nu. \tag{2.73}$$

In Euclidean space with cartesian coordinates, g is the unit matrix as shown in Example 2.5. Indices of arbitrary tensors can be moved up and down via the relations

$$T^{\mu}{}_{\nu} = g_{\nu\rho}\ T^{\mu\rho} \tag{2.74}$$

and

$$T^{\mu}{}_{\nu} = g^{\mu\rho}\ T_{\rho\nu}\ , \tag{2.75}$$

where $g^{\mu\rho}$ is the inverse metric:

$$g^{\mu\rho} g_{\nu\rho} = \delta^{\mu}{}_{\nu}\ . \tag{2.76}$$

■ **Example 2.6** We present two tensor operations. Tensors can be multiplied, and this gives us a product that has the union set of indices, for example:

$$A^{\mu\nu} B_\rho = C^{\mu\nu}{}_\rho . \tag{2.77}$$

The order of multiplication of A and B plays a role. Therefore, such a product is only meaningful for tensors with a certain symmetry, for example, the product tensor of two vectors:

$$v^\mu w^\nu = C^{\mu\nu}. \tag{2.78}$$

Here C is a symmetric tensor, i.e.,

$$C^{\mu\nu} = C^{\nu\mu}. \tag{2.79}$$

Only tensors with the same rank can be added:

$$A^\mu{}_\nu + B^\rho{}_\sigma = C^\alpha{}_\beta . \tag{2.80}$$

The equation

$$A^\mu{}_\nu + B^{\rho\sigma}{}_\tau \ =? \quad C^{\alpha\beta}{}_\tau \tag{2.81}$$

violates the rank requirement and is therefore undefined. For a detailed discussion of tensors, see [8]. ■

2.3 Base manifold and tangent space

Now that we have seen an overview of the tensor formalism, we will consider the spaces on which these tensors are operating. A tensor can be considered as a function, for example:

$$T^\mu{}_\nu \ : \mathbb{R}^4 \to \mathbb{R}^2, \tag{2.82}$$

which maps a 4-vector to a two-dimensional tensor field:

$$[ct, X, Y, Z] \to T^\mu{}_\nu \ (ct, X, Y, Z), \tag{2.83}$$

where the two indices of the tensor indicate that the image map is two-dimensional. We speak of a "tensor field" in cases where a continuous argument range is mapped to a continuous image range that is different from the argument set. For example, T could be an electromagnetic field that is defined at each point of 4-space. If the set of arguments is not Euclidean, we require that, at each point of the argument set, a local neighborhood exists that is homomorphous to an open subset of $\mathbb{R}^n$ where n is the dimension of the argument set. This is then called a *manifold*. Applying multiple tensor functions to a manifold means that several *maps* of the manifold exist. It is further required that the manifold is differentiable, because we want to apply the differential calculus later. Assume that a point P is located within the valid local range of two different coordinate systems. The manifold is then differentiable in P, if the Jacobian of the transformation between both coordinate systems is of rank n, with n being the dimension of the manifold. For definitions of scalar products, lengths, angles and volumes we need a metric structure for "measurements", and this requires the existence of a metric tensor. A differentiable manifold with a metric tensor is called a *Riemannian manifold*.

■ **Example 2.7** In Fig. 2.4 an example of a 2-dimensional manifold is given: the surface of the Earth, which has a non-Euclidean geometry. For large triangles on the Earth's surface, the sum of the angles is not 180°. A small region can be mapped to a flat area where Euclidean geometry is re-established. This can be done for each point of a manifold within a neighborhood, but not globally for the whole manifold. ■

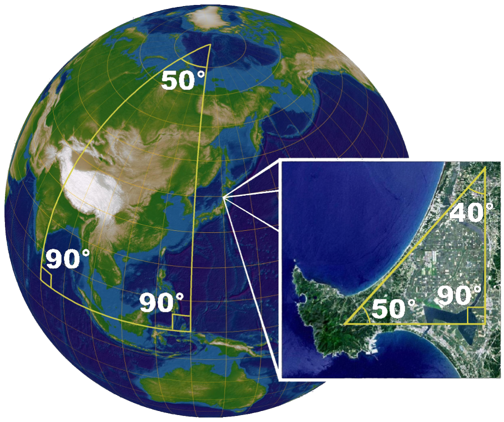


Figure 2.4: 2-dimensional manifold and mapping of a section to a plane segment.

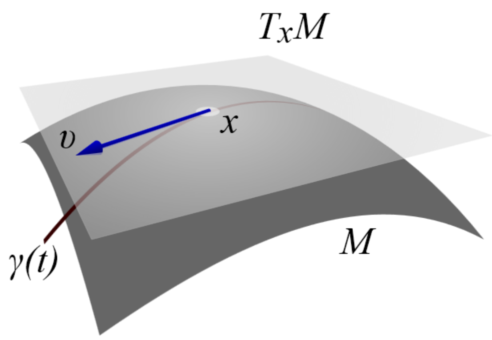


Figure 2.5: Tangential vector v to a 2-dimensional manifold M.

At each point of such a manifold a *tangent space* can be defined. This is a flat $\mathbb{R}^n$ space with the same dimension as the manifold. An example of a 2-dimensional manifold and tangent space is depicted in Fig. 2.5. The manifold is denoted by M, and the tangent space at the point x by T_xM. Such a tangent space (a plane) exists, for example, for the motion of mass points along an orbital curve $\gamma(t)$.

The manifold can be covered by points with local neighborhoods, and these points have corresponding tangent spaces. The set of all tangent spaces is called the *tangent bundle*. Changing the coordinate systems within the manifold means that the mapping from the manifold to the tangent space has to be redefined. The metric of the manifold can be used to define a scalar product in the tangent space.

We now want to make the definition of tangent space independent of the choice of coordinates. The tangent space T_xM at a point x in the manifold can be identified with the space of directional derivative operators along curves through x. The partial derivatives $\frac{\partial}{\partial x^\mu} = \partial_\mu$ represent a suitable basis for the vector space of directional derivatives, which we can therefore safely identify with the tangent space.

Consider two manifolds M and N and a function $F: M \to N$ for a mapping of points of M to points in N. In M and N no differentiation is defined. However, we can define *coordinate charts* from the manifolds to their corresponding tangent spaces. These are the functions denoted by ϕ and ψ in Fig. 2.6. The coordinate charts allow us to construct a map between the tangent spaces:

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n. \tag{2.84}$$

With the aid of this construct, we can define a partial derivative of f indirectly via the tangent spaces. For a point x^μ in $\mathbb{R}^m$ (the mapped point x of M) we define:

$$\frac{\partial f}{\partial x^\mu} := \frac{\partial}{\partial x^\mu}(\psi \circ f \circ \phi^{-1})(x^\mu). \tag{2.85}$$

In many application cases we have a curve in the manifold M described by a parameter λ. This could be the motion of a mass point that is dependent on time. Similarly to (2.85), we can define the derivative of function f according to λ by using the chain rule:

$$\frac{df}{d\lambda} := \frac{dx^\mu}{d\lambda}\partial_\mu f. \tag{2.86}$$

As can be seen, there is a summation over the indices μ, and the ∂_μ can be considered to be a basis of the tangent space. This approach can be found in some mathematical textbooks (for example, [13]).

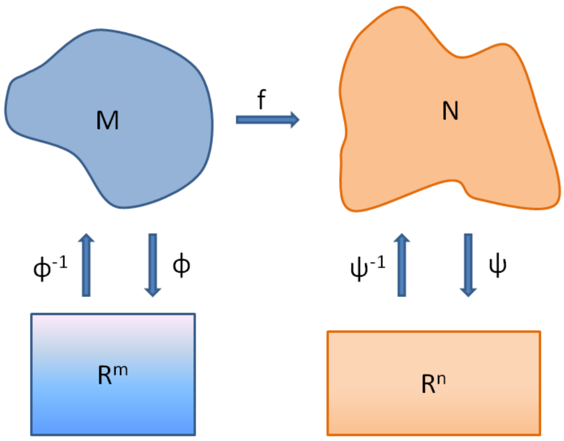


Figure 2.6: Mapping between two manifolds and tangent spaces.

n-forms

There is a special class of tensors, called *n-forms*, that comprise all completely anti-symmetric covariant tensors. In an n-dimensional space, there are 0-forms, 1-forms, ..., n-forms. All forms higher than n are zero by the antisymmetry requirement. A 2-form F can be constructed, for example, from two 1-forms (co-vectors) a and b:

$$F_{\mu\nu} = \frac{1}{2}(a_\mu b_\nu - a_\nu b_\mu). \tag{2.87}$$

By index raising, this can be rewritten in the following form:

$$F'^{\mu\nu} = \frac{1}{2}(a'^\mu b'^\nu - a'^\nu b'^\mu) \tag{2.88}$$

with

$$a'^\mu = g^{\mu\nu} a_\nu, \text{ etc.} \tag{2.89}$$

By having a square bracket represent an antisymmetric index permutation:

$$[\mu\nu] \to \mu\nu - \nu\mu, \tag{2.90}$$

we can also write this as

$$F_{\mu\nu} = \frac{1}{2} a_{[\mu} b_{\nu]}. \tag{2.91}$$

In general, we can define

$$T_{[\mu_1\mu_2\ldots\mu_n]} = \frac{1}{n!}(T_{\mu_1\mu_2\ldots\mu_n} + \text{alternating sum over permutations of } \mu_1 \ldots \mu_n). \tag{2.92}$$

The antisymmetric tensor may contain further indices that are not permuted.

■ **Example 2.8** Consider a tensor $T_{\mu\nu\rho\sigma}{}^{\tau}$ that is antisymmetric in the first three indices. We then have

$$T_{[\mu\nu\rho]\sigma}{}^{\tau} = \frac{1}{6}(T_{\mu\nu\rho\sigma}{}^{\tau} - T_{\mu\rho\nu\sigma}{}^{\tau} + T_{\rho\mu\nu\sigma}{}^{\tau} - T_{\nu\mu\rho\sigma}{}^{\tau} + T_{\nu\rho\mu\sigma}{}^{\tau} - T_{\rho\nu\mu\sigma}{}^{\tau}). \tag{2.93}$$

The antisymmetry of the first two indices allows us to simplify this expression to

$$\begin{aligned} T_{[\mu\nu\rho]\sigma}{}^{\tau} &= \frac{1}{6}\left(T_{\mu\nu\rho\sigma}{}^{\tau} - (-T_{\mu\nu\rho\sigma}{}^{\tau}) + T_{\rho\mu\nu\sigma}{}^{\tau} - (-T_{\rho\mu\nu\sigma}{}^{\tau}) + T_{\nu\rho\mu\sigma}{}^{\tau} - (-T_{\nu\rho\mu\sigma}{}^{\tau})\right) \\ &= \frac{1}{3}\left(T_{\mu\nu\rho\sigma}{}^{\tau} + T_{\rho\mu\nu\sigma}{}^{\tau} + T_{\nu\rho\mu\sigma}{}^{\tau}\right). \end{aligned} \tag{2.94}$$

This is the sum of indices μ, ν, ρ cyclically permuted. ■

Given a p-form a and q-form b, we define the *exterior product* (or *wedge product*) using the $\wedge$ (*wedge*) operator:

$$(a \wedge b)_{\mu_1\ldots\mu_{p+q}} := \frac{(p+q)!}{p!q!} a_{[\mu_1\ldots\mu_p} b_{\mu_{p+1}\ldots\mu_{p+q}]}. \tag{2.95}$$

For example, the wedge product of two 1-forms is

$$(a \wedge b)_{\mu\nu} = 2a_{[\mu} b_{\nu]} = a_\mu b_\nu - a_\nu b_\mu. \tag{2.96}$$

The wedge product is associative:

$$(a \wedge (b+c))_{\mu\nu} = (a \wedge b)_{\mu\nu} + (a \wedge c)_{\mu\nu}. \tag{2.97}$$

Mathematicians like to omit the indices if it is clear that an equation is written for forms, thus the last equation can also be written as

$$a \wedge (b+c) = a \wedge b + a \wedge c, \tag{2.98}$$

in short-hand notation. Another property is that wedge products are not commutative. For a p-form a and a q-form b:

$$a \wedge b = (-1)^{pq} b \wedge a, \tag{2.99}$$

and for a 1-form:

$$a \wedge a = 0. \tag{2.100}$$

These features may be why the name "exterior product" was selected for a generalization of a vector product in three dimensions.

An important operation on forms is the application of the *Hodge dual*. First, we have to define the *Levi-Civita symbol* in n dimensions:

$$\varepsilon_{\mu_1\dots\mu_n} = \begin{cases} 1 & \text{if } \mu_1\dots\mu_n \text{ is an even permutation of } 0,\dots(n-1), \\ -1 & \text{if } \mu_1\dots\mu_n \text{ is an odd permutation of } 0,\dots(n-1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.101}$$

The determinant of a matrix can be expressed by this symbol. If $M^{\mu}_{\mu'}$ is an $n\times n$ matrix, the determinant $|M|$ obeys the relation

$$\varepsilon_{\mu'_1\dots\mu'_n}|M| = \varepsilon_{\mu_1\dots\mu_n} M^{\mu_1}_{\mu'_1}\cdots M^{\mu_n}_{\mu'_n} \tag{2.102}$$

or, when restricted to one permutation on the left side:

$$|M| = \varepsilon_{\mu_1\dots\mu_n} M^{\mu_1}_{1}\cdots M^{\mu_n}_{n}. \tag{2.103}$$

The Levi-Civita symbol is defined in every coordinate system in the same way, and thus does not undergo coordinate transformations. Therefore, it is not a tensor. The symbol is totally antisymmetric, i.e., when any two indices are interchanged, the sign changes. All elements where one index appears twice are zero, because the index set must be a permutation.

The Levi-Civita symbol can also be defined with upper indices, in the same way. Then the determinant (2.102/2.103) takes the form

$$\varepsilon^{\mu'_1\dots\mu'_n}|M| = \varepsilon^{\mu_1\dots\mu_n} M^{\mu'_1}_{\mu_1}\cdots M^{\mu'_n}_{\mu_n} \tag{2.104}$$

or

$$|M| = \varepsilon^{\mu_1\dots\mu_n} M^{1}_{\mu_1}\cdots M^{n}_{\mu_n}. \tag{2.105}$$

We can construct a tensor from the Levi-Civita symbol by multiplying it by the square root of the modulus of the metric (we have to take the modulus, because the metric is negative definite in Minkowski space). To show this, we start with the transformation equation of the metric tensor

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu}, \tag{2.106}$$

and apply the determinant. The product rule for determinants allows us to write this as

$$|g_{\mu'\nu'}| = \left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| \left|\frac{\partial x^{\nu}}{\partial x^{\nu'}}\right| |g_{\mu\nu}| = \left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right|^2 |g_{\mu\nu}| \tag{2.107}$$

or

$$\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| = \sqrt{\frac{|g_{\mu'\nu'}|}{|g_{\mu\nu}|}}, \tag{2.108}$$

where the left side represents the determinant of the Jacobian. By using the special case

$$M^{\mu'}_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}}, \tag{2.109}$$

and inserting it into (2.104), we obtain

$$\varepsilon^{\mu'_1\dots\mu'_n}\left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| = \varepsilon^{\mu_1\dots\mu_n}\frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_n}}{\partial x^{\mu_n}}. \tag{2.110}$$

The determinant of the inverse Jacobian is

$$\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|^{-1}. \tag{2.111}$$

Therefore, by inserting (2.108) into (2.110), we obtain

$$\varepsilon^{\mu_1'\dots\mu_n'}\frac{1}{\sqrt{|g_{\mu'\nu'}|}} = \varepsilon^{\mu_1\dots\mu_n}\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_n'}}{\partial x^{\mu_n}}\frac{1}{\sqrt{|g_{\mu\nu}|}}. \tag{2.112}$$

We see that $\varepsilon^{\mu_1\dots\mu_n}/\sqrt{|g|}$ transforms like a tensor, and therefore is a tensor, by definition. The corresponding covariant tensor transforms as

$$\varepsilon_{\mu_1'\dots\mu_n'}\sqrt{|g_{\mu'\nu'}|} = \varepsilon_{\mu_1\dots\mu_n}\frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}}\cdots\frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}\sqrt{|g_{\mu\nu}|}. \tag{2.113}$$

Indices can be raised and lowered as usual by multiplication with metric elements.

With this behavior of the Levi-Civita symbol in mind, we define the Hodge-Dual of a tensorial form in the following way. We assume an n-dimensional manifold, a p-dimensional sub-manifold $p < n$, and a tensor p-form A, and then define

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}} := \frac{1}{p!}|g|^{-1/2}\,\varepsilon^{\nu_1\dots\nu_p}{}_{\mu_1\dots\mu_{n-p}}A_{\nu_1\dots\nu_p}. \tag{2.114}$$

The tilde superscript $\widetilde{}$ is called the *Hodge dual operator*. In the mathematical literature this is typically denoted by an asterisk as prefix-operator ($*A$), but this is a very misleading notation, and therefore we prefer the tilde superscript. The Hodge dual can be rewritten using a Levi-Civita symbol with only covariant components:

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}|g|^{-1/2}\,g^{\nu_1\sigma_1}\cdots g^{\nu_p\sigma_p}\,\varepsilon_{\sigma_1\dots\sigma_p\mu_1\dots\mu_{n-p}}A_{\nu_1\dots\nu_p}. \tag{2.115}$$

In this book, we will mostly use a somewhat simpler form when a contravariant tensor is transformed into a covariant tensor and vice versa. The factors $g^{\nu_1\sigma_1}$, etc., can be used to raise the indices of $A_{\nu_1\dots\nu_p}$:

$$\widetilde{A}_{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}|g|^{-1/2}\,\varepsilon_{\nu_1\dots\nu_n}A^{\nu_1\dots\nu_p}, \tag{2.116}$$

$$\widetilde{A}^{\mu_1\dots\mu_{n-p}} = \frac{1}{p!}|g|^{1/2}\,\varepsilon^{\nu_1\dots\nu_n}A_{\nu_1\dots\nu_p}, \tag{2.117}$$

where the sign of the exponent of $|g|$ has been changed according to (2.113). As an example, in four-dimensional space we use $n = 4, p = 2$. Then the Hodge duals of the A form are

$$\widetilde{A}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\,\varepsilon_{\mu\nu\sigma\rho}A^{\sigma\rho}, \tag{2.118}$$

$$\widetilde{A}^{\mu\nu} = \frac{1}{2}|g|^{1/2}\,\varepsilon^{\mu\nu\sigma\rho}A_{\sigma\rho}. \tag{2.119}$$

The Hodge dual $\widetilde{A}$ is linearly independent of the original form A. We will use the Hodge dual when deriving the theorems of Cartan geometry and the field equations of ECE theory.

2.4 Differentiation

We have already used some types of differentiation in the preceding sections, but only in the "standard" way within Euclidean spaces. Now we will extend this to curved spaces (manifolds) and to the calculus of p-forms.

2.4.1 Covariant differentiation

We have already used partial derivatives of tensors and parametrized derivatives. This, however, is not sufficient to define a general type of derivative in curved spaces of manifolds. Partial derivatives depend on the coordinate system. What we need is a "generally covariant" derivative that keeps its form under coordinate transformations and passes into the partial derivative for Euclidean spaces.

To retain linearity, the covariant derivative should have the form of a partial derivative plus a linear transformation. The latter corrects the partial derivative in such a way that covariance is ensured. The linear transformation depends on the coordinate indices. We start with the following definition for the *covariant derivative* of an arbitrary vector field V^ν:

$$D_\mu V^\nu := \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \tag{2.120}$$

where the $\Gamma^\nu_{\mu\lambda}$ are functions and are called the *connection coefficients* or *Christoffel symbols*. In contrast to an ordinary partial derivative, the covariant derivative of a vector component V^ν depends on all other components via the sum with the connection coefficients (observe the summation convention!). The covariant derivative has tensor properties by definition; therefore, Eq. (2.120) is a tensor equation that transduces a (1,0) tensor into a (1,1) tensor. This allows us to apply the transformation rules for tensors:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} D_\mu V^\nu = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\frac{\partial}{\partial x^\mu} V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \right). \tag{2.121}$$

On the other hand, we can apply the transformation to Eq. (2.120) directly:

$$D_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'}. \tag{2.122}$$

The single terms on the right side transform as follows:

$$\begin{aligned} \partial_{\mu'} V^{\nu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial x^{\nu'}}{\partial x^\nu} V^\nu \right) \\ &= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\nu} V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \frac{\partial}{\partial x^\mu} V^\nu, \end{aligned} \tag{2.123}$$

$$\Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} = \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda, \tag{2.124}$$

where the product rule has been applied in the first term. Eqs. (2.122) and (2.121) can be equated, and then the term with the partial derivative of V^ν cancels out and we obtain

$$\Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda. \tag{2.125}$$

Here we have replaced the dummy index ν by λ in the term with the mixed partial derivative. This is a common operation for tensor equations. Another common operation is multiplying a tensor equation by an indexed term and summing over one or more free indices (i.e., making the previously independent index into a dummy index). Multiplying the last equation by $\frac{\partial x^\lambda}{\partial x^{\lambda'}}$ then gives

$$\Gamma^{\nu'}_{\mu'\lambda'} V^\lambda = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} V^\lambda. \tag{2.126}$$

Eq. (2.126) holds for any vector V^λ; therefore, it must hold for the coefficients of V^λ directly, and we obtain the transformation equation for the connection coefficients:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{2.127}$$

The last term prevents the Gammas from transforming as tensors. Therefore, indices of Gammas cannot be raised and lowered through multiplication with metric elements, and we need not put too much effort into maintaining the order of upper and lower indices.

So far, we have investigated covariant derivatives of a contravariant vector (Eq. (2.120)). The theory can be extended to covariant vectors of 1-forms ω_ν:

$$D_\mu \omega_\nu := \partial_\mu \omega_\nu + \bar{\Gamma}^\lambda_{\mu\nu}\omega_\lambda, \tag{2.128}$$

where $\bar{\Gamma}$ is a connection coefficient that is a priori different from Γ. It can be shown [8] that, for consistency reasons, $\bar{\Gamma}$ is the same as Γ except for the sign:

$$\bar{\Gamma}^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu}. \tag{2.129}$$

Please note that the summation indices are different between (2.120) and (2.128). Now that we have a covariant derivative for contravariant and covariant components, the covariant derivative for arbitrary (k,m) tensors can be defined as follows:

$$\begin{aligned} D_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m} := {} & \partial_\sigma T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_m} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\ldots\nu_m} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\mu_3\ldots\mu_k}{}_{\nu_1\ldots\nu_m} + \ldots \\ & - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_m} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1\ldots\mu_k}{}_{\nu_1\lambda\nu_3\ldots\nu_m} - \ldots \end{aligned} \tag{2.130}$$

Through the application of the covariant derivative, a (k,m) tensor is transformed into a (k,m+1) tensor. It is also possible to take the covariant derivative of a scalar function. Since no indices are defined for the connection in this case, we define the covariant derivative for a scalar function ϕ simply as

$$D_\mu \phi := \partial_\mu \phi. \tag{2.131}$$

As we have seen, the connection coefficients are not tensors. It is, however, easy to turn them into tensors by taking the antisymmetric sum of the lower indices, for example:

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \tag{2.132}$$

This tensor is called the *torsion tensor*. When applying the transformation (2.127) for the difference of Gammas, the last term vanishes because the order in the mixed partial derivative is arbitrary. The torsion tensor is antisymmetric by definition. In four dimensions it can be written out for each index λ as

$$(T^\lambda{}_{\mu\nu}) = \begin{bmatrix} 0 & T^\lambda{}_{01} & T^\lambda{}_{02} & T^\lambda{}_{03} \\ -T^\lambda{}_{01} & 0 & T^\lambda{}_{12} & T^\lambda{}_{13} \\ -T^\lambda{}_{02} & -T^\lambda{}_{12} & 0 & T^\lambda{}_{23} \\ -T^\lambda{}_{03} & -T^\lambda{}_{13} & -T^\lambda{}_{23} & 0 \end{bmatrix}. \tag{2.133}$$

There are six independent components per λ. We will see later that this is one of the basis elements of Cartan geometry. A connection that is symmetric in its lower indices is torsion-free.

For completeness, we give the definition of the *Riemann curvature tensor*, which is also defined by the connection coefficients, but in a more complicated manner:

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}. \tag{2.134}$$

This tensor is antisymmetric in its last two indices. If it is written in pure covariant form, $R_{\lambda\rho\mu\nu} = g_{\tau\lambda}\, R^\tau{}_{\rho\mu\nu}$, and the manifold is torsion-free, the Riemann tensor is also antisymmetric in its first two indices. However, this property will not be used in Cartan geometry.

2.4.2 Metric compatibility and parallel transport

A fundamental property of vectors in physics is that they must be independent of their coordinate representation. From Euclidean space, we know that a rotation of a vector leaves its length and orientation against other vectors constant. In curved manifolds, however, this is not necessarily the case. Whether the length of a vector is preserved depends on the metric tensor. A parallel transport of a vector is depicted in Fig. 2.7. On a spherical surface, a vector is parallel transported from the north pole to a point on the equator in two ways: 1) it is moved directly along a meridian (red; right); and 2) it is moved first along another meridian and then along an equatorial latitude (left; blue). Obviously, the results are different, so this naive procedure is not compatible with a spherical manifold.

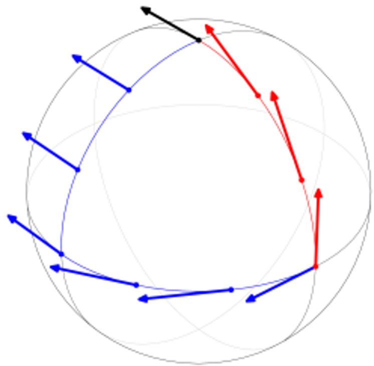


Figure 2.7: Parallel transport of a vector on a sphere.

Let us formalize the process to define parallel transport in a compatible way. A path is a displacement of a vector V^ν whose coordinates are parameterized, say by a parameter λ:

$$V^\nu = V^\nu(\lambda) \ \text{ at point } x^\nu(\lambda). \tag{2.135}$$

This can be considered as moving the vector (which is a tensor) along a predefined path. We define the covariant derivative along the path by

$$\frac{D}{d\lambda} := \frac{dx^\mu}{d\lambda} D_\mu, \tag{2.136}$$

where $\frac{dx^\mu}{d\lambda}$ is the tangent vector of the path. This gives us a method for specifying a *parallel transport* of V. This transport condition is satisfied if the covariant derivative along the path vanishes:

$$\frac{D\,V^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu V^\nu = \frac{dx^\mu}{d\lambda}(\partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho) = 0. \tag{2.137}$$

Since the tangent vector cannot vanish (we would not have a path anymore), it follows that the covariant derivative of the tensor must vanish:

$$D_\mu V^\nu = 0. \tag{2.138}$$

This is the condition for parallel transport. It is satisfied if and only if the covariant derivative along a path vanishes. This holds for any tensor. In particular we can choose the metric tensor and require it to be parallel transported:

$$D_\sigma g_{\mu\nu} = 0. \tag{2.139}$$

This requirement is called *metric compatibility*. The connection is also metrically compatible because it is contained in the covariant derivative, which means that the metric tensor is covariantly constant everywhere and can be parallel transported. If this requirement were omitted, we would have difficulties defining meaningful physics in a manifold because, for example, norms of vectors would not be constant but change during translations or rotations.

■ **Example 2.9** We show that the inner product of two vectors is preserved if the vectors can be parallel transported. The inner product of vectors V^μ and W^ν is $g_{\mu\nu}V^\mu W^\nu$. Its covariant path derivative is

$$\frac{D}{d\lambda}\left(g_{\mu\nu}V^\mu W^\nu\right) = \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)W^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}W^\nu\right) = 0, \tag{2.140}$$

because all three tensors are parallel transported, by definition. In the same way, one can prove that, if $g_{\mu\nu}$ can be parallel transported, then so can its inverse $g^{\mu\nu}$:

$$\begin{aligned} 0 = \frac{D}{d\lambda}g_{\mu\nu} &= \frac{D}{d\lambda}\left(g_{\mu\sigma}\, g_{\rho\nu}\, g^{\rho\sigma}\right) \\ &= \frac{D}{d\lambda}\left(g_{\mu\sigma}\right)g_{\rho\nu}\, g^{\rho\sigma} + g_{\mu\sigma}\frac{D}{d\lambda}\left(g_{\rho\nu}\right)g^{\rho\sigma} + g_{\mu\sigma}\, g_{\rho\nu}\frac{D}{d\lambda}\left(g^{\rho\sigma}\right). \end{aligned} \tag{2.141}$$

The first two terms in the last line vanish by definition, and consequently the third term has to vanish as well. ■

The concept of parallel transport allows us to find the equation for geodesics. A *geodesic* is the generalization of a straight line in Euclidean space. Mass points without external forces move along these generalized lines. In a curved manifold, the motion follows the curving of space and therefore is not a straight line. We can find the geodesics equation by requiring that the path parallel-transport its own tangent vector. This is analogous to flat space where the tangent vector is parallel to its line vector. From (2.137) we have

$$\frac{D}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{2.142}$$

which can be written as

$$\frac{dx^\mu}{d\lambda}D_\mu\frac{dx^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda}\left(\frac{\partial}{\partial x^\mu}\frac{dx^\nu}{d\lambda} + \Gamma^\nu_{\mu\rho}\frac{dx^\rho}{d\lambda}\right) = 0 \tag{2.143}$$

and, after the replacement of $\frac{\partial}{\partial x^\mu}$ with $\frac{\partial\lambda}{\partial x^\mu}\frac{d}{d\lambda}$, it simplifies to

$$\frac{d^2x^\nu}{d\lambda^2} + \Gamma^\nu_{\mu\rho}\frac{dx^\mu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0, \tag{2.144}$$

which is the *geodesic equation*. In flat space, the Gammas vanish and Newton's law, $\ddot{x} = 0$, for an unconstrained motion is regained.

Given a path in the manifold, covariant derivatives can be used to describe the deviation of a tensor from being parallel transported. Consider a round-trip as depicted in Fig. 2.8. A tensor is moved counter-clockwise along its covariant tangent vector D_μ, then D_ν, and afterward back to its starting point in reverse order. In the case where the tensor is parallel transportable, all derivatives vanish. However, this will not be the case in general. The *commutator* of two covariant derivatives is defined by

$$[D_\mu, D_\nu] := D_\mu D_\nu - D_\nu D_\mu, \tag{2.145}$$

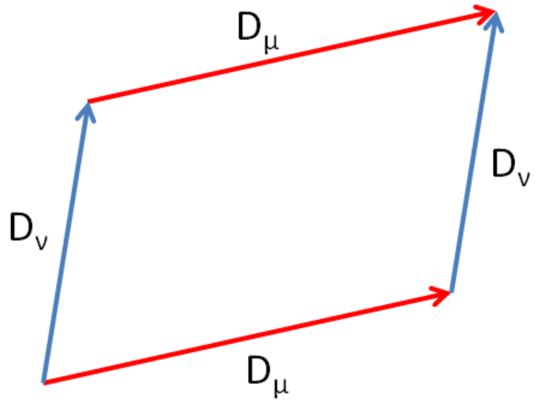


Figure 2.8: Closed loop for the composition of two covariant derivatives.

and describes the difference between the two orderings of the paths with respect to the covariant derivative. We can apply this to a vector V^ρ and evaluate the terms:

$$\begin{aligned}
[D_\mu, D_\nu]V^\rho &= D_\mu D_\nu V^\rho - D_\nu D_\mu V^\rho \\
&= \partial_\mu (D_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} D_\nu V^\sigma \\
&\quad - \partial_\nu (D_\mu V^\rho) + \Gamma^\lambda_{\nu\mu} D_\lambda V^\rho - \Gamma^\rho_{\nu\sigma} D_\mu V^\sigma \\
&= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
&\quad + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda \\
&\quad - \partial_\nu \partial_\mu V^\rho - (\partial_\nu \Gamma^\rho_{\mu\sigma}) V^\sigma - \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho + \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} V^\sigma \\
&\quad - \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda \\
&= \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right) V^\sigma - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) D_\lambda V^\rho .
\end{aligned} \tag{2.146}$$

By comparing the last line with the definitions of the curvature tensor (2.134) and the torsion tensor (2.132), we see that it can be written as

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} \, D_\lambda V^\rho . \tag{2.147}$$

Interestingly, the commutator of covariant derivatives of a vector depends linearly on the vector itself and its tangent vector, where the coefficients are the curvature and torsion tensors. In the case of no torsion, there would be no dependence on a derivative of V^ρ at all. The action of $[D_\mu, D_\nu]$ can be applied to a tensor of arbitrary rank. The general form is

$$\begin{aligned}
[D_\rho, D_\sigma] X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_m} &= R^{\mu_1}{}_{\lambda\rho\sigma} X^{\lambda \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_m} + R^{\mu_2}{}_{\lambda\rho\sigma} X^{\mu_1 \lambda \dots \mu_k}{}_{\nu_1 \dots \nu_m} + \cdots \\
&\quad - R^\lambda{}_{\nu_1\rho\sigma} X^{\mu_1 \dots \mu_k}{}_{\lambda \nu_2 \dots \nu_m} - R^\lambda{}_{\nu_2\rho\sigma} X^{\mu_1 \dots \mu_k}{}_{\nu_1 \lambda \dots \nu_m} - \cdots \\
&\quad - T^\lambda{}_{\rho\sigma} \, D_\lambda X^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_m} .
\end{aligned} \tag{2.148}$$

We have seen that the curvature and torsion tensors depend on the connection coefficients directly. To describe the geometry of a manifold, one must know these coefficients. The geometry is typically defined by a coordinate transformation. However, there is no direct way to derive the connection coefficients from the coordinate transformation equations. In Example 2.8 we have seen that the metric tensor can be derived from the Jacobian, which contains the derivatives of the coordinate transformations. Therefore, what we need is a relation between the metric and the connection, from which the connection coefficients can be derived, when the metric is known. Such a relation is given by the metric compatibility condition:

$$D_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} g_{\mu\lambda} = 0. \tag{2.149}$$

For a space of four dimensions, this tensor equation represents $4^3 = 64$ single equations. The first of them (for the diagonal metric elements) read:

$$\begin{aligned} &\frac{\partial}{\partial x^0} g_{00} - 2\Gamma^0_{00}\, g_{00} = 0 \\ &-\Gamma^1_{00}\, g_{11} - \Gamma^0_{01}\, g_{00} = 0 \\ &-\Gamma^2_{00}\, g_{22} - \Gamma^0_{02}\, g_{00} = 0 \\ &-\Gamma^3_{00}\, g_{33} - \Gamma^0_{03}\, g_{00} = 0 \\ &\qquad\qquad \dots . \end{aligned} \tag{2.150}$$

You should keep in mind that the metric is symmetric, and therefore not all equations are linearly independent. It is difficult to see how many independent equations remain. Computer algebra (code available at [11]) tells us that one half (24 equations) are dependent on the other 24 equations. Therefore, we can predefine 24 Gammas arbitrarily. One solution, for example, is

$$\begin{aligned} &\Gamma^0_{00} = \frac{\frac{\partial}{\partial x^0}\, g_{00}}{2\, g_{00}} \\ &\Gamma^0_{01} = -\frac{g_{11}}{g_{00}} A_{25} \\ &\Gamma^0_{02} = -\frac{g_{22}}{g_{00}} A_{43} \\ &\Gamma^0_{03} = -\frac{g_{33}}{g_{00}} A_{40} \\ &\Gamma^0_{10} = \frac{\frac{\partial}{\partial x^1}\, g_{00}}{2\, g_{00}} \\ &\dots \end{aligned} \tag{2.151}$$

with

$$\begin{aligned} &\Gamma^1_{00} = A_{25} \\ &\Gamma^2_{00} = A_{43} \\ &\Gamma^3_{00} = A_{40} \\ &\dots , \end{aligned} \tag{2.152}$$

where the A_i are the predefined parameters, and they may even be functions of x^μ. From the first equation of (2.150), it can be seen that assuming $\Gamma^0_{00} = 0$ is not a good choice, because this would impose the restriction $\frac{\partial g_{00}}{\partial x^0} = 0$ on the metric a priori. Therefore, the diagonal elements of the lower pair of indices of Gamma do not vanish in general. By comparing the solutions for Γ^0_{01} and Γ^0_{10} in (2.151) we can see that the Gammas are not symmetric in the lower indices.

Having found the connection coefficients, we can construct the curvature and torsion tensors (2.134) and (2.132). While the coefficients go into the curvature tensor as is, the torsion tensor depends only on the antisymmetric part of the Gammas. Each 2-tensor or connection coefficient can be split into a symmetric and antisymmetric part:

$$\Gamma^\rho_{\mu\nu} = \Gamma^{\rho(S)}_{\mu\nu} + \Gamma^{\rho(A)}_{\mu\nu} \tag{2.153}$$

with

$$\Gamma^{\rho(S)}_{\mu\nu} = \Gamma^{\rho(S)}_{\nu\mu}, \tag{2.154}$$
$$\Gamma^{\rho(A)}_{\mu\nu} = -\Gamma^{\rho(A)}_{\nu\mu}.$$

For the torsion tensor we have

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda(A)}_{\mu\nu} - \Gamma^{\lambda(A)}_{\nu\mu} = 2\,\Gamma^{\lambda(A)}_{\mu\nu}, \tag{2.155}$$

since the symmetric part does not enter the torsion. This motivates the imposition of additional antisymmetry requirements on the Gammas, instead of allowing 24 elements to be chosen arbitrarily. Consequently, in addition to the metric compatibility equation (2.149), we define 24 extra equations,

$$\Gamma^{\rho}_{\mu\nu} = -\Gamma^{\rho}_{\nu\mu}, \tag{2.156}$$

for all pairs $\mu \neq \nu$ with $\mu > \nu$. This reduces the number of free solution parameters from 24 to 4 (see computer algebra code [120]). The situation becomes quite complicated when non-diagonal elements in the metric are present [121]. Alternatively, we could even force a purely symmetric connection by requiring that

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}. \tag{2.157}$$

Then there would be no free parameters and all Gammas would be uniquely defined, with 24 of them turning out to be zero. However, in this case, torsion would be zero and we would run into irretrievable conflicts with geometrical laws, as we will see in subsequent sections. There is a reason for leaving a certain variability in the connection: the theorems of Cartan geometry have to be satisfied, which imposes additional conditions on curvature and torsion, and thereby on the connection.

For completeness, we will now describe how the symmetric connection coefficients are computed in Einsteinian general relativity. We start with Eq. (2.149), and write this equation three times with permuted indices:

$$\partial_\sigma g_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu} g_{\mu\lambda} = 0, \tag{2.158}$$
$$\partial_\mu g_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} g_{\nu\lambda} = 0,$$
$$\partial_\nu g_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\sigma\lambda} = 0.$$

Subtracting the second and third equations from the first and using the symmetry of the connection gives

$$\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma} - \partial_\nu g_{\sigma\mu} + 2\,\Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma} = 0, \tag{2.159}$$

and multiplying the resulting equation by $g^{\sigma\rho}$ gives

$$(\Gamma^{\lambda}_{\mu\nu} g_{\lambda\sigma})\, g^{\sigma\rho} = \Gamma^{\lambda}_{\mu\nu}(g_{\lambda\sigma}\, g^{\sigma\rho}) = \Gamma^{\lambda}_{\mu\nu}\delta^{\rho}_{\lambda} = \Gamma^{\rho}_{\mu\nu} \tag{2.160}$$

for the Gamma term. From (2.159) it then follows that

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{2.161}$$

The symmetric connection is determined completely by the metric, in accordance with our earlier result from the single equation of metric compatibility.

A diagonal metric was used for all derivations in this section. These derivations remain valid if non-diagonal elements are added, but the solutions become much more complex. Imposing additional symmetry or antisymmetry conditions on the connection may lead to results differing from those for a diagonal metric.

It should be noted that the metric of a given geometry of a manifold is not unique, and depends on the choice of coordinate system. Recalling the examples above, Euclidean space can be described by cartesian or spherical coordinates that lead to different metric tensors. However, the spacetime structure is the same, only the numerical addressing of points changes, as do the coordinates of vectors. However, the vectors as physical objects (position and length) remain the same.

■ **Example 2.10** We compute the connection for the spherical coordinate system (r, θ, ϕ) for three cases: general connection, antisymmetrized connection, and symmetrized connection. This example is available as Maxima code [119]. The metric tensor is from Example 2.5:

$$(g_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{bmatrix}. \tag{2.162}$$

Since the metric is not time-dependent, indices run from 1 to 3. This gives $3^3 = 27$ equations from metric compatibility (2.149), and the first equations are

$$\begin{aligned} -2\,\Gamma^1_{11} &= 0 \\ -\Gamma^2_{11}\,r^2 - \Gamma^1_{12} &= 0 \\ -\Gamma^3_{11}\,r^2\sin^2(\theta) - \Gamma^1_{13} &= 0 \\ -\Gamma^2_{11}\,r^2 - \Gamma^1_{12} &= 0 \\ 2r - 2\,\Gamma^2_{12}\,r^2 &= 0 \\ \ldots . \end{aligned} \tag{2.163}$$

The solution (obtained by computer algebra) contains nine free parameters $A_1, \ldots, A_9$. There are 27 solutions in total. Some of them are

$$\begin{aligned} &\Gamma^1_{11} = 0 \\ &\Gamma^1_{13} = -A_9\,r^2\sin^2(\theta) \\ &\Gamma^1_{31} = 0 \\ &\Gamma^1_{33} = -A_3\,r^2\sin^2(\theta) \\ &\Gamma^2_{12} = \frac{1}{r} \\ &\Gamma^2_{21} = -\frac{A_4}{r^2} \\ &\Gamma^3_{23} = \frac{\cos(\theta)}{\sin(\theta)} \\ &\Gamma^3_{32} = A_2 \\ &\ldots . \end{aligned} \tag{2.164}$$

With 9 additional antisymmetry conditions, the solutions are

$$
\begin{aligned}
&\Gamma^1_{11} = 0 \\
&\Gamma^1_{13} = \Gamma^1_{31} = 0 \\
&\Gamma^1_{23} = -\Gamma^1_{32} = -A_{10} \\
&\Gamma^2_{12} = -\Gamma^2_{21} = \frac{1}{r} \\
&\Gamma^3_{23} = -\Gamma^3_{32} = \frac{\cos(\theta)}{\sin(\theta)} \\
&\dots .
\end{aligned}
\tag{2.165}
$$

There is only one free parameter A_{10} left. A certain similarity to the general solution is retained, but with antisymmetry. If symmetric connection coefficients are enforced, most Gammas are zero. The only non-zero coefficients are

$$
\begin{aligned}
&\Gamma^1_{22} = -r \\
&\Gamma^1_{33} = -r\sin^2(\theta) \\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \\
&\Gamma^2_{33} = -\cos(\theta)\sin(\theta) \\
&\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
&\Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos(\theta)}{\sin(\theta)}.
\end{aligned}
\tag{2.166}
$$

This example is often found in textbooks of general relativity. If all coordinates have the physical dimension of length, then the connection coefficients have the same physical dimension. In this example we have angles and lengths, and therefore the physical dimensions differ. ■

2.4.3 Exterior derivative

So far, we have dealt with covariant derivatives of tensors, and we have seen that a partial derivative of a tensor does not conserve tensor properties. We now want to extend the concept of derivatives to n-forms, like the antisymmetric forms that we introduced in Section 2.3. Therefore, we need to define a derivative for n-forms that conserves antisymmetry and tensor properties. A partial derivative for one coordinate generates an additional index in a tensor, and thus a p-form is extended to a (p+1)-form by the definition

$$(d \wedge A)_{\mu_1 \dots \mu_{p+1}} := (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}. \tag{2.167}$$

This (p+1)-form is a tensor, irrespective of what A is. The simplest exterior derivative is that of a scalar function $\phi(x_\mu)$:

$$(d \wedge \phi)_\mu = \partial_\mu \phi, \tag{2.168}$$

in other words, it is the gradient of ϕ. Another example is the definition of the electromagnetic field in tensor form $F_{\mu\nu}$ as a 2-form (see Example 2.11, below). It is derived as an exterior derivative of a 1-form, the vector potential A_μ:

$$F_{\mu\nu} := (d \wedge A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.169}$$

The tensor character of exterior derivatives can be seen by applying the transformation law (2.123) to a (0,1) tensor V, for example:

$$\begin{aligned}\frac{\partial}{\partial x^{\mu'}}V_{\nu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}V_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu}}{\partial x^{\nu'}}V_{\nu}\right) \\ &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu}}{\partial x^{\mu}\partial x^{\nu'}}V_{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial}{\partial x^{\mu}}V_{\nu}.\end{aligned} \tag{2.170}$$

The first term in the second line, which should not appear in a tensor transformation, can be rewritten as

$$\frac{\partial^2 x^{\nu}}{\partial x^{\mu'}\partial x^{\nu'}}V_{\nu}, \tag{2.171}$$

which is now symmetric in μ' and ν'. Since the exterior derivative contains only antisymmetric sums of both indices, all of these terms vanish because partial derivatives are commutable. Therefore, $d \wedge V_{\nu}$ transforms like a tensor, and so do all n-forms.

An important property of an exterior derivative is that the result of its double application is zero:

$$d \wedge (d \wedge A) = 0. \tag{2.172}$$

The reason is the same as above: the partial derivatives are commutable, summing to zero in all antisymmetric sums.

■ **Example 2.11** We present Maxwell's homogeneous field equations in form notation and then transform them into their well-known vector expression (see computer algebra code [122]). The homogeneous laws are the Gauss law and the Faraday law. In tensor notation they are condensed into one equation:

$$d \wedge F = 0 \tag{2.173}$$

or, with indices,

$$(d \wedge F)_{\mu\nu\rho} = 0. \tag{2.174}$$

Because F is a 2-form, the exterior derivative of F is a 3-form. The electromagnetic field tensor is antisymmetric and defined by the contravariant tensor

$$F^{\mu\nu} = \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}, \tag{2.175}$$

where E^i are the components of the electric field (for example, $E^1 = E_X,\ E^2 = E_Y$, etc.) and B^i are the components of the magnetic field. To be able to apply the exterior derivative, we first have to transform this tensor into covariant form. Since classical electrodynamics takes place in a Euclidean space, we use the Minkowski metric to lower the indices:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.176}$$

The covariant field tensor then becomes

$$F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -cB^3 & cB^2 \\ -E^2 & cB^3 & 0 & -cB^1 \\ -E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}. \tag{2.177}$$

When we compare it to the contravariant form, we see that only the signs of the electric field components have changed. After working out the exterior derivative for $\mu = 0,\ \nu = 1,\ \rho = 2$, we obtain

$$(d \wedge F)_{012} = \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} - \partial_0 F_{21} - \partial_1 F_{02} - \partial_2 F_{10}. \tag{2.178}$$

Because F is antisymmetric, the negative summands are equal to the positive summands with reversed sign so that we have

$$(d \wedge F)_{012} = 2(\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01}), \tag{2.179}$$

which is twice the cyclic sum of indices. Since (μ, ν, ρ) must be a subset of (0, 1, 2, 3), only the combinations

(0, 1, 2)
(0, 1, 3)
(0, 2, 3)
(1, 2, 3)

are possible, leading to four equations for $d \wedge F$. Setting $F_{01} = E_X$ etc. leads to the four equations:

$$\begin{aligned} 2(c\partial_0 B^3 + \partial_1 E^2 - \partial_2 E^1) &= 0 \\ 2(-c\partial_0 B^2 + \partial_1 E^3 - \partial_3 E^1) &= 0 \\ 2(c\partial_0 B^1 + \partial_2 E^3 - \partial_3 E^2) &= 0 \\ 2(c\partial_1 B^1 + c\partial_2 B^2 + c\partial_3 B_3) &= 0 \end{aligned} \tag{2.180}$$

or, when written with cartesian components and simplified:

$$\begin{aligned} \partial_t B_Z + \partial_X E_Y - \partial_Y E_X &= 0 \\ \partial_t B_Y - \partial_X E_Z + \partial_Z E_X &= 0 \\ \partial_t B_X + \partial_Y E_Z - \partial_Z E_Y &= 0 \\ \partial_X B_X + \partial_Y B_Y + \partial_Z B_Z &= 0 \end{aligned} \tag{2.181}$$

with $\partial_0 = 1/c \cdot \partial_t$. By comparing these equations with the curl operator,

$$\boldsymbol{\nabla} \times \mathbf{V} = \begin{bmatrix} \partial_Y V_Z - \partial_Z V_Y \\ -\partial_X V_Z + \partial_Z V_X \\ \partial_X V_Y - \partial_Y V_X \end{bmatrix}, \tag{2.182}$$

we see that the first three equations of (2.181) contain the third, second and first lines of this operator, and can be written in vector form:

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}, \tag{2.183}$$

which is the Faraday law. The fourth equation of (2.181) is the Gauss law:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{2.184}$$

We conclude this example with the hint that the inhomogeneous Maxwell equations (Coulomb law and Ampère-Maxwell law) cannot be written as exterior tensor derivatives due to the current terms. In those cases, a formulation similar to the one in the next example has to be used. ■

■ **Example 2.12** As an example involving the Hodge dual (see computer algebra code [123]), we derive the homogeneous Maxwell equations from a tensor notation containing the Hodge dual of the electromagnetic field tensor, which was introduced in the preceding example. In tensor notation, the equation is

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0, \tag{2.185}$$

and involves the Hodge dual of the 4 x 4 field tensor, which is defined as follows:

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & -E^3 & E^2 \\ cB^2 & E^3 & 0 & -E^1 \\ cB^3 & -E^2 & E^1 & 0 \end{bmatrix}. \tag{2.186}$$

Indices are raised using the Minkowski metric (2.176):

$$\widetilde{F}^{\mu\nu} = \eta^{\mu\kappa}\eta^{\nu\rho}\widetilde{F}_{\kappa\rho}. \tag{2.187}$$

Therefore, the covariant Hodge dual is

$$\widetilde{F}^{\mu\nu} = \begin{bmatrix} 0 & cB^1 & cB^2 & cB^3 \\ -cB^1 & 0 & -E^3 & E^2 \\ -cB^2 & E^3 & 0 & -E^1 \\ -cB^3 & -E^2 & E^1 & 0 \end{bmatrix}, \tag{2.188}$$

for example:

$$\widetilde{F}_{01} = \frac{1}{2}(\varepsilon_{0123}F^{23} + \varepsilon_{0132}F^{32}) = F^{23} \tag{2.189}$$

and

$$\widetilde{F}^{01} = \eta^{00}\eta^{11}\widetilde{F}_{01} = -\widetilde{F}_{01}. \tag{2.190}$$

The homogeneous laws of classical electrodynamics are obtained by choosing the appropriate indices, as shown below. The Gauss law is obtained by choosing

$$\nu = 0, \tag{2.191}$$

which gives us

$$\partial_1\widetilde{F}^{10} + \partial_2\widetilde{F}^{20} + \partial_3\widetilde{F}^{30} = 0. \tag{2.192}$$

In vector notation this is

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0. \tag{2.193}$$

The Faraday law of induction is obtained by choosing

$$\nu = 1,2,3 \tag{2.194}$$

and consists of three component equations:

$$\begin{aligned} \partial_0\widetilde{F}^{01} + \partial_2\widetilde{F}^{21} + \partial_3\widetilde{F}^{31} &= 0 \\ \partial_0\widetilde{F}^{02} + \partial_1\widetilde{F}^{12} + \partial_3\widetilde{F}^{32} &= 0 \\ \partial_0\widetilde{F}^{03} + \partial_1\widetilde{F}^{13} + \partial_2\widetilde{F}^{23} &= 0. \end{aligned} \tag{2.195}$$

They can be condensed into one vector equation:

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}. \tag{2.196}$$

The differential form, tensor and vector notations are summarized as follows:

$$d \wedge F = 0 \;\rightarrow\; \partial_\mu \widetilde{F}^{\mu\nu} = 0 \;\rightarrow\; \left\{ \begin{array}{l} \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \\ \frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}. \end{array} \right. \tag{2.197}$$

The homogeneous laws of classical electrodynamics are most elegantly represented by the differential form notation, but most usefully represented by the vector notation. ■

Exterior covariant derivative

So far, we have seen that exterior derivatives are antisymmetric sums of partial derivatives applied to n-forms. The question now is what happens if we want to combine the concepts of exterior derivative and covariant derivative. This would be a generalization of these concepts that would be more appropriate for curved manifolds, because covariant derivatives play an important role in their description, for example, in defining commutators (see Section 2.4.2). We can define an exterior covariant derivative by creating an (n+1)-form from an n-form A:

$$D \wedge A := (D \wedge A)_{\mu_1 \dots \mu_{n+1}} = D_{[\mu} A_{\nu_1 \cdots \nu_n]}. \tag{2.198}$$

For a 1-form A_ν, this then is

$$\begin{aligned} D \wedge A = (D \wedge A)_{\mu\nu} = D_{[\mu} A_{\nu]} &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - \partial_\nu A_\mu + \Gamma^\lambda_{\nu\mu} A_\lambda \\ &= \partial_{[\mu} A_{\nu]} - (\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}) A_\lambda \end{aligned} \tag{2.199}$$

and, with the definition (2.132) of the torsion tensor, it can be written as

$$D \wedge A = \partial_{[\mu} A_{\nu]} - T^\lambda{}_{\mu\nu} A_\lambda. \tag{2.200}$$

Since the right side is a tensor, $D \wedge A$ is a tensor. This equation can be written in form notation:

$$D \wedge A = d \wedge A - T\, A. \tag{2.201}$$

We will extend this concept further in the next chapter.

2.5 Cartan geometry

Having reviewed the basics of Riemannian geometry, including torsion, we now approach the central point of this book: Cartan geometry. This geometry will be the mathematical foundation of all fields of physics, as we will see.

2.5.1 Tangent space, tetrads and metric

Riemannian geometry has given us nearly all of the tools that we need for Cartan geometry, which is the basis of ECE theory. We continue on this path by focusing on tangent spaces. In Section 2.1 we dealt with coordinate transformations in the base manifold. The tangent space at a point x in the base manifold was introduced as a Minkowski space of the same dimension for the local neighborhood of x. A vector V^μ defined in the base manifold can be transformed into the corresponding vector (denoted by V^a) in tangent space, by a transformation matrix q. We will use Latin indices to denote vectors and tensors in tangent space. This is similar to the introduction of the transformation matrix α in Eqs. (2.46 ff.), but with the difference that this transformation takes place between two different spaces. The basic transformation is

$$V^a = q^a{}_\mu V^\mu \tag{2.202}$$

with transformation matrix elements $q^a{}_\mu$. This is the basis of Cartan geometry, and q is called the *tetrad*. q transforms between the base manifold and tangent space. The inverse transformation is $q^{-1} = (q^\mu{}_a)$, and produces a vector in the base manifold:

$$V^\mu = q^\mu{}_a V^a. \tag{2.203}$$

If the metric of the tangent space η_{ab} is transformed to the base manifold (this is a (0,2) tensor), the result must be the metric of the base manifold $g_{\mu\nu}$, by definition:

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu\, \eta_{ab}, \tag{2.204}$$

and inversely:

$$\eta_{ab} = \frac{1}{n}\, q^\mu{}_a q^\nu{}_b\, g_{\mu\nu}, \tag{2.205}$$

where n is the dimension of the base manifold. Since q is a coordinate transformation, the product of q and its inverse has to be the unit matrix:

$$\mathbf{q}\mathbf{q}^{-1} = \mathbf{1}. \tag{2.206}$$

When written in component form, it becomes

$$q^a{}_\mu q^\nu{}_a = \delta^\nu_\mu, \tag{2.207}$$
$$q^a{}_\mu q^\mu{}_b = \delta^a_b. \tag{2.208}$$

The sum of the diagonal elements of (2.206), called the *trace*, is the dimension of the spaces between which the transformation takes place:

$$q^a{}_\mu q^\mu{}_a = n. \tag{2.209}$$

Since this kind of summed product will occur often in our calculations, it would be convenient for the result to be unity. Therefore, we will set the following definition:

$$q^a{}_\mu q^\mu{}_a := 1, \tag{2.210}$$

and then introduce a scaling factor of $1/\sqrt{n}$ to the tetrad elements and of $\sqrt{n}$ to the inverse tetrad elements:

$$q^a{}_\mu \rightarrow \frac{1}{\sqrt{n}}\, q^a{}_\mu, \tag{2.211}$$
$$q^\mu{}_a \rightarrow \sqrt{n}\, q^\mu{}_a, \tag{2.212}$$

so that conditions (2.204) and (2.205) will remain satisfied.

■ **Example 2.13** We consider the transformation to spherical polar coordinates, by starting with Eq. (2.58), the transformation matrix from Example 2.4:

$$\alpha = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.213}$$

The inverse transformation is

$$\boldsymbol{\alpha}^{-1} = \begin{bmatrix} \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta) \\ \frac{\cos(\phi)\cos(\theta)}{r} & \frac{\sin(\phi)\cos(\theta)}{r} & -\frac{\sin(\theta)}{r} \\ -\frac{\sin(\phi)}{r\sin(\theta)} & \frac{\cos(\phi)}{r\sin(\theta)} & 0 \end{bmatrix}, \tag{2.214}$$

as can be seen from computer algebra code [124]. To make this transformation a tetrad from a cartesian base manifold to a Euclidian tangent space with spherical polar coordinates, we have to set

$$\mathbf{q} = \frac{1}{\sqrt{3}}\,\boldsymbol{\alpha}, \tag{2.215}$$

$$\mathbf{q}^{-1} = \sqrt{3}\,\boldsymbol{\alpha}^{-1}. \tag{2.216}$$

This then gives us

$$\mathbf{q}\mathbf{q}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.217}$$

which is the unit matrix, as required. ■

2.5.2 Derivatives in tangent space

We will now explore the differential calculus in tangent space to see how it is connected to that in the base manifold. The tangent space at a point x is a Euclidian space, and we could argue that this allows us to use ordinary differentiation. To define a derivative, we have to construct infinitesimal transitions from the neighborhood of x. For a point $y \neq x$, however, another tangent space is defined, because of the definition of tangent spaces. Therefore, the curved structure of the base manifold has to be respected when defining the derivatives in tangent space. In the base manifold, we defined the covariant derivative in a way that respects this curved structure (see Eq. (2.120)):

$$D_\mu V^\nu := \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \tag{2.218}$$

where the partial derivatives ∂_μ and the connection coefficients $\Gamma^\nu_{\mu\lambda}$ are operating on a vector V^λ in the base manifold. We can use the same definition for a vector V^a in tangent space, but the connection coefficients will be different:

$$D_\mu V^a := \partial_\mu V^a + \omega^a{}_{\mu b} V^b. \tag{2.219}$$

The role of the connection coefficients is taken over by other coefficients called *spin connections* $\omega^a{}_{\mu b}$. These have the same number of indices as the Γ's but transform in the tangent space; therefore, they have two Latin indices. The name "spin connection" comes from the fact that they can be used to define covariant derivatives of spinors (which is impossible using the Γ connection coefficients). The derivative D_μ itself is defined with respect to the base manifold, and therefore has a Greek index. This index also has to be present in the spin connection to maintain the indices as required for a tensor expression.

Covariant derivatives of a mixed index tensor are defined so that the indices of tangent space are accompanied by a spin connection, and the indices of the base manifold are accompanied by a Christoffel connection, for example:

$$D_\mu V^a{}_\nu = \partial_\mu V^a{}_\nu + \omega^a{}_{\mu b} V^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} V^a{}_\lambda \tag{2.220}$$

or

$$D_\mu X^{ab}{}_{c\nu} = \partial_\mu X^{ab}{}_{c\nu} + \omega^a{}_{\mu d} X^{db}{}_{c\nu} + \omega^b{}_{\mu d} X^{ad}{}_{c\nu} - \omega^d{}_{\mu c} X^{ab}{}_{d\nu} - \Gamma^\lambda{}_{\mu\nu} X^{ab}{}_{c\lambda}\,. \tag{2.221}$$

In (2.221), d and λ are dummy indices. The summations over lower (covariant) indices have a minus sign for both the spin connection and the Christoffel connection terms. The spin connections are not tensors, as is also the case for the Γ connections. However, the expressions with covariant derivatives are tensors.

2.5.3 Exterior derivatives in tangent space

In Section 2.4.3 we introduced exterior derivatives, which are n-forms based on covariant derivatives. We can interpret a mixed-index tensor V^a_μ as a vector-valued 1-form, where a is the index of the vector component. V^a is a short notation for this 1-form. The concept of antisymmetric n-forms was introduced in Section 2.3. An exterior derivative of n-forms was introduced in Section 2.4.3, where a p-form was extended to a (p+1)-form by introducing the antisymmetric derivative operator $d\wedge$ (see Eq. (2.167)):

$$(d\wedge A)_{\mu_1\dots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\dots\mu_{p+1}]}. \tag{2.222}$$

We can extend this concept to the tangent space. First, the definition of the covariant derivative can be extended to mixed-index tensors by giving A one or more indices of tangent space:

$$(d\wedge A^b)_{\mu_1\dots\mu_{p+1}} := (p+1)\partial_{[\mu_1} A^b{}_{\mu_2\dots\mu_{p+1}]}. \tag{2.223}$$

This definition stands on its own, but in curved manifolds it becomes important to define a covariant exterior derivative of p-forms by basing this definition on the covariant derivative operator D_μ. In form notation, this kind of covariant derivative is written as

$$(D\wedge A^b)_{\mu_1\dots\mu_{p+1}} := (p+1)D_{[\mu_1} A^b{}_{\mu_2\dots\mu_{p+1}]}, \tag{2.224}$$

where the D's on the right side are the “usual" covariant derivatives of coordinate index μ_1, etc., as shown in (2.220), for example. A is a tensor that can have an arbitrary number of Greek and Latin indices, as before. The lower Greek indices define the p-form. Using indexless notation, we can also write

$$D\wedge A := (p+1)D_{[\mu_1} A_{\mu_2\dots\mu_{p+1}]}. \tag{2.225}$$

We will come back to this short-hand notation later. For example, Eq. (2.220) with exterior covariant derivative and coordinate indices $\mu \in \{0,1,2\}$ reads:

$$\begin{aligned} D\wedge V^a &= (D\wedge V^a)_{\mu\nu} \\ &= 2\,(D_0 V^a_1 + D_1 V^a_2 + D_2 V^a_0 - D_1 V^a_0 - D_2 V^a_1 - D_0 V^a_2\,) \\ &= 2\,(D_0(V^a_1 - V^a_2\,) + D_1(V^a_2 - V^a_0\,) + D_2(V^a_0 - V^a_1\,))\,, \end{aligned} \tag{2.226}$$

where the “normal” covariant derivatives continue to follow the same definition as before, for example:

$$D_0 V^a_1 = \partial_0 V^a_1 + \omega^a{}_{0b} V^b_1 - \Gamma^\lambda{}_{01} V^a_\lambda\,. \tag{2.227}$$

The antisymmetry of the 2-form (2.226) requires that

$$(D\wedge V^a)_{\mu\nu} = -(D\wedge V^a)_{\nu\mu}, \tag{2.228}$$

from which it follows that interchanging the indices μ and ν gives the negative result of (2.226). That this is the case can be seen directly from the second line of the equation.

2.5.4 Tetrad postulate

Since the tangent space is uniquely related to the base manifold via the tetrad matrix $q^a{}_\mu$, the Γ connections of the base manifold and the spin connections of the tangent space are related to each other. To see how this is the case, we use *metric compatibility*, which is the requirement that a vector must be the same when described in different coordinate systems. This is necessary for physical uniqueness; otherwise, we would be dealing with a kind of mathematics that is not related to physical objects and processes. We introduced this concept in Section 2.4.2 for vectors in the base manifold, and here we extend it to the tangent space in Cartan geometry.

With this in mind, we can represent a covariant derivative of a tangent vector in two different ways. Denoting the orthonormal unit vectors in the base manifold by $\hat{e}_\nu$, and those of the tangent space by $\hat{e}_a$, we can write

$$DV = D_\mu V^\nu = (\partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda)\hat{e}_\nu \tag{2.229}$$

and

$$DV = D_\mu V^a = (\partial_\mu V^a + \omega^a{}_{\mu b} V^b)\hat{e}_a, \tag{2.230}$$

for the same vector DV. The latter case is also referred to as a *mixed basis*, because the derivative relates to the manifold, as before. This equation can be transformed into the base manifold coordinates by transforming the coordinates V^a and the unit vectors $\hat{e}_a$ according to the rules introduced in Section 2.5.1 and with the renaming of dummy indices:

$$\begin{aligned} D_\mu V^a &= \left(\partial_\mu V^a + \omega^a{}_{\mu b} V^b\right)\hat{e}_a \\ &= \left(\partial_\mu (q^a{}_\nu V^\nu) + \omega^a{}_{\mu b} q^b{}_\lambda V^\lambda\right) q^\sigma{}_a \hat{e}_\sigma \\ &= q^\sigma{}_a \left(q^a{}_\nu \partial_\mu V^\nu + V^\nu \partial_\mu q^a{}_\nu + \omega^a{}_{\mu b} q^b{}_\lambda V^\lambda\right)\hat{e}_\sigma \\ &= \left(\partial_\mu V^\nu + q^\nu{}_a V^\lambda \partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^\nu{}_a q^b{}_\lambda V^\lambda\right)\hat{e}_\nu. \end{aligned} \tag{2.231}$$

Comparing this with Eq. (2.229) immediately leads to

$$\boxed{\Gamma^\nu{}_{\mu\lambda} = q^\nu{}_a \partial_\mu q^a{}_\lambda + q^\nu{}_a q^b{}_\lambda \omega^a{}_{\mu b}.} \tag{2.232}$$

Multiplying this equation by $q^\lambda{}_c$ and applying the same rules as above gives

$$q^\lambda{}_c \Gamma^\nu{}_{\mu\lambda} = q^\nu{}_a \omega^a{}_{\mu c} + q^\lambda{}_c q^\nu{}_a \partial_\mu q^a{}_\lambda, \tag{2.233}$$

and multiplying it by $q^b{}_\nu$ gives

$$q^b{}_\nu q^\lambda{}_c \Gamma^\nu{}_{\mu\lambda} = \omega^b{}_{\mu c} + q^\lambda{}_c \partial_\mu q^b{}_\lambda, \tag{2.234}$$

which, after the renaming of indices, becomes

$$\boxed{\omega^a{}_{\mu b} = q^a{}_\nu q^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - q^\lambda{}_b \partial_\mu q^a{}_\lambda.} \tag{2.235}$$

We have now obtained the relations between both types of connections that we needed. Knowing one of them and the tetrad matrix allows us to compute the other connection.

We can further multiply Eq. (2.232) by $q^c{}_\nu$, which (after applying the rules) gives us

$$q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} = \partial_\mu q^a{}_\lambda + q^b{}_\lambda \omega^a{}_{\mu b}. \tag{2.236}$$

As can be seen by comparison with (2.220), these are exactly the terms of the covariant derivative of the tensor $q^a{}_\nu$ in a mixed basis. It follows that

$$\boxed{D_\mu q^a{}_\nu = 0.} \tag{2.237}$$

This is called the *tetrad postulate*. It states that the covariant derivative of all tetrad elements vanishes.

This is a consequence of metric compatibility, which we discussed at the beginning of this section. As was shown earlier in Eq. (2.139), metric compatibility in the base manifold is defined by an analogous equation for the metric:

$$D_\sigma g_{\mu\nu} = 0. \tag{2.238}$$

If the space is Euclidean, we have

$$D_\sigma \eta_{\mu\nu} = 0 \tag{2.239}$$

for the Minkowski metric (2.176). Since this is also the metric for the tangent space, we can apply the corresponding definition of the covariant derivative:

$$D_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega^c{}_{\mu a} \eta_{cb} - \omega^c{}_{\mu b} \eta_{ac} = 0. \tag{2.240}$$

The Minkowski metric lowers the Latin indices of the spin connections so that we have

$$-\omega_{a\mu b} - \omega_{b\mu a} = 0 \tag{2.241}$$

or

$$\omega_{a\mu b} = -\omega_{b\mu a}. \tag{2.242}$$

Metric compatibility provides the property of antisymmetry to the spin connections. Notice that antisymmetry is defined only if the related indices are at the same lower or upper positions. Despite this antisymmetry, the spin connection is not a tensor, as is also the case for the Γ connection. The symmetry properties of the Γ connection were discussed in Section 2.4.2.

■ **Example 2.14** We compute some spin connection examples from Eq. (2.235), for which we need a given geometry defined by a tetrad and the Christoffel connection coefficients. We will use Example 2.13, where we considered a transformation to spherical polar coordinates. We interpret this in such a way that the polar coordinates of the base manifold are transformed into cartesian coordinates of the tangent space. According to Eqs. (2.213) and (2.215), the tetrad matrix is

$$\mathbf{q} = \frac{1}{\sqrt{3}} \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{2.243}$$

The spin connections for spherical polar coordinates were computed in Example 2.10 for three cases:

1. a general connection;
2. a connection antisymmetrized in the non-diagonal lower indices; and
3. a symmetric Christoffel connection (used in Einsteinian relativity).

The functions for the Γ's from that example have to be inserted into Eq. (2.235), together with the tetrad elements of (2.243). Please notice that both tetrad and inverse tetrad elements occur in (2.235). The $q^a{}_\nu$ are the elements of (2.243), and the $q^\nu{}_a$ are those of the inverted tetrad matrix, essentially Eq. (2.214). The calculation is lengthy and has been automated through computer algebra code [125]. The results for Case 1 (the general connection) are, for example:

$$\omega^{(1)}{}_{1(1)} = 0, \tag{2.244}$$
$$\omega^{(1)}{}_{1(2)} = -\sin(\theta)\,(A_9\, r \sin(\theta) + A_8 \cos(\theta)),$$
$$\omega^{(1)}{}_{1(3)} = A_8 \sin(\phi) \sin(\theta)^2 - A_9 \sin(\phi)\, r \cos(\theta) \sin(\theta) - \frac{A_7 \cos(\phi)}{r}.$$

The A's are constants contained in the Γ's. They have different physical units, otherwise there would be problems in summation. In order to make it easier to distinguish between Latin and Greek indices, the numbers for Latin indices have been set in parentheses. For Case 2 (as described above), the results are simpler:

$$\omega^{(1)}{}_{1(1)} = 0, \tag{2.245}$$
$$\omega^{(1)}{}_{1(2)} = \frac{A_{10} \cos(\theta)}{r^2 \sin(\theta)},$$
$$\omega^{(1)}{}_{1(3)} = -\frac{A_{10} \sin(\phi)}{r^2},$$

and in Case 3 (symmetric Christoffel connection), all spin connections vanish:

$$\omega^a{}_{\mu b} = 0, \tag{2.246}$$

indicating that there is no spin connection for a geometry without torsion. The antisymmetry holds even for the case where a and b are indices at different positions (upper and lower), because the metric in tangent space is the unit matrix (which also makes raising and lowering of indices trivial). The antisymmetry has been checked using computer algebra code [125], and it is always

$$\omega^a{}_{\mu b} = -\omega^b{}_{\mu a}, \tag{2.247}$$

as required. ■

2.5.5 Evans lemma

We now come to some aspects of Cartan geometry that are of particular relevance to ECE theory. The tetrad postulate can be modified to give a second order differential equation for the tetrad elements. This equation is a wave equation and is fundamental to many areas of physics. The tetrad postulate (2.237) can be augmented with an additional derivative:

$$D^\mu (D_\mu q^a{}_\nu) = 0. \tag{2.248}$$

We introduced a covariant derivative with an upper index in order to make μ a summation (dummy) index. Because the expression in the parentheses is a scalar function due to the tetrad postulate, we need not bother with how this derivative is defined, it reduces to a partial derivative by definition. This allows us to write

$$\partial^\mu (D_\mu q^a{}_\nu) = 0 \tag{2.249}$$

or

$$\partial^\mu (\partial_\mu q^a{}_\nu + \omega^a{}_{\mu b} q^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda) = 0. \tag{2.250}$$

In a manifold with 4-vectors $[ct, X, Y, Z]$, the covariant form of the partial derivative is defined in the usual way:

$$[\partial_0,\, \partial_1,\, \partial_2,\, \partial_3] = \left[\frac{1}{c}\frac{\partial}{\partial t},\, \frac{\partial}{\partial X},\, \frac{\partial}{\partial Y},\, \frac{\partial}{\partial Z}\right], \tag{2.251}$$

while the contravariant form of the partial derivative is defined with the sign changed for the spatial derivatives:

$$\left[\partial^0,\, \partial^1,\, \partial^2,\, \partial^3\right] = \left[\frac{1}{c}\frac{\partial}{\partial t},\, -\frac{\partial}{\partial X},\, -\frac{\partial}{\partial Y},\, -\frac{\partial}{\partial Z}\right]. \tag{2.252}$$

Therefore, $\partial^\mu \partial_\mu$ is the d'Alembert operator

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial Z^2}. \tag{2.253}$$

From Eq. (2.250) it then follows that

$$\Box\, q^a{}_\nu + G^a{}_\nu = 0, \tag{2.254}$$

which is a wave equation with the tensor function

$$G^a{}_\nu = \partial^\mu(\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu(\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda). \tag{2.255}$$

This equation can be transformed into an eigenvalue equation by requiring $G^a{}_\nu$ to be split into a tetrad part and a scalar function R:

$$G^a{}_\mu = R\, q^a{}_\nu \tag{2.256}$$

with

$$R = q^\nu{}_a \left(\partial^\mu(\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu(\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda)\right). \tag{2.257}$$

R contains only dummy indices and is a scalar function. This allows (2.254) to be written as

$$\boxed{\Box\, q^a{}_\nu + R\, q^a{}_\nu = 0.} \tag{2.258}$$

This equation is called the *Evans lemma*. It is a generally covariant eigenvalue equation, in which R plays the role of a curvature, as we will see in later chapters. The entire field of generally covariant quantum mechanics is based on this equation, which is highly non-linear, because R depends on the eigenfunction $q^a{}_\nu$ and the Christoffel and spin connections. In later chapters, for first approximations, we will often assume that R is a constant.

2.5.6 Maurer-Cartan structure equations

The torsion and curvature tensors of Riemannian geometry can be transformed to 2-forms of Cartan geometry simply by defining

$$T^a{}_{\mu\nu} := q^a{}_\kappa T^\kappa{}_{\mu\nu}\,, \tag{2.259}$$

$$R^a{}_{b\mu\nu} := q^a{}_\rho q^\sigma{}_b R^\rho{}_{\sigma\mu\nu}. \tag{2.260}$$

Multiplication with tetrad elements replaces some Greek indices with Latin indices of the tangent space, so the torsion and curvature tensors defined by Eqs. (2.132) and (2.134) become 2-forms of torsion and curvature. To these forms two foundational relations apply, which will be derived in this section, using the proof described in [14].

We first define forms of the Christoffel and spin connections similarly to (2.259) and (2.260):

$$\Gamma^a{}_{\mu\nu} := q^a{}_\lambda \, \Gamma^\lambda{}_{\mu\nu}, \tag{2.261}$$

$$\omega^a{}_{\mu\nu} := q^b{}_\nu \, \omega^a{}_{\mu b}. \tag{2.262}$$

These are both 2-forms as well. The tetrad postulate (2.237) can be formulated by inserting these definitions into (2.236):

$$\Gamma^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu + \omega^a{}_{\mu\nu}. \tag{2.263}$$

Inserting the definition of torsion,

$$T^\kappa{}_{\mu\nu} := \Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu}, \tag{2.264}$$

into (2.259) gives

$$T^a{}_{\mu\nu} = q^a{}_\kappa (\Gamma^\kappa{}_{\mu\nu} - \Gamma^\kappa{}_{\nu\mu}) = \Gamma^a{}_{\mu\nu} - \Gamma^a{}_{\nu\mu}, \tag{2.265}$$

and inserting relation (2.263) gives

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu\nu} - \omega^a{}_{\nu\mu}. \tag{2.266}$$

Using the $\wedge$ operator for antisymmetric forms (which was introduced in Example 2.8 and expanded in Section 2.4.3), this can be written as

$$(T^a)_{\mu\nu} = (d \wedge q^a)_{\mu\nu} + (\omega^a{}_b \wedge q^b)_{\mu\nu} \tag{2.267}$$

or, in short form notation:

$$\boxed{T^a = d \wedge q^a + \omega^a{}_b \wedge q^b,} \tag{2.268}$$

which is called the *first Maurer-Cartan structure equation*.

The Riemann curvature tensor is defined by

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}. \tag{2.269}$$

We define an additional 1-form of the Christoffel connection:

$$\Gamma^a{}_{\mu b} := q^a{}_\lambda q^\nu{}_b \Gamma^\lambda{}_{\mu\nu}, \tag{2.270}$$

and from (2.263) we have

$$\Gamma^a{}_{\mu b} = q^\nu{}_b (\partial_\mu q^a{}_\nu + \omega^a{}_{\mu\nu}). \tag{2.271}$$

The curvature form (2.260) can then be written as

$$R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a{}_{\nu b} - \partial_\nu \Gamma^a{}_{\mu b} + \Gamma^a{}_{\mu c}\Gamma^c{}_{\nu b} - \Gamma^a{}_{\nu c}\Gamma^c{}_{\mu b}. \tag{2.272}$$

This is an antisymmetric 2-form that in form notation reads:

$$R^a{}_b = d \wedge \Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b. \tag{2.273}$$

The first term on the right side is

$$d \wedge \Gamma^a{}_b = (d \wedge d \wedge q^a) q_b + d \wedge \omega^a{}_b = d \wedge \omega^a{}_b, \tag{2.274}$$

because of the rule that $d \wedge d \wedge a = 0$ for any form a. The second term of (2.273) is

$$\Gamma^a{}_c \wedge \Gamma^c{}_b = (q_c\, d \wedge q^a + \omega^a{}_c) \wedge (q_b\, d \wedge q^c + \omega^c{}_b). \tag{2.275}$$

The terms with the exterior derivative can be written with full indices as $q^\nu{}_c \partial_\mu q^a{}_\nu$, for example, and the product is summed over by the dummy index ν.

Applying the Leibniz rule gives us

$$q^\lambda{}_c \partial_\mu q^a{}_\lambda + q^a{}_\lambda \partial_\mu q^\lambda{}_c = \partial_\mu (q^\lambda{}_c q^a{}_\lambda) = \partial_\mu \delta^a_c = 0, \tag{2.276}$$

and therefore:

$$q^\lambda{}_c \partial_\mu q^a{}_\lambda = -q^a{}_\lambda \partial_\mu q^\lambda{}_c. \tag{2.277}$$

The summations on the left and right sides can be contracted to functions:

$$q^a{}_c = -q^a{}_c. \tag{2.278}$$

It then follows that

$$q^a{}_c = 0, \tag{2.279}$$

$$q^\nu{}_c \partial_\mu q^a{}_\nu = 0. \tag{2.280}$$

Therefore, from (2.275):

$$\Gamma^a{}_c \wedge \Gamma^c{}_b = \omega^a{}_c \wedge \omega^c{}_b, \tag{2.281}$$

and with (2.274), we obtain from (2.273):

$$\boxed{R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b,} \tag{2.282}$$

which is called the *second Maurer-Cartan structure equation*. Using the definition of the exterior covariant derivative (2.198), the Maurer-Cartan structure equations can be written in the following form:

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b, \tag{2.283}$$

$$R^a{}_b = D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{2.284}$$

■ **Example 2.15** The validity of the structure equations is demonstrated by another example of the transformation to spherical polar coordinates. The tetrad was discussed in Example 2.13, and the spin connections were discussed in Example 2.14. Two versions of the Gamma connections are used: a general, asymmetric connection, and an antisymmetrized connection, as described in Example 2.14. If we know the Gamma connection, we can compute the torsion form:

$$T^a{}_{\nu\mu} = q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \right), \tag{2.285}$$

and the Riemann form:

$$R^a{}_{b\mu\nu} = q^a{}_\sigma q^\mu{}_b \left(\partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\rho} \right). \tag{2.286}$$

This has been done using computer algebra code [126], and the antisymmetry of the form elements in the two last indices has also been verified:

$$T^a{}_{\mu\nu} = -T^a{}_{\nu\mu}\,, \tag{2.287}$$
$$R^a{}_{b\mu\nu} = -R^a{}_{b\nu\mu}. \tag{2.288}$$

For example, with the antisymmetrized connection we find that

$$T^{(2)}{}_{11} = 0 \tag{2.289}$$
$$T^{(2)}{}_{13} = \frac{2\cos(\phi)\sin(\theta)}{\sqrt{3}} + \frac{2A_{10}\sin(\phi)\cos(\theta)}{\sqrt{3}r} \tag{2.290}$$
$$T^{(2)}{}_{31} = -\frac{2\cos(\phi)\sin(\theta)}{\sqrt{3}} - \frac{2A_{10}\sin(\phi)\cos(\theta)}{\sqrt{3}r} \tag{2.291}$$
$$R^{(1)}{}_{(3)11} = 0 \tag{2.292}$$
$$R^{(1)}{}_{(3)13} = -\frac{A_{10}{}^2\sin(\phi)\cos(\theta)}{r^3\sin(\theta)} \tag{2.293}$$
$$R^{(1)}{}_{(3)31} = \frac{A_{10}{}^2\sin(\phi)\cos(\theta)}{r^3\sin(\theta)}. \tag{2.294}$$

Now that all elements of the torsion and curvature forms have been computed, we are ready to evaluate the right sides of the structure equations (2.283) and (2.284), which in indexed form can be written as

$$D_\mu q^a{}_\nu - D_\nu q^a{}_\mu = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu \tag{2.295}$$

and

$$D_\mu \omega^a{}_{\nu b} - D_\nu \omega^a{}_{\mu b} = \partial_\mu \omega^a{}_{\nu b} - \partial_\nu \omega^a{}_{\mu b} + \omega^a{}_{\mu c}\omega^c{}_{\nu b} - \omega^a{}_{\nu c}\omega^c{}_{\mu b}. \tag{2.296}$$

The covariant derivatives have been resolved according to their definitions, for each permutation of (μ,ν). When the indices run over *all* values 1, 2, this does not matter because the antisymmetry property sets all quantities with equal indices, for example $(\mu,\nu) = (1,1)$, to zero. In the computer algebra code [126], it is shown that the right sides of the structure equations are equal to the definitions, (2.285) and (2.286), of the torsion and curvature forms. In addition, it is shown that re-computing the torsion and curvature tensors from their 2-forms gives the original tensors (2.264) and (2.269):

$$T^\rho{}_{\mu\nu} = q^\rho{}_a T^a{}_{\mu\nu}\,, \tag{2.297}$$
$$R^\sigma{}_{\rho\mu\nu} = q^\sigma{}_a\, q^b{}_\rho\, R^a{}_{b\mu\nu}. \tag{2.298}$$

■

3. Fundamental theorems of Cartan geometry

Our knowledge has now reached the point where we can explore fundamental theorems of Cartan geometry. They have been known for a long time and are a part of standard courses (for example, see [8]). These theorems are most elegantly presented in form notation, but proofs necessitate the use of tensor notation.

3.1 Cartan-Bianchi identity

The first theorem that we will discuss is the *Cartan-Bianchi identity* [15], which is also known as the *first Bianchi identity* or simply the *Bianchi identity*, in Riemannian geometry without torsion. We have added Cartan's name to stress that this theorem connects torsion and curvature in Cartan geometry. In form notation, it reads:

$$D \wedge T^a = R^a{}_b \wedge q^b. \tag{3.1}$$

This is an equation of 3-forms. To prove this equation, we recast the left side into the right side. Inserting the definition of the exterior covariant derivative gives, for the left side:

$$(D \wedge T^a)_{\mu\nu\rho} = (d \wedge T^a)_{\mu\nu\rho} + (\omega^a{}_b \wedge T^b)_{\mu\nu\rho}. \tag{3.2}$$

Since this is an antisymmetric 3-form, we can write in commutator notation (see Section 2.3):

$$D_{[\mu} T^a{}_{\nu\rho]} = \partial_{[\mu} T^a{}_{\nu\rho]} + \omega^a{}_{[\mu b} T^b{}_{\nu\rho]} \,. \tag{3.3}$$

In Example 2.8, we had seen that the six index permutations of a 3-form can be reduced to three cyclic permutations of the indices, by use of antisymmetry properties. Therefore, we obtain

$$\begin{aligned} D_{[\mu} T^a{}_{\nu\rho]} =& \partial_\mu T^a{}_{\nu\rho} + \partial_\nu T^a{}_{\rho\mu} + \partial_\rho T^a{}_{\mu\nu} \\ &+ \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} + \omega^a{}_{\rho b} T^b{}_{\mu\nu} \,. \end{aligned} \tag{3.4}$$

Please notice that the lower b index of the spin connection is not included in the permutations, because it is a Latin index of tangent space.

Inserting the definition of torsion,

$$T^a{}_{\nu\mu} = \Gamma^a{}_{\mu\nu} - \Gamma^a{}_{\nu\mu} = q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right), \tag{3.5}$$

then leads to

$$\begin{aligned} D_{[\mu}T^a{}_{\nu\rho]} =& \partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] + \partial_\nu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right)\right] \\ &+ \partial_\rho \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right)\right] \\ &+ \omega^a{}_{\mu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + \omega^a{}_{\nu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right) \\ &+ \omega^a{}_{\rho b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right). \end{aligned} \tag{3.6}$$

With the help of the Leibniz theorem, the first term in brackets can be written as

$$\partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] = \left(\partial_\mu q^a{}_\lambda\right) \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \tag{3.7}$$

Applying the tetrad postulate (2.236) in the following form:

$$\partial_\mu q^a{}_\lambda = q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} - q^b{}_\lambda \omega^a{}_{\mu b} \tag{3.8}$$

then gives

$$\begin{aligned} \partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] =& \left(q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} - q^b{}_\lambda \omega^a{}_{\mu b}\right) \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) \\ &+ q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \end{aligned} \tag{3.9}$$

Adding the first and fourth terms of (3.6) causes the terms with $\omega^a{}_{\mu b}$ to cancel out:

$$\begin{aligned} &\partial_\mu \left[q^a{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right)\right] + \omega^a{}_{\mu b} q^b{}_\lambda \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) \\ &= q^a{}_\sigma \Gamma^\sigma{}_{\mu\lambda} \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right). \end{aligned} \tag{3.10}$$

After combining all terms of (3.6), we obtain

$$\begin{aligned} D_{[\mu}T^a{}_{\nu\rho]} =& \\ & q^a{}_\sigma \Gamma^\sigma{}_{\mu\lambda} \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + q^a{}_\lambda \left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right) \\ &+ q^a{}_\sigma \Gamma^\sigma{}_{\nu\lambda} \left(\Gamma^\lambda{}_{\rho\mu} - \Gamma^\lambda{}_{\mu\rho}\right) + q^a{}_\lambda \left(\partial_\nu \Gamma^\lambda{}_{\rho\mu} - \partial_\nu \Gamma^\lambda{}_{\mu\rho}\right) \\ &+ q^a{}_\sigma \Gamma^\sigma{}_{\rho\lambda} \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) + q^a{}_\lambda \left(\partial_\rho \Gamma^\lambda{}_{\mu\nu} - \partial_\rho \Gamma^\lambda{}_{\nu\mu}\right). \end{aligned} \tag{3.11}$$

Rearranging the sum gives us

$$\begin{aligned} D_{[\mu}T^a{}_{\nu\rho]} =& \\ & q^a{}_\lambda \left[\left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho}\right) + q^a{}_\lambda \left(\partial_\nu \Gamma^\lambda{}_{\rho\mu} - \partial_\rho \Gamma^\lambda{}_{\nu\mu}\right) + q^a{}_\lambda \left(\partial_\rho \Gamma^\lambda{}_{\mu\nu} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right)\right] \\ &+ q^a{}_\sigma \left[\Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\rho\nu}\right]. \end{aligned} \tag{3.12}$$

Next, in the first line, we replace the dummy index λ with σ:

$$\begin{aligned} D_{[\mu}T^a{}_{\nu\rho]} =& \\ & q^a{}_\sigma \left[\left(\partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho}\right) + q^a{}_\sigma \left(\partial_\nu \Gamma^\sigma{}_{\rho\mu} - \partial_\rho \Gamma^\sigma{}_{\nu\mu}\right) + q^a{}_\sigma \left(\partial_\rho \Gamma^\sigma{}_{\mu\nu} - \partial_\mu \Gamma^\sigma{}_{\rho\nu}\right)\right. \\ &\left.+ \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} - \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\lambda} \Gamma^\lambda{}_{\rho\mu} - \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\nu\mu} + \Gamma^\sigma{}_{\rho\lambda} \Gamma^\lambda{}_{\mu\nu} - \Gamma^\sigma{}_{\mu\lambda} \Gamma^\lambda{}_{\rho\nu}\right]. \end{aligned} \tag{3.13}$$

After some renumbering, this expression is comparable to the definition of the Riemann tensor (2.269):

$$R^{\sigma}{}_{\rho\mu\nu} := \partial_\mu \Gamma^{\sigma}{}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\rho}. \tag{3.14}$$

As we can see, (3.13) contains the cyclic sum of the Riemann tensor:

$$D_{[\mu} T^a{}_{\nu\rho]} = q^a{}_\sigma\, R^{\sigma}{}_{[\rho\mu\nu]} = q^a{}_\sigma\, R^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.15}$$

According to the procedure that we used in Eqs. (2.270 - 2.273), the Riemann tensor can be written as a 2-Form:

$$R^a{}_{b\nu\rho} = q^a{}_\sigma q^\mu{}_b\, R^{\sigma}{}_{\mu\nu\rho}. \tag{3.16}$$

To bring the right side of (3.15) into this form, we extend the Riemann tensor by a unity term, according to rule (2.207):

$$q^\tau{}_b q^b{}_\mu = \delta^\tau_\mu, \tag{3.17}$$

and re-associate the products:

$$q^a{}_\sigma\, R^{\sigma}{}_{\mu\nu\rho} = R^a{}_{\mu\nu\rho} = R^a{}_{\tau\nu\rho}(q^\tau{}_b\, q^b{}_\mu)\delta^\tau_\mu = (R^a{}_{\tau\nu\rho} q^\tau{}_b)\; q^b{}_\tau \delta^\tau_\mu = R^a{}_{b\nu\rho}\, q^b{}_\mu. \tag{3.18}$$

Reintroducing the cyclic sum gives us

$$D_{[\mu} T^a{}_{\nu\rho]} = q^b{}_{[\mu}\, R^a{}_{b\nu\rho]} = R^a{}_{b[\mu\nu}\, q^b{}_{\rho]}, \tag{3.19}$$

which in form notation is the Cartan-Bianchi identity (3.1):

$$\boxed{D \wedge T^a = R^a{}_b \wedge q^b.} \tag{3.20}$$

■ **Example 3.1** We check the Cartan-Bianchi identity by computing all required elements according to Example 2.15 (the transformation from cartesian to spherical polar coordinates). By following Eq. (3.19), we can write the Cartan-Bianchi identity (3.20) in indexed form:

$$D_\mu T^a{}_{\nu\rho} + D_\nu T^a{}_{\rho\mu} + D_\rho T^a{}_{\mu\nu} = R^a{}_{b\mu\nu}\, q^b{}_\rho + R^a{}_{b\nu\rho}\, q^b{}_\mu + R^a{}_{b\rho\mu}\, q^b{}_\nu. \tag{3.21}$$

Resolving the covariant derivatives according to (3.4) ultimately gives

$$\begin{aligned} &\partial_\mu T^a{}_{\nu\rho} + \partial_\nu T^a{}_{\rho\mu} + \partial_\rho T^a{}_{\mu\nu} + \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} + \omega^a{}_{\rho b} T^b{}_{\mu\nu} \\ &= R^a{}_{b\mu\nu}\, q^b{}_\rho + R^a{}_{b\nu\rho}\, q^b{}_\mu + R^a{}_{b\rho\mu}\, q^b{}_\nu, \end{aligned} \tag{3.22}$$

for each index triple (μ,ν,ρ). The left and right sides of this equation have been computed using computer algebra code [127], and comparison has shown that both sides are equal. We note that this result is obtained for both forms of Gamma connections (unconstrained and symmetrized). The torsion tensor is the same for both forms, but the Gamma and spin connections are different. The Cartan-Bianchi identity holds, irrespective of this difference. ■

3.2 Cartan-Evans identity

In the preceding section, it has been shown that the Cartan-Bianchi identity is a rigorous identity of the Riemannian manifold, in which ECE theory is defined. The *Cartan-Evans identity* [16–18] is a new identity of differential geometry, and is the counterpart of the Cartan-Bianchi identity, in dual-tensor representation. In later chapters, the ECE field equations will be shown to be identical to the Cartan-Bianchi and the Cartan-Evans identities. The Cartan-Bianchi identity is valid in the Riemannian manifold, and Cartan geometry in the Riemannian manifold is well known to be equivalent to Riemann geometry, which is considered to be the geometry of natural philosophy (physics). The same holds for the Cartan-Evans identity, which reads:

$$D \wedge \widetilde{T}^a = \widetilde{R}^a{}_b \wedge q^b. \tag{3.23}$$

The concept of the Hodge dual was introduced at the end of Section 2.3, and use of the Hodge dual for Maxwell's equations has already been discussed in Example 2.12. In this section, we will introduce a Hodge dual connection for use in the covariant Hodge dual derivative, and then work out the proof of the Cartan-Evans identity analogously to the proof of the Cartan-Bianchi identity.

As discussed in previous sections, only the antisymmetric part of the Christoffel connection is essential for Cartan geometry. By restricting the connection to the antisymmetric part, we can use Eq. (2.114) to define the Hodge dual of the Christoffel connection:

$$\Lambda^\lambda{}_{\mu\nu} := \widetilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\beta}, \tag{3.24}$$

where $|g|^{-1/2}$ (the inverse square root of the modulus of the determinant of the metric) is a weighting factor that turns the Levi-Civita symbol $\varepsilon_{\alpha\beta\mu\nu}$ into the totally antisymmetric unit tensor (see Section 2.3, starting at Eq. (2.101)). In (3.24), the Levi-Civita symbol appears with mixed upper and lower indices. Therefore, we have to raise the first two indices in accordance with (2.115):

$$\Lambda^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}g^{\rho\alpha}g^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^\lambda{}_{\alpha\beta}. \tag{3.25}$$

Since the totally antisymmetric tensor (based on the Levi-Civita symbol) does not change its form in any coordinate transformation, we can use the metric of Minkowski space $\eta_{\mu\nu}$ with $|g| = 1$:

$$\Lambda^\lambda{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\eta^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^\lambda{}_{\alpha\beta}. \tag{3.26}$$

In this way, a new connection $\Lambda^\lambda{}_{\mu\nu}$ is defined. It is well known that the connection does not transform as a tensor under the general coordinate transformation, but the antisymmetry in its lower two indices means that its Hodge dual may be defined for each upper index of the connection, as in the equation above. The antisymmetry of the connection is the basis for the Cartan-Evans identity, a new and fundamental identity of differential geometry. In ECE theory, it will be used to derive the homogeneous field equations, analogously to the derivations of the homogeneous Maxwell equations in Example 2.12. Note carefully that the torsion is a tensor, but the connection is not a tensor. The same is true for the Hodge duals of the torsion and the connection.

The fundamental commutator equation of Riemannian geometry was derived as Eq. (2.147):

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T^\lambda{}_{\mu\nu}\, D_\lambda V^\rho, \tag{3.27}$$

which holds for any vector V^ρ of the base manifold. We now take the Hodge duals of both sides of

Eq. (3.27) using

$$[D_\mu, D_\nu]_{\mathrm{HD}} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}[D_\alpha, D_\beta], \tag{3.28}$$

$$\widetilde{R}^\rho{}_{\sigma\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}R^\rho{}_{\sigma\alpha\beta}, \tag{3.29}$$

$$\widetilde{T}^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}T^\lambda{}_{\alpha\beta}. \tag{3.30}$$

This gives us

$$[D_\alpha, D_\beta]_{\mathrm{HD}}V^\rho = \widetilde{R}^\rho{}_{\sigma\alpha\beta}V^\sigma - \widetilde{T}^\lambda{}_{\alpha\beta}\,D_\lambda V^\rho. \tag{3.31}$$

Next, we relabel indices in Eq. (3.31):

$$[D_\mu, D_\nu]_{\mathrm{HD}}V^\rho = \widetilde{R}^\rho{}_{\sigma\mu\nu}V^\sigma - \widetilde{T}^\lambda{}_{\mu\nu}\,D_\lambda V^\rho. \tag{3.32}$$

The left side of this equation is defined by

$$[D_\mu, D_\nu]_{\mathrm{HD}}V^\rho := D_\mu(D_\nu V^\rho) - D_\nu(D_\mu V^\rho), \tag{3.33}$$

where the covariant derivatives must be used with the Hodge dual connection (which was defined in Eq. (3.24)):

$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^\rho{}_{\mu\lambda}V^\lambda, \tag{3.34}$$

$$D_\nu V^\rho = \partial_\nu V^\rho + \Lambda^\rho{}_{\nu\lambda}V^\lambda. \tag{3.35}$$

We then work out the algebra for the torsion and curvature according to Eqs. (2.132, 2.134):

$$\widetilde{T}^\lambda{}_{\mu\nu} = \Lambda^\lambda{}_{\mu\nu} - \Lambda^\lambda{}_{\nu\mu}, \tag{3.36}$$

$$\widetilde{R}^\lambda{}_{\mu\nu\rho} = \partial_\mu\Lambda^\lambda{}_{\nu\rho} - \partial_\nu\Lambda^\lambda{}_{\mu\rho} + \Lambda^\lambda{}_{\mu\sigma}\Lambda^\sigma{}_{\nu\rho} - \Lambda^\lambda{}_{\nu\sigma}\Lambda^\sigma{}_{\mu\rho}. \tag{3.37}$$

These are the Hodge dual torsion and curvature tensors of the Riemannian manifold.

We will now prove the Cartan Evans identity. We start with two forms of this identity:

$$D \wedge \widetilde{T}^a = \widetilde{R}^a{}_b \wedge q^b \tag{3.38}$$

or

$$d \wedge \widetilde{T}^a + \omega^a{}_b \wedge \widetilde{T}^b = \widetilde{R}^a{}_b \wedge q^b. \tag{3.39}$$

In tensorial notation, in the Riemannian manifold, Eqs. (3.38, 3.39) become

$$D_\mu\widetilde{T}^a{}_{\nu\rho} + D_\rho\widetilde{T}^a{}_{\mu\nu} + D_\nu\widetilde{T}^a{}_{\rho\mu} = \widetilde{R}^a{}_{\mu\nu\rho} + \widetilde{R}^a{}_{\rho\mu\nu} + \widetilde{R}^a{}_{\nu\rho\mu}, \tag{3.40}$$

which can be written with permutation brackets as

$$D_{[\mu}\widetilde{T}^a{}_{\nu\rho]} = \widetilde{R}^a{}_{[\mu\nu\rho]} = q^a{}_\sigma\,\widetilde{R}^\sigma{}_{[\mu\nu\rho]}. \tag{3.41}$$

This equation is formally identical to (3.15) with the following correspondences:

$$\begin{aligned} T &\to \widetilde{T}, \\ R &\to \widetilde{R}, \\ \Gamma &\to \Lambda. \end{aligned} \tag{3.42}$$

Therefore, the proof of the Cartan-Evans identity can proceed analogously to that of the Cartan-Bianchi identity in the previous section. Starting with the equivalent of the left side of Eq. (3.15),

$$D_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} = \partial_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} + \omega^{a}{}_{[\mu b}\widetilde{T}^{b}{}_{\nu\rho]}\,, \tag{3.43}$$

it follows that this expression is equal to its right-side equivalent of (3.15):

$$q^{a}{}_{\sigma}\,\widetilde{R}^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.44}$$

It follows from Eqs. (3.40 / 3.41), which are the counterpart of (3.19), that

$$D_{[\mu}\widetilde{T}^{a}{}_{\nu\rho]} = q^{a}{}_{\sigma}\widetilde{R}^{\sigma}{}_{[\mu\nu\rho]}. \tag{3.45}$$

In form notation, this is the Cartan-Evans identity:

$$\boxed{D\wedge\widetilde{T}^{a} = \widetilde{R}^{a}{}_{b}\wedge q^{b}.} \tag{3.46}$$

In the proof of the Cartan-Bianchi identity, the tetrad postulate (2.237) was used. For the Cartan-Evans identity, the version with the Λ connection has to be used:

$$\partial_{\mu}q^{a}{}_{\lambda} + q^{b}{}_{\lambda}\,\omega^{a}{}_{\mu b} - q^{a}{}_{\nu}\Lambda^{\nu}{}_{\mu\lambda} = 0. \tag{3.47}$$

In this instance, the spin connection ω depends on the Λ connection, not the Γ connection. (It would have been best to use a different symbol for ω, but we remained with ω for convenience.)

In summary, all geometrical elements for the Cartan-Evans identity are obtained from the following equation set:

$$\Lambda^{\lambda}{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\eta^{\rho\alpha}\eta^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^{\lambda}{}_{\alpha\beta}, \tag{3.48}$$

$$\omega^{a}{}_{\mu b} = q^{a}{}_{\nu}q^{\lambda}{}_{b}\Lambda^{\nu}{}_{\mu\lambda} - q^{\lambda}{}_{b}\partial_{\mu}q^{a}{}_{\lambda}, \tag{3.49}$$

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \Lambda^{\lambda}_{\mu\nu} - \Lambda^{\lambda}_{\nu\mu}, \tag{3.50}$$

$$\widetilde{R}^{\lambda}{}_{\mu\nu\rho} = \partial_{\mu}\Lambda^{\lambda}_{\nu\rho} - \partial_{\nu}\Lambda^{\lambda}_{\mu\rho} + \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\nu\rho} - \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\mu\rho}. \tag{3.51}$$

Alternatively, the Hodge duals of curvature and torsion can be computed from the original quantities (based on the Γ connection):

$$\widetilde{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}R^{\rho}{}_{\sigma\alpha\beta}, \tag{3.52}$$

$$\widetilde{T}^{\lambda}{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\varepsilon^{\alpha\beta}{}_{\mu\nu}T^{\lambda}{}_{\alpha\beta}. \tag{3.53}$$

The 2-forms of $\widetilde{T}^{a}$ and $\widetilde{R}^{a}{}_{b}$ are obtainable in the usual way, by multiplication with tetrad elements:

$$\widetilde{R}^{a}{}_{b\mu\nu} = q^{a}{}_{\rho}q^{\sigma}{}_{b}\widetilde{R}^{\rho}{}_{\sigma\mu\nu}, \tag{3.54}$$

$$\widetilde{T}^{a}{}_{\mu\nu} = q^{a}{}_{\lambda}\widetilde{T}^{\lambda}{}_{\mu\nu}\,. \tag{3.55}$$

One of the novel inferences of the Cartan-Evans identity is that a Hodge dual connection is possible in the Riemannian manifold in four dimensions. This is a basic discovery, and may be developed in pure mathematics using any type of manifold. However, whether this type of pure mathematics generalization could become relevant to physics is not immediately apparent.

■ **Example 3.2** Analogously to Example 3.1, we check the Cartan-Evans identity by computing all required elements according to Example 2.15 (the transformation from cartesian to spherical polar coordinates). As we see from (3.45), the Cartan-Evans identity (3.46) can be written in indexed form:

$$D_\mu \widetilde{T}^a{}_{\nu\rho} + D_\nu \widetilde{T}^a{}_{\rho\mu} + D_\rho \widetilde{T}^a{}_{\mu\nu} = \widetilde{R}^a{}_{b\mu\nu}\, q^b{}_\rho + \widetilde{R}^a{}_{b\nu\rho}\, q^b{}_\mu + \widetilde{R}^a{}_{b\rho\mu}\, q^b{}_\nu. \tag{3.56}$$

Resolving the covariant derivatives according to (3.4) ultimately gives us

$$\begin{aligned} &\partial_\mu \widetilde{T}^a{}_{\nu\rho} + \partial_\nu \widetilde{T}^a{}_{\rho\mu} + \partial_\rho \widetilde{T}^a{}_{\mu\nu} + \omega^a{}_{\mu b} \widetilde{T}^b{}_{\nu\rho} + \omega^a{}_{\nu b} \widetilde{T}^b{}_{\rho\mu} + \omega^a{}_{\rho b} \widetilde{T}^b{}_{\mu\nu} \\ &= \widetilde{R}^a{}_{b\mu\nu}\, q^b{}_\rho + \widetilde{R}^a{}_{b\nu\rho}\, q^b{}_\mu + \widetilde{R}^a{}_{b\rho\mu}\, q^b{}_\nu, \end{aligned} \tag{3.57}$$

for each index triple (μ, ν, ρ). The left and right sides of this equation have been computed using computer algebra code [128, 129]. There is, however, a difference. While the Cartan-Bianchi identity holds for any dimension n of Riemannian space, introducing the Hodge dual for the Cartan-Evans identity constrains the dimension of the dual 2-forms to $n-2$. Therefore, to obtain comparable equations for both identities, we have to use $n = 4$, which gives us 2-forms of the Hodge duals, as well. We have to extend the transformation matrix α (Eq. (2.213)) by the 0-component (time coordinate), resulting in

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{3.58}$$

The time coordinate remains unaltered by the transformation. The n-dimensional metric tensor $\mathbf{g}$ can be computed from the tetrad by using (2.204):

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu \eta_{ab}. \tag{3.59}$$

We obtain the metric tensor:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}, \tag{3.60}$$

and the modulus of the determinant is

$$|g| = r^4 \sin^2\theta. \tag{3.61}$$

The Levi-Civita symbol $\varepsilon_{\alpha\beta\mu\nu}$ in four dimensions can be computed by using the formula:

$$\varepsilon_{a_0,a_1,a_2,a_3} = \mathrm{sig}(a_3 - a_0)\ \mathrm{sig}(a_3 - a_1)\ \mathrm{sig}(a_3 - a_2)\ \mathrm{sig}(a_2 - a_0)\ \mathrm{sig}(a_2 - a_1)\ \mathrm{sig}(a_1 - a_0). \tag{3.62}$$

We now have all of the elements that we need to evaluate Eqs. (3.48 - 3.51). These elements allow us to evaluate both sides of Eq. (3.56) by following the same approach that was used in Example 3.1. We do this in two ways, using computer algebra. The first method repeats the calculations of Example 3.1 (Cartan-Bianchi identity), but in four dimensions [128]. An interesting result is that the Gamma connection, obtained with additional antisymmetry conditions, has only 4 free parameters. This is similar to Einstein's theory, where the symmetric metric is only determined up to 4 parameters, which can then be chosen freely and represent "free choice of coordinates". In

Cartan geometry, the metric is uniquely determined by the tetrad. The "free choice" appears in the connections. Therefore, this choice is also present in torsion and curvature, and in the fundamental theorems. The following are some results from computer algebra code [128]:

$$
\begin{aligned}
\Gamma^0{}_{12} &= A_4 r^2 \\
\Gamma^3{}_{01} &= -\frac{A_2}{r^2 \sin^2\theta} \\
\omega^{(2)}{}_{1(3)} &= \frac{A_3 \cos\phi}{r^2} \\
T^2{}_{13} &= \frac{2A_3}{r^2} \\
R^0{}_{213} &= \frac{A_1 r \sin\theta - A_2 \cos\theta}{\sin\theta}.
\end{aligned} \tag{3.63}
$$

As in Example 3.1, comparison of both sides of the Cartan-Evans identity shows that both sides are equal, in this case for four dimensions.

The second method uses computer algebra code [129] to compute the Hodge dual connections Λ and ω and the tensors $\widetilde{T}$, $\widetilde{R}$, and their corresponding 2-forms. We obtain, for example, for the Hodge dual connections and tensors:

$$
\begin{aligned}
\Lambda^0{}_{03} &= \frac{A_4}{\sin^2\theta} \\
\Lambda^3{}_{01} &= \frac{\cos\theta}{r^2 \sin^3\theta} \\
\omega^{(2)}{}_{1(3)} &= \frac{\sin\phi\,\left(r^2\cos\theta\sin^2\theta + A_2\sin\theta - A_1 r\cos\theta\right)}{r^3\sin\theta} \\
\widetilde{T}^2{}_{13} &= 0 \\
\widetilde{T}^2{}_{02} &= -\frac{2A_3}{r^4\sin^2\theta} \\
\widetilde{R}^0{}_{213} &= 0 \\
\widetilde{R}^0{}_{202} &= -\frac{2A_2 r^2\cos\theta\sin\theta - A_2{}^2}{r^4\sin^4\theta}.
\end{aligned} \tag{3.64}
$$

After inserting these quantities into both sides of the Cartan-Evans identity, we found that both sides are equal, thus the identity holds in the chosen example. ■

3.3 Alternative forms of the Cartan-Bianchi and Cartan-Evans identities

3.3.1 Cartan-Evans identity

We have shown that the Cartan-Evans identity is based on the fundamental definitions of the Hodge duals of torsion and curvature, and combines any three of their elements in a cyclic permutation (Eq. (3.45)). By using the definition

$$\widetilde{T}^a{}_{\mu\nu} = q^a{}_\lambda \widetilde{T}^\lambda{}_{\mu\nu}\,, \tag{3.65}$$

and the Leibniz rule, we obtain

$$D_\mu \widetilde{T}^a{}_{\nu\rho} = (D_\mu q^a{}_\kappa)\widetilde{T}^\kappa{}_{\nu\rho} + q^a{}_\kappa D_\mu \widetilde{T}^\kappa{}_{\nu\rho}\,. \tag{3.66}$$

Furthermore, by using the tetrad postulate,

$$D_\mu q^a{}_\kappa = 0, \tag{3.67}$$

we find that

$$D_\mu \widetilde{T}^a{}_{\nu\rho} = q^a{}_\kappa D_\mu \widetilde{T}^\kappa{}_{\nu\rho} \,. \tag{3.68}$$

It then follows that

$$D_\mu \widetilde{T}^\kappa{}_{\nu\rho} + D_\nu \widetilde{T}^\kappa{}_{\rho\mu} + D_\rho \widetilde{T}^\kappa{}_{\mu\nu} = \widetilde{R}^\kappa{}_{\mu\nu\rho} + \widetilde{R}^\kappa{}_{\nu\rho\mu} + \widetilde{R}^\kappa{}_{\rho\mu\nu}, \tag{3.69}$$

which is the Cartan-Evans identity written for the base manifold only. This equation may be rewritten as

$$D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}. \tag{3.70}$$

The easiest way to see this is to take a particular example:

$$D_1 \widetilde{T}^\kappa{}_{23} + D_3 \widetilde{T}^\kappa{}_{12} + D_2 \widetilde{T}^\kappa{}_{31} = \widetilde{R}^\kappa{}_{123} + \widetilde{R}^\kappa{}_{312} + \widetilde{R}^\kappa{}_{231}, \tag{3.71}$$

and then take Hodge dual terms with upper indices according to Eq. (2.117), so that the constant factors cancel out. For the Levi-Civita symbol, the following relation holds:

$$\varepsilon^{\mu\nu\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}, \tag{3.72}$$

so that the sign change also cancels out. Furthermore, for a double Hodge dual of a tensor T, the relation

$$\widetilde{\widetilde{T}} = \pm T \tag{3.73}$$

is valid, so that any sign change of this kind also cancels out. We then take the Hodge dual of (3.71) term by term. The Levi-Civita symbol effects that in the expressions

$$\varepsilon^{\mu\nu\alpha\beta} T^\kappa{}_{\alpha\beta} \tag{3.74}$$

the index pairs $(\mu\nu)$ and $(\alpha\beta)$ are mutually exclusive:

$$\begin{aligned} &\mu \neq \nu, \quad \alpha \neq \beta, \\ &\mu \notin \{\alpha, \beta\}, \\ &\nu \notin \{\alpha, \beta\}. \end{aligned} \tag{3.75}$$

The final result that we obtain for the Hodge dual example, (3.71), is

$$D_1 T^{\kappa 01} + D_2 T^{\kappa 02} + D_3 T^{\kappa 03} = R^\kappa{}_1{}^{01} + R^\kappa{}_2{}^{02} + R^\kappa{}_3{}^{03}, \tag{3.76}$$

which is an example of Eq. (3.70), the alternative form of the Cartan-Evans identity:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}.} \tag{3.77}$$

3.3.2 Cartan-Bianchi identity

Eq. (3.77) is the most useful format of the Cartan-Evans identity. The Cartan-Bianchi identity can also be rewritten in this format. From Eq. (3.19) it follows that

$$D_\mu T^\kappa{}_{\nu\rho} + D_\nu T^\kappa{}_{\rho\mu} + D_\rho T^\kappa{}_{\mu\nu} = R^\kappa{}_{\mu\nu\rho} + R^\kappa{}_{\nu\rho\mu} + R^\kappa{}_{\rho\mu\nu}, \tag{3.78}$$

which is identical to (3.69), except that these are the original tensors instead of the Hodge duals. Therefore, the same derivation as above leads to the alternative form of the Cartan-Bianchi-identity:

$$\boxed{D_\mu \widetilde{T}^{\kappa\mu\nu} = \widetilde{R}^\kappa{}_\mu{}^{\mu\nu}.} \tag{3.79}$$

It should be noted that in the above *contravariant* forms of both identities, the Hodge duals and the original tensors are interchanged, compared to the covariant forms (3.19) and (3.45).

3.3.3 Consequences of the identities

In this section we will investigate the implications of the antisymmetry of the Gamma connection in Cartan geometry. The Gamma connection, at least, has to have antisymmetric parts with

$$\Gamma^{\lambda}{}_{\mu\nu} = -\Gamma^{\lambda}{}_{\nu\mu}. \tag{3.80}$$

If $\mu = \nu$, the commutator vanishes, as do the torsion and curvature tensors. If there are only symmetric parts in the connection:

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} \neq ?\, 0, \tag{3.81}$$

then torsion vanishes, leading to the special case of (3.77):

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} = 0. \tag{3.82}$$

It has been shown by computer algebra [19, 20] that all of the metrics of the Einstein field equation in the presence of matter give the erroneous result:

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq ?\, 0, \tag{3.83}$$

$$D_{\mu} T^{\kappa\mu\nu} = ?\, 0. \tag{3.84}$$

This contradicts basic properties of Cartan geometry, a superset of Riemannian geometry, and therefore Eq. (3.77) is a constraint for theories like Einsteinian relativity, which are based on Riemannian geometry. This error has been perpetuated uncritically for nearly a hundred years and has enabled a flawed Standard Model cosmology, which should be replaced with a cosmology that includes torsion, in accordance with the principles of the scientific method.

3.4 Further identities

There are additional identities that are not as significant to the field equations of ECE theory, but which represent new insights into Cartan geometry. They were developed as part of ECE theory, and we present them here, partially without proofs (which can be found in the referenced papers).

3.4.1 Evans torsion identity (first Evans identity)

The *Evans torsion identity* [17], which contains only torsion terms, can be derived from the Cartan-Bianchi identity. In explicit form, the Evans torsion identity reads:

$$T^{\kappa}{}_{\lambda\nu}\, T^{\lambda}{}_{\sigma\mu} + T^{\kappa}{}_{\lambda\mu}\, T^{\lambda}{}_{\nu\sigma} + T^{\kappa}{}_{\lambda\sigma}\, T^{\lambda}{}_{\mu\nu} = 0, \tag{3.85}$$

and in short form with permutation brackets:

$$\boxed{T^{\kappa}{}_{\lambda[\nu}\, T^{\lambda}{}_{\sigma\mu]} = 0.} \tag{3.86}$$

This identity can be rewritten in form notation as

$$T^{\kappa}{}_{\lambda} \wedge T^{\lambda} = 0, \tag{3.87}$$

or (after multiplying by $q^{a}{}_{\kappa}$) as

$$\boxed{T^{a}{}_{\lambda} \wedge T^{\lambda} = 0.} \tag{3.88}$$

Here $T^{a}{}_{\lambda}$ is a 1-form and T^{λ} is a 2-form, giving us a 3-form on the left side. The proof, which consists mainly of inserting the definition of torsion into the Cartan-Bianchi identity, can be found in [17].

3.4.2 Jacobi identity

The Jacobi identity [21] is an exact identity used in field theory and general relativity. It is an operator identity that applies to covariant derivatives and group generators [22] alike. It is very rarely proven in detail, so we are providing the following complete proof. The *Jacobi identity* is a permuted sum of three covariant derivatives:

$$\boxed{[D_\rho,[D_\mu,D_\nu]]+[D_\nu,[D_\rho,D_\mu]]+[D_\mu,[D_\nu,D_\rho]]=0.} \tag{3.89}$$

For the proof, we expand the commutators on the left side:

$$\begin{aligned} L.S. =&[D_\rho,D_\mu D_\nu-D_\nu D_\mu]+[D_\nu,D_\rho D_\mu-D_\mu,D_\rho]+[D_\mu,D_\nu D_\rho-D_\rho D_\nu] \\ =&D_\rho(D_\mu D_\nu-D_\nu D_\mu)-(D_\mu D_\nu-D_\nu D_\mu)D_\rho \\ &+D_\nu(D_\rho D_\mu-D_\mu,D_\rho)-(D_\rho D_\mu-D_\mu,D_\rho)D_\nu \\ &+D_\mu(D_\nu D_\rho-D_\rho D_\nu)-(D_\nu D_\rho-D_\rho D_\nu)D_\mu. \end{aligned} \tag{3.90}$$

This expansion is regarded as an expansion by algebra, and it sums to zero:

$$\begin{aligned} L.S. =&D_\rho D_\mu D_\nu-D_\rho D_\nu D_\mu-D_\mu D_\nu D_\rho+D_\nu D_\mu D_\rho \\ &+D_\nu D_\rho D_\mu-D_\nu D_\mu D_\rho-D_\rho D_\mu D_\nu+D_\mu D_\rho D_\nu \\ &+D_\mu D_\nu D_\rho-D_\mu D_\rho D_\nu-D_\nu D_\rho D_\mu+D_\rho D_\nu D_\mu \\ =&0. \end{aligned} \tag{3.91}$$

Q.E.D. The Jacobi identity can also be written in an alternative form:

$$\boxed{[[D_\mu,D_\nu],D_\rho]+[[D_\rho,D_\mu],D_\nu]+[[D_\nu,D_\rho],D_\mu]=0.} \tag{3.92}$$

3.4.3 Bianchi-Cartan-Evans identity

Einsteinian general relativity uses the second Bianchi identity, which is obtained from the covariant derivative of the first Bianchi identity. In that derivation, general relativity omits torsion, but the same procedure can be applied to the Cartan-Bianchi identity of Cartan geometry, which contains both torsion and curvature. The result is the *Bianchi-Cartan-Evans identity* [23–25]:

$$\boxed{D_\mu D_\lambda T^\kappa{}_{\nu\rho}+D_\rho D_\lambda T^\kappa{}_{\mu\nu}+D_\nu D_\lambda T^\kappa{}_{\rho\mu}=D_\mu R^\kappa{}_{\lambda\nu\rho}+D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}.} \tag{3.93}$$

The proof was first carried out in UFT Paper 88 [23] which is the most read paper of ECE theory. Two variants of this identity can be produced by cyclic permutation of (μ,ν,ρ). Eq. (3.93) is the correct "second Bianchi identity", because it is augmented with torsion. In Einsteinian theory, the incorrect version,

$$D_\mu R^\kappa{}_{\lambda\nu\rho}+D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}=?\,0, \tag{3.94}$$

is used. This follows from (3.93) by arbitrarily omitting torsion. The Einstein field equation is derived from this erroneously truncated "second Bianchi identity" [23]. Therefore, all solutions of the Einstein field equation are inconsistent.

It should be noted that the Bianchi-Cartan-Evans identity gives no information beyond what is provided by the Cartan-Bianchi identity, because it is derived from the latter by differentiation and is therefore not independent.

3.4.4 Jacobi-Cartan-Evans identity

The Jacobi identity can be used to derive the Jacobi-Cartan-Evans identity. When the terms of the Cartan-Bianchi identity are inserted into the Jacobi identity (3.89), we get the relation

$$
\begin{aligned}
&\left([D_\rho,[D_\mu,D_\nu]]+[D_\nu,[D_\rho,D_\mu]]+[D_\mu,[D_\nu,D_\rho]]\right)V^\kappa \\
&=\left(D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda \\
&\quad-\left(T^\lambda{}_{\mu\nu}\,[D_\rho,D_\lambda]+T^\lambda{}_{\rho\mu}\,[D_\nu,D_\lambda]+T^\lambda{}_{\nu\rho}\,[D_\mu,D_\lambda]\right)V^\kappa \\
&=0,
\end{aligned}
\tag{3.95}
$$

in which the Jacobi identity has been applied to an arbitrary vector V^κ of the base manifold. Because the Jacobi identity sums to zero, we obtain the equation

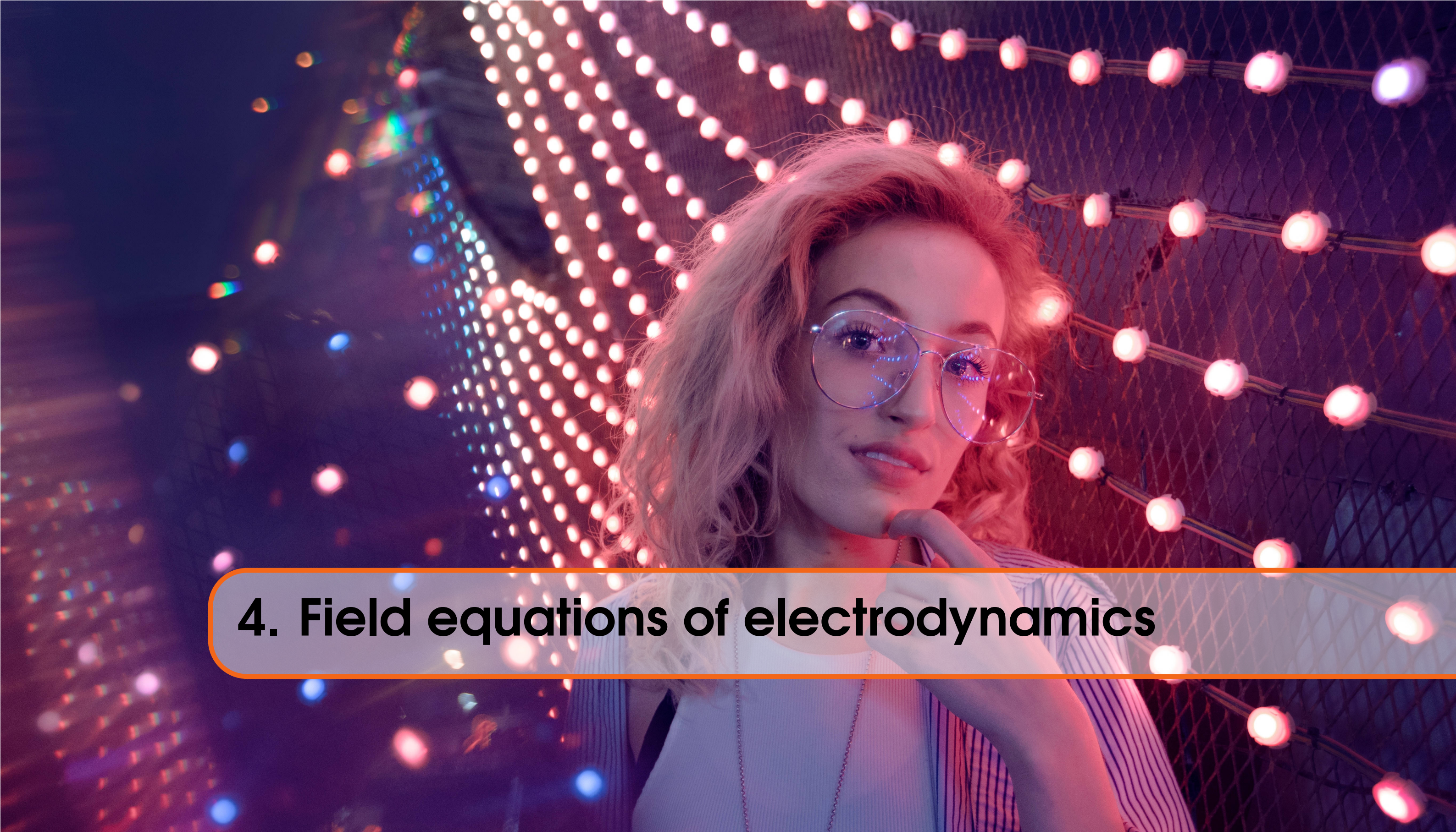
$$
\begin{aligned}
&\left(D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda \\
&=\left(T^\lambda{}_{\mu\nu}\,[D_\rho,D_\lambda]+T^\lambda{}_{\rho\mu}\,[D_\nu,D_\lambda]+T^\lambda{}_{\nu\rho}\,[D_\mu,D_\lambda]\right)V^\kappa.
\end{aligned}
\tag{3.96}
$$

Further transformations, described in [25], give

$$
\boxed{D_\rho R^\kappa{}_{\lambda\mu\nu}+D_\nu R^\kappa{}_{\lambda\rho\mu}+D_\mu R^\kappa{}_{\lambda\nu\rho}=T^\alpha{}_{\mu\nu}\,R^\kappa{}_{\lambda\rho\alpha}+T^\alpha{}_{\rho\mu}\,R^\kappa{}_{\lambda\nu\alpha}+T^\alpha{}_{\nu\rho}\,R^\kappa{}_{\lambda\mu\alpha},}
\tag{3.97}
$$

which is called the *Jacobi-Cartan-Evans identity*. This will be used in an advanced version of ECE theory, the ECE2 theory that is introduced in Chapter 6.

Part Two: Electrodynamics



4. Field equations of electrodynamics

In the preceding chapter, we developed the mathematical methodology for ECE theory: Cartan geometry and its most important theorems. Now we switch our focus to physics. We first describe how physical quantities are obtained from geometrical quantities. We start by defining suitable axioms, and then derive the field equations of electromagnetism, as well as the wave equation. In this way, electrodynamics is transformed into an axiomatic, mathematically correct theory. We also discuss how the spin connections extend classical electromagnetism to a theory of general relativity, and how these novel concepts are consistent with existing laws of physics and supported empirically.

4.1 Axioms

In order to obtain physical quantities, we first have to define how geometry is transformed into physics. We do this through two fundamental axioms that relate to potentials and electromagnetic fields.

The first axiom states that the electromagnetic potential is proportional to the Cartan tetrad. Thus, the geometry of spacetime is directly equated to physical quantities. The potential contains the same indices as the tetrad, and we use the 4-vector potential A_μ in relativistic notation. However, the tetrad $q^a{}_\mu$ is (formally) a matrix and contains the polarization index a. Therefore, the potential is extended to matrix form with two indices: $A^a{}_\mu$. This is the main formal difference from classical electrodynamics: all electromagnetic ECE quantities have a polarization index, extending the definition range by one dimension. We will see later how we can reduce these quantities to one polarization direction, if required.

The first axiom is formally defined by

$$A^a{}_\mu := A^{(0)} q^a{}_\mu, \tag{4.1}$$

where we have introduced a factor of proportionality $A^{(0)}$. Since the tetrad is dimensionless, $A^{(0)}$ must have the physical units of a vector potential, which are Vs/m or $T \cdot m$. The product $c \cdot A^{(0)}$ can be considered as a primordial voltage, where c is the velocity of light. In detailed form, the complete ECE potential reads:

$$A^a{}_\mu = \begin{bmatrix} A^{(0)}{}_0 & A^{(0)}{}_1 & A^{(0)}{}_2 & A^{(0)}{}_3 \\ A^{(1)}{}_0 & A^{(1)}{}_1 & A^{(1)}{}_2 & A^{(1)}{}_3 \\ A^{(2)}{}_0 & A^{(2)}{}_1 & A^{(2)}{}_2 & A^{(2)}{}_3 \\ A^{(3)}{}_0 & A^{(3)}{}_1 & A^{(3)}{}_2 & A^{(3)}{}_3 \end{bmatrix} = A^{(0)} \begin{bmatrix} q^{(0)}{}_0 & q^{(0)}{}_1 & q^{(0)}{}_2 & q^{(0)}{}_3 \\ q^{(1)}{}_0 & q^{(1)}{}_1 & q^{(1)}{}_2 & q^{(1)}{}_3 \\ q^{(2)}{}_0 & q^{(2)}{}_1 & q^{(2)}{}_2 & q^{(2)}{}_3 \\ q^{(3)}{}_0 & q^{(3)}{}_1 & q^{(3)}{}_2 & q^{(3)}{}_3 \end{bmatrix}. \tag{4.2}$$

The rows number the polarization indices and the columns number the coordinate indices. The zeroth component of the potential is the scalar potential ϕ, which also gets a polarization index:

$$A^a{}_0 = \frac{\phi^a}{c}. \tag{4.3}$$

The above equations use the mixed-index notation (contravariant and covariant). The coordinates of vectors, however, correspond to contravariant indices. Therefore, we transform the coordinate indices by the Minkowski metric, Eq. (2.39):

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{4.4}$$

in the usual way:

$$A^{a\mu} = \eta^{\mu\nu} A^a{}_\nu. \tag{4.5}$$

The diagonal form of the Minkowski metric leads to

$$\begin{aligned} A^{(0)0} &= \eta^{00} A^{(0)}{}_0 = A^{(0)}{}_0, \\ A^{(0)1} &= \eta^{11} A^{(0)}{}_1 = -A^{(0)}{}_1, \\ A^{(0)2} &= \eta^{22} A^{(0)}{}_2 = -A^{(0)}{}_2, \\ &\cdots, \end{aligned} \tag{4.6}$$

so that the potential components with coordinates 1,2,3 are changed in sign:

$$A^{a\mu} = \begin{bmatrix} A^{(0)0} & A^{(0)1} & A^{(0)2} & A^{(0)3} \\ A^{(1)0} & A^{(1)1} & A^{(1)2} & A^{(1)3} \\ A^{(2)0} & A^{(2)1} & A^{(2)2} & A^{(2)3} \\ A^{(3)0} & A^{(3)1} & A^{(3)2} & A^{(3)3} \end{bmatrix} = \begin{bmatrix} A^{(0)}{}_0 & A^{(0)}{}_1 & A^{(0)}{}_2 & A^{(0)}{}_3 \\ A^{(1)}{}_0 & -A^{(1)}{}_1 & -A^{(1)}{}_2 & -A^{(1)}{}_3 \\ A^{(2)}{}_0 & -A^{(2)}{}_1 & -A^{(2)}{}_2 & -A^{(2)}{}_3 \\ A^{(3)}{}_0 & -A^{(3)}{}_1 & -A^{(3)}{}_2 & -A^{(3)}{}_3 \end{bmatrix}. \tag{4.7}$$

This is the form of the potential that we will use most of the time.

The second axiom concerns electromagnetic force fields. The relativistic electromagnetic field tensor $F^{\mu\nu}$ was introduced in Example 2.11 and its Hodge dual was introduced in Example 2.12. These are 2-index tensors, comprising the electric and magnetic fields. Therefore, the electromagnetic field of ECE theory also has to have two coordinate indices. The second axiom defines the electromagnetic field tensor of ECE theory to be proportional to the Cartan torsion $T^a{}_{\mu\nu}$:

$$F^a{}_{\mu\nu} := A^{(0)} T^a{}_{\mu\nu}. \tag{4.8}$$

Because torsion has a polarization index, the electromagnetic field has to have one, too. It is a 3-index quantity, an indexed antisymmetric 2-form of Cartan geometry. In classical electromagnetism, the electromagnetic field is a derivative of the potential; therefore, it has units of [potential]/[length] = $Vs/m^2 = T$. Since geometrical torsion has units of $1/m$, the constant of proportionality $A^{(0)}$ is the same for both the potential and the field. The polarization index can be seen as a vector index augmenting the electric field **E**, and the magnetic field (i.e., induction) **B**. Therefore, we have fields $\mathbf{E}^a$ and $\mathbf{B}^a$, which are components of the ECE electromagnetic tensor field:

$$F^{a\mu\nu} = \begin{bmatrix} F^{a00} & F^{a01} & F^{a02} & F^{a03} \\ F^{a10} & F^{a11} & F^{a12} & F^{a13} \\ F^{a20} & F^{a21} & F^{a22} & F^{a23} \\ F^{a30} & F^{a31} & F^{a32} & F^{a33} \end{bmatrix} = \begin{bmatrix} 0 & -E^{a1} & -E^{a2} & -E^{a3} \\ E^{a1} & 0 & -cB^{a3} & cB^{a2} \\ E^{a2} & cB^{a3} & 0 & -cB^{a1} \\ E^{a3} & -cB^{a2} & cB^{a1} & 0 \end{bmatrix}. \tag{4.9}$$

The Hodge dual of the classical electromagnetic field $F^{\mu\nu}$ was computed in Example 2.12. As demonstrated before, this has to be augmented with a polarization or tangent space index a, leading to

$$\widetilde{F}^{a\mu\nu} = \begin{bmatrix} 0 & cB^{a1} & cB^{a2} & cB^{a3} \\ -cB^{a1} & 0 & -E^{a3} & E^{a2} \\ -cB^{a2} & E^{a3} & 0 & -E^{a1} \\ -cB^{a3} & -E^{a2} & E^{a1} & 0 \end{bmatrix}. \tag{4.10}$$

We have chosen the contravariant versions of F and $\widetilde{F}$, because these correspond to the electric and magnetic vector components, and are needed for deriving the field equations in vector form.

The ECE potential is a Cartan 1-form and the ECE electromagnetic field is a Cartan 2-form. The presence of a polarization index a makes both of them vector-valued. In summary, the fundamental ECE axioms are

$$A^a{}_\mu := A^{(0)} q^a{}_\mu, \tag{4.11}$$

$$F^a{}_{\mu\nu} := A^{(0)} T^a{}_{\mu\nu}. \tag{4.12}$$

4.2 Field equations

In this section we will derive the ECE field equations in a form that corresponds to Maxwell's equations, which are listed immediately below.

Gauss' law:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{4.13}$$

Faraday's law of induction:

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{4.14}$$

Coulomb's law:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{4.15}$$

Ampère-Maxwell's law:

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}, \tag{4.16}$$

where ρ is the electrical charge density and **J** is the current density. The ECE field equations will be shown to be identical to the Cartan-Bianchi identity and the Cartan-Evans identity. We will convert these theorems of geometry into physical laws by multiplying them by the factor $A^{(0)}$.

4.2.1 Field equations in covariant tensor form

From Eqs. (3.20) and (3.46), in form notation:

$$D \wedge T^a = R^a{}_b \wedge q^b, \tag{4.17}$$
$$D \wedge \widetilde{T}^a = \widetilde{R}^a{}_b \wedge q^b, \tag{4.18}$$

after inserting the ECE Axioms (4.11) and (4.12), we obtain

$$D \wedge F^a = R^a{}_b \wedge A^b, \tag{4.19}$$
$$D \wedge \widetilde{F}^a = \widetilde{R}^a{}_b \wedge A^b. \tag{4.20}$$

These are the field equations, written as 3-forms on both sides. We will see that these two equations lead to the equivalent of Maxwell's equations (4.13 - 4.16). To show this equivalence, we start by rewriting the field equations with full indices:

$$(D \wedge F^a)_{\mu\nu\rho} = (R^a{}_b \wedge A^b)_{\mu\nu\rho}, \tag{4.21}$$
$$(D \wedge \widetilde{F}^a)_{\mu\nu\rho} = (\widetilde{R}^a{}_b \wedge A^b)_{\mu\nu\rho}, \tag{4.22}$$

or, with the wedge operation carried out, in tensor form:

$$D_{[\mu} F^a{}_{\nu\rho]} = R^a{}_{b[\mu\nu} A^b{}_{\rho]}, \tag{4.23}$$
$$D_{[\mu} \widetilde{F}^a{}_{\nu\rho]} = \widetilde{R}^a{}_{b[\mu\nu} A^b{}_{\rho]}. \tag{4.24}$$

The covariant exterior derivative $D\wedge$ was defined by Eqs. (3.2, 3.3):

$$(D \wedge F^a)_{\mu\nu\rho} = (d \wedge F^a)_{\mu\nu\rho} + (\omega^a{}_b \wedge F^b)_{\mu\nu\rho} \tag{4.25}$$

or, written as a cyclic sum:

$$D_{[\mu} F^a{}_{\nu\rho]} = \partial_{[\mu} F^a{}_{\nu\rho]} + \omega^a{}_{[\mu b} F^b{}_{\nu\rho]}\,. \tag{4.26}$$

Inserting this derivative into the first field equation (4.23) gives (in form notation):

$$d \wedge F^a + \omega^a{}_b \wedge F^b = R^a{}_b \wedge A^b, \tag{4.27}$$

and bringing the spin connection term to the right side gives:

$$d \wedge F^a = R^a{}_b \wedge A^b - \omega^a{}_b \wedge F^b. \tag{4.28}$$

This equation has a form similar to the first two field equations of classical electrodynamics in Example 2.11, where they are condensed into one equation (2.173):

$$d \wedge F = 0. \tag{4.29}$$

These two field equations (the Gauss law and the Faraday law) are homogeneous, i.e., there are no current terms on the right side. In contrast, the right side of (4.28) is not zero. This indicates that a current may exist for the Gauss and Faraday laws in their generalized form in ECE theory. This is called the *homogeneous current* and denoted by j. It corresponds to magnetic charges and currents, whose existence is not universally accepted.

In ECE theory, the homogeneous current has to be augmented with a polarization index, so that Eq. (4.28) can be written as

$$d \wedge F^a = j^a \tag{4.30}$$

with the definition of the homogeneous current being

$$j^a := R^a{}_b \wedge A^b - \omega^a{}_b \wedge F^b. \tag{4.31}$$

For the second field equation (4.24), which contains the Hodge duals $\widetilde{F}$ and $\widetilde{R}$, we have to use the spin connection of the Λ connection as defined by Eq. (3.49). For clarity, we add an index (Λ) here:

$$d \wedge \widetilde{F}^a = \widetilde{R}^a{}_b \wedge A^b - \omega_{(\Lambda)}{}^a{}_b \wedge \widetilde{F}^b. \tag{4.32}$$

This equation defines the other pair of generalized Maxwell equations, the Coulomb and the Ampère-Maxwell laws. We find that

$$d \wedge \widetilde{F}^a = \mu_0 J^a \tag{4.33}$$

with the definition

$$J^a := \frac{1}{\mu_0}\left(\widetilde{R}^a{}_b \wedge A^b - \omega_{(\Lambda)}{}^a{}_b \wedge \widetilde{F}^b\right), \tag{4.34}$$

which we call the *inhomogeneous current*. This corresponds to the well-known electrical 4-current density. In addition, we have introduced the vacuum permeability $\mu_0 = 4\pi \cdot 10^{-7} \frac{\mathrm{Vs}}{\mathrm{Am}}$ in order to obtain J^a in the usual units of A/m^2. The 0-component of J^a is the electric charge density ρ, augmented by a polarization index. For consistency, we will also use the factor $1/\mu_0$ in the homogeneous current (4.31).

Please note that the current densities here are 3-forms. For example, in tensor notation, Eq. (4.34) reads:

$$\begin{aligned}(J^a)_{\mu\nu\rho} &= \frac{1}{\mu_0}\left(\widetilde{R}^a{}_{b[\mu\nu}A^b{}_{\rho]} - \omega_{(\Lambda)}{}^a{}_{[\mu b}\widetilde{F}^b{}_{\nu\rho]}\right). \\ &= \frac{1}{\mu_0}\Big(\widetilde{R}^a{}_{b\mu\nu}A^b{}_\rho + \widetilde{R}^a{}_{b\nu\rho}A^b{}_\mu + \widetilde{R}^a{}_{b\rho\mu}A^b{}_\nu \\ &\quad -\omega_{(\Lambda)}{}^a{}_{\mu b}\widetilde{F}^b{}_{\nu\rho} - \omega_{(\Lambda)}{}^a{}_{\nu b}\widetilde{F}^b{}_{\rho\mu} - \omega_{(\Lambda)}{}^a{}_{\rho b}\widetilde{F}^b{}_{\mu\nu}\Big).\end{aligned} \tag{4.35}$$

The standard current density, however, is a 1-form, because the current density $\mathbf{J}$ is a vector with only one coordinate index. Therefore, j^a and J^a are types of generalized currents that cannot be easily correlated to known quantities. However, we will see in the next section that the contravariant formulation reduces them to 1-forms, as desired.

So far, we have seen that we can write the ECE field equations in form notation with covariant tensors:

$$d \wedge F^a = \mu_0\, j^a, \tag{4.36}$$

$$d \wedge \widetilde{F}^a = \mu_0\, J^a. \tag{4.37}$$

For formal symmetry, we have added a factor of μ_0 to the homogeneous current j^a. This is arbitrary and only changes the units of j^a, which of course are different from those of J^a.

4.2.2 Field equations in contravariant tensor form

In this section, the field equations are translated into vector form so that they will be familiar to engineers and physicists. We start with the alternative forms of the Cartan-Bianchi and Cartan-Evans identities, Eqs. (3.79) and (3.77):

$$D_\mu \widetilde{T}^{\kappa\mu\nu} = \widetilde{R}^\kappa{}_\mu{}^{\mu\nu}, \tag{4.38}$$

$$D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}. \tag{4.39}$$

These equations were given in the base manifold, but can be transformed to tangent space. The κ index is replaced with the a index, by multiplying the equations by $q^a{}_\kappa$ and applying the tetrad postulate:

$$D_\mu \widetilde{T}^{a\mu\nu} = \widetilde{R}^a{}_\mu{}^{\mu\nu}, \tag{4.40}$$

$$D_\mu T^{a\mu\nu} = R^a{}_\mu{}^{\mu\nu}. \tag{4.41}$$

Multiplying by the factor $A^{(0)}$ then gives us the second form of field equations:

$$D_\mu \widetilde{F}^{a\mu\nu} = A^{(0)}\widetilde{R}^a{}_\mu{}^{\mu\nu}, \tag{4.42}$$

$$D_\mu F^{a\mu\nu} = A^{(0)} R^a{}_\mu{}^{\mu\nu}. \tag{4.43}$$

Please notice that the roles of the original and the Hodge dual equations have been interchanged, compared to (4.36) and (4.37). The first equation, corresponding to the first pair of Maxwell equations, is based on the Hodge dual equation, and the second equation, corresponding to the third and fourth Maxwell equations, contains the original torsion and curvature. There is no wedge product in the equations but there is a summation over the coordinate parameter μ. These are tensor equations with a contraction.

Next, we apply the definition of the covariant derivative, similarly to how we applied it in the preceding section:

$$D_\mu F^{a\nu\rho} = \partial_\mu F^{a\nu\rho} + \omega^a{}_{\mu b} F^{b\nu\rho}, \tag{4.44}$$

which leads to

$$\partial_\mu \widetilde{F}^{a\mu\nu} = A^{(0)}\widetilde{R}^a{}_\mu{}^{\mu\nu} - \omega_{(\Lambda)}{}^a{}_{\mu b} \widetilde{F}^{b\mu\nu}, \tag{4.45}$$

$$\partial_\mu F^{a\mu\nu} = A^{(0)} R^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} F^{b\mu\nu}. \tag{4.46}$$

The first equation is similar to Eq. (2.185), and represents the first two Maxwell equations of classical electrodynamics, in Hodge dual formulation. Therefore, we can again interpret the right sides of the equations as homogeneous and inhomogeneous currents, respectively. Similarly to (4.36, 4.37), we can write

$$\boxed{\partial_\mu \widetilde{F}^{a\mu\nu} = \mu_0 j^{a\nu},} \tag{4.47}$$

$$\boxed{\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu},} \tag{4.48}$$

with

$$j^{a\nu} := \frac{1}{\mu_0}\left(A^{(0)}\widetilde{R}^a{}_\mu{}^{\mu\nu} - \omega_{(\Lambda)}{}^a{}_{\mu b} \widetilde{F}^{b\mu\nu}\right), \tag{4.49}$$

$$J^{a\nu} := \frac{1}{\mu_0}\left(A^{(0)} R^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} F^{b\mu\nu}\right). \tag{4.50}$$

Now we have arrived at 1-forms for the currents, which can be set into correspondence with classical expressions. The covariant 4-current density is written in tangent space as

$$(J^a)_\mu = \begin{bmatrix} J^a{}_0 \\ J^a{}_1 \\ J^a{}_2 \\ J^a{}_3 \end{bmatrix}. \tag{4.51}$$

To use the components in their usual contravariant form, we have to raise the coordinate indices with the Minkowski metric, which causes a sign change in the space components:

$$(J^a)^\nu = \begin{bmatrix} J^{a0} \\ J^{a1} \\ J^{a2} \\ J^{a3} \end{bmatrix} = \begin{bmatrix} J^a{}_0 \\ -J^a{}_1 \\ -J^a{}_2 \\ -J^a{}_3 \end{bmatrix}. \tag{4.52}$$

The 0-component is defined by

$$J^{a0} = c\rho^a. \tag{4.53}$$

The right side of J^{a0} has units of $\frac{m}{s}\frac{C}{m^3} = \frac{C}{m^2 s} = \frac{A}{m^2}$, which are the same current density units that are used for the spatial components.

We conclude this section with the hint that the currents are geometrical quantities, and not externally imposed as in Maxwell's theory. The field equations are fully geometrical, and no terms have been added to define an external energy-momentum (as has been done in Einstein's general relativity). Since the currents depend on the fields F, for which the equations have to be solved, we have an intrinsic nonlinearity. This is similar to a known case involving Ohm's law, where the current density is assumed to be proportional to the electric field via the conductivity σ, which is generally a tensor. In most cases, σ is assumed to be a scalar quantity, and Ohm's law is used for the current term in classical electrodynamics:

$$\mathbf{J} = \sigma \mathbf{E}. \tag{4.54}$$

By comparing this with Eq. (4.50), we see that the conductivity takes the role of a constant scalar spin connection. However, the ECE field equations are valid in a curving and twisting spacetime, and this goes far beyond the Minkowski space of Maxwell's equations and special relativity.

4.2.3 Field equations in vector form

In Example 2.12, we showed how the Gauss and Faraday laws are derived from a tensor equation of the Hodge dual of the classical electromagnetic field, $\widetilde{F}^{\mu\nu}$. It is easy to extend this procedure to the field equation (4.47):

$$\partial_\mu \widetilde{F}^{a\mu\nu} = \mu_0 j^{a\nu}. \tag{4.55}$$

According to Eq. (4.10), the field is an antisymmetric tensor, consisting of electric and magnetic field components. In Examples 2.11 and 2.12, the field tensor was given in electric field units of V/m, for convenience. We have the freedom of choice for these units, and here we use the units of the magnetic field (Tesla), so that we obtain the same constants in the vector equations as we do in Maxwell's equations. This means that we have to make the following replacements:

$$E^\mu \rightarrow E^\mu/c,$$
$$cB^\mu \rightarrow B^\mu.$$

In addition, the fields have to be augmented with the polarization index a:

$$\widetilde{F}^{a\mu\nu} = \begin{bmatrix} \widetilde{F}^{a00} & \widetilde{F}^{a01} & \widetilde{F}^{a02} & \widetilde{F}^{a03} \\ \widetilde{F}^{a10} & \widetilde{F}^{a11} & \widetilde{F}^{a12} & \widetilde{F}^{a13} \\ \widetilde{F}^{a20} & \widetilde{F}^{a21} & \widetilde{F}^{a22} & \widetilde{F}^{a23} \\ \widetilde{F}^{a30} & \widetilde{F}^{a31} & \widetilde{F}^{a32} & \widetilde{F}^{a33} \end{bmatrix} = \begin{bmatrix} 0 & B^{a1} & B^{a2} & B^{a3} \\ -B^{a1} & 0 & -E^{a3}/c & E^{a2}/c \\ -B^{a2} & E^{a3}/c & 0 & -E^{a1}/c \\ -B^{a3} & -E^{a2}/c & E^{a1}/c & 0 \end{bmatrix}. \tag{4.56}$$

The homogeneous field equations are obtained by specific selections of indices, in the following way. Eq. (4.55) consists of four equations ($\nu = 0,1,2,3$), each with four summands of μ on the left side. The case $\mu = \nu$ leads to diagonal elements of F, so these terms can be omitted. The generalized Gauss law is obtained by choosing $\nu = 0$, which leads to

$$\partial_1 \widetilde{F}^{a10} + \partial_2 \widetilde{F}^{a20} + \partial_3 \widetilde{F}^{a30} = -\partial_1 B^{a1} - \partial_2 B^{a2} - \partial_3 B^{a3} = \mu_0 j^{a0}. \tag{4.57}$$

In vector notation this is

$$\boldsymbol{\nabla} \cdot \mathbf{B}^a = -\mu_0 j^{a0}. \tag{4.58}$$

The Faraday law of induction is obtained by choosing $\nu = 1,2,3$, and consists of three component equations:

$$\begin{aligned} &\partial_0 \widetilde{F}^{a01} + \partial_2 \widetilde{F}^{a21} + \partial_3 \widetilde{F}^{a31} = \mu_0 j^{a1}, \\ &\partial_0 \widetilde{F}^{a02} + \partial_1 \widetilde{F}^{a12} + \partial_3 \widetilde{F}^{a32} = \mu_0 j^{a2}, \\ &\partial_0 \widetilde{F}^{a03} + \partial_1 \widetilde{F}^{a13} + \partial_2 \widetilde{F}^{a23} = \mu_0 j^{a3}. \end{aligned} \tag{4.59}$$

According to (4.56), they can be written as

$$\begin{aligned} &\partial_0 B^{a1} + \partial_2 E^{a3}/c - \partial_3 E^{a2}/c = \mu_0 j^{a1}, \\ &\partial_0 B^{a2} - \partial_1 E^{a3}/c + \partial_3 E^{a1}/c = \mu_0 j^{a2}, \\ &\partial_0 B^{a3} + \partial_1 E^{a2}/c - \partial_2 E^{a1}/c = \mu_0 j^{a3}. \end{aligned} \tag{4.60}$$

Taking into account that $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$, these equations can be condensed into one vector equation, which is

$$\frac{\partial \mathbf{B}^a}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E}^a = c\,\mu_0 \mathbf{j}^a. \tag{4.61}$$

We see that these "homogeneous" equations are not actually homogeneous, because there is a magnetic charge density j^{a0} and a magnetic current vector $\mathbf{j}^a$. However, in nearly all practical applications, we will set

$$j^{a0} = 0, \tag{4.62}$$

$$\mathbf{j}^a = \mathbf{0}. \tag{4.63}$$

The Coulomb and Ampère-Maxwell laws are derived in a completely analogous way from Eq. (4.48):

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu}. \tag{4.64}$$

According to Eq. (4.9), the contravariant field tensor (in units of Tesla) is

$$F^{a\mu\nu} = \begin{bmatrix} F^{a00} & F^{a01} & F^{a02} & F^{a03} \\ F^{a10} & F^{a11} & F^{a12} & F^{a13} \\ F^{a20} & F^{a21} & F^{a22} & F^{a23} \\ F^{a30} & F^{a31} & F^{a32} & F^{a33} \end{bmatrix} = \begin{bmatrix} 0 & -E^{a1}/c & -E^{a2}/c & -E^{a3}/c \\ E^{a1}/c & 0 & -B^{a3} & B^{a2} \\ E^{a2}/c & B^{a3} & 0 & -B^{a1} \\ E^{a3}/c & -B^{a2} & B^{a1} & 0 \end{bmatrix}. \tag{4.65}$$

We obtain the generalized Coulomb law by choosing $\nu = 0$:

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \mu_0 J^{a0}, \tag{4.66}$$

which (in vector notation) is

$$\boldsymbol{\nabla}\cdot\mathbf{E}^a = c\,\mu_0 J^{a0}. \tag{4.67}$$

Because the 0-component of the current density is the charge density (see Eq. (4.53)), this equation can also be written as

$$\boldsymbol{\nabla}\cdot\mathbf{E}^a = \frac{\rho^a}{\varepsilon_0}. \tag{4.68}$$

The Ampère-Maxwell law is obtained by choosing $\nu = 1,2,3$, and consists of three component equations:

$$\begin{aligned}
&\partial_0 F^{a01} + \partial_2 F^{a21} + \partial_3 F^{a31} = \mu_0 J^{a1},\\
&\partial_0 F^{a02} + \partial_1 F^{a12} + \partial_3 F^{a32} = \mu_0 J^{a2},\\
&\partial_0 F^{a03} + \partial_1 F^{a13} + \partial_2 F^{a23} = \mu_0 J^{a3},
\end{aligned} \tag{4.69}$$

which with the aid of (4.65) become

$$\begin{aligned}
&-\partial_0 E^{a01}/c + \partial_2 B^{a3} - \partial_3 B^{a2} = \mu_0 J^{a1},\\
&-\partial_0 E^{a02}/c - \partial_1 B^{a3} + \partial_3 B^{a1} = \mu_0 J^{a2},\\
&-\partial_0 E^{a03}/c + \partial_1 F^{a2} - \partial_2 B^{a1} = \mu_0 J^{a3}.
\end{aligned} \tag{4.70}$$

As before, they can be condensed into one vector equation:

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}^a}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B}^a = \mu_0\,\mathbf{J}^a. \tag{4.71}$$

Ultimately, we arrive at the Maxwell-like field equations, in vector form:

$$\boldsymbol{\nabla}\cdot\mathbf{B}^a = -\mu_0\, j^{a0}, \tag{4.72}$$

$$\frac{\partial \mathbf{B}^a}{\partial t} + \boldsymbol{\nabla}\times\mathbf{E}^a = c\,\mu_0\,\mathbf{j}^a, \tag{4.73}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}^a = \frac{\rho^a}{\varepsilon_0}, \tag{4.74}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}^a}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B}^a = \mu_0\,\mathbf{J}^a, \tag{4.75}$$

which correspond, according to Eqs. (4.13 - 4.16), to the Gauss law, the Faraday law, the Coulomb law and the Ampère-Maxwell law. These equations are valid in a generally covariant spacetime. Because of the four values for the polarization index a, the equation system consists of $4\cdot 8 = 32$ equations. There are only $4\cdot 6 = 24$ variables. It is known, however, that the Gauss law is dependent on the Faraday law, and that the Coulomb law is dependent on the Ampère-Maxwell law. Therefore, there are only 24 independent equations, and the equation system is uniquely defined. In addition, the equations for each a index are separable. The meaning of the polarization index will be clarified next through two detailed examples.

4.2.4 Examples of ECE field equations

The Coulomb law in Cartan geometry

■ **Example 4.1** In Chapters 2 and 3 we demonstrated, through examples, how all elements of a given tetrad can be calculated within Cartan geometry. Now we extend this method to physical fields.

One of the simplest and most important elements of electrodynamics is the Coulomb potential. In 4-vector notation, the potential is the 0-component

$$A^0 = \frac{\phi(r)}{c} = \frac{1}{c}\frac{q_e}{4\pi\varepsilon_0 r}, \tag{4.76}$$

where q_e is the central point charge and r is the radial coordinate of a spherical coordinate system

$$(X^\mu) = \begin{bmatrix} t \\ r \\ \theta \\ \phi \end{bmatrix}. \tag{4.77}$$

According to Eq. (4.2), the potential corresponds to the first diagonal element of the tetrad:

$$\phi(r) = c\,A^{(0)} q^{(0)}{}_0. \tag{4.78}$$

Inserting the potential into the q matrix gives

$$(q^a{}_\mu) = \frac{1}{2}\frac{(A^a{}_\mu)}{A^{(0)}} = \frac{1}{A^{(0)}}\begin{bmatrix} \frac{\phi(r)}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.79}$$

which is a singular matrix. Cartan geometry, however, is only defined with non-singular tetrads (see Section 2.5.1). Therefore, a vector potential is necessarily required, in addition to a scalar potential. We choose the simplest form, a constant vector potential, which gives no magnetostatic field. The final form of the tetrad then becomes

$$(q^a{}_\mu) = \frac{1}{2}\begin{bmatrix} \frac{C_0}{r} & 0 & 0 & 0 \\ 0 & -C_1 & 0 & 0 \\ 0 & 0 & -C_2 & 0 \\ 0 & 0 & 0 & -C_3 \end{bmatrix}, \tag{4.80}$$

where

$$C_0 = \frac{q_e}{A^{(0)}\,c\,4\pi\varepsilon_0} \tag{4.81}$$

and the C_i are arbitrary constants for $i = 1,2,3$. For simplicity of results, we assume $C_i > 0$ and omit the factors $A^{(0)}$ and c. Then, the vector potential becomes

$$\mathbf{A} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}. \tag{4.82}$$

We will first reiterate the relevant equations, and then show how Cartan geometry can be applied.

Metric compatibility (2.149):

$$D_\sigma g_{\mu\nu} = \partial_\sigma g_{\mu\nu} - \Gamma^\lambda{}_{\sigma\mu} g_{\lambda\nu} - \Gamma^\lambda{}_{\sigma\nu} g_{\mu\lambda} = 0 \tag{4.83}$$

with an explicit antisymmetry requirement for all non-diagonal Γ elements (2.156):

$$\Gamma^\rho{}_{\mu\nu} = -\Gamma^\rho{}_{\nu\mu}. \tag{4.84}$$

The metric (2.204):

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu\, \eta_{ab}, \tag{4.85}$$

$$g^{\mu\nu} = \frac{1}{n}\, q^\mu{}_a q^\nu{}_b\, \eta^{ab}. \tag{4.86}$$

The spin connection (2.235):

$$\omega^a{}_{\mu b} = q^a{}_\nu q^\lambda{}_b \Gamma^\nu{}_{\mu\lambda} - q^\lambda{}_b \partial_\mu q^a{}_\lambda. \tag{4.87}$$

The Λ connection and its spin connection (3.48 – 3.49):

$$\Lambda^\lambda{}_{\mu\nu} = \frac{1}{2}|g|^{-1/2}\eta^{\rho\alpha}\eta^{\sigma\beta}\varepsilon_{\rho\sigma\mu\nu}\Gamma^\lambda{}_{\alpha\beta}, \tag{4.88}$$

$$\omega_{(\Lambda)}{}^a{}_{\mu b} = q^a{}_\nu q^\lambda{}_b \Lambda^\nu{}_{\mu\lambda} - q^\lambda{}_b \partial_\mu q^a{}_\lambda. \tag{4.89}$$

The curvature (2.134) and torsion (2.132) tensors:

$$R^\lambda{}_{\mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}, \tag{4.90}$$

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \tag{4.91}$$

their 2-forms (2.260) and torsion (2.259):

$$R^a{}_{b\mu\nu} = q^a{}_\rho q^\sigma{}_b R^\rho{}_{\sigma\mu\nu}, \tag{4.92}$$

$$T^a_{\mu\nu} = q^a{}_\lambda T^\lambda{}_{\mu\nu}\ , \tag{4.93}$$

with contravariant forms:

$$R^a{}_b{}^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}R^a{}_{b\rho\sigma}, \tag{4.94}$$

$$T^{a\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}T^a_{\rho\sigma}\ . \tag{4.95}$$

We now evaluate the equations by using the tetrad (4.80) (the Maxima code can be found in [130]). This gives Γ connections with four unspecified parameters D_1 to D_4:

$$\Gamma^0{}_{01} = \frac{1}{r} \tag{4.96}$$

$$\Gamma^0{}_{10} = -\frac{1}{r}$$

$$\Gamma^0{}_{12} = \frac{D_4 {C_2}^2 r^2}{{C_0}^2}$$

$$\Gamma^0{}_{13} = -\frac{D_3 {C_1}^2 r^2}{{C_0}^2}$$

$$\dots\ .$$

It is possible to set the D_i to zero:

$$D_1 = D_2 = D_3 = D_4 = 0. \tag{4.97}$$

Then, only three non-vanishing connections remain:

$$\Gamma^0{}_{01} = \frac{1}{r} \tag{4.98}$$

$$\Gamma^0{}_{10} = -\frac{1}{r} \tag{4.99}$$

$$\Gamma^1{}_{00} = \frac{C_0^2}{C_1^2 r^3}. \tag{4.100}$$

The first two connections are antisymmetric, while the third connection is a diagonal element that does not contribute to torsion.

After applying Eq. (4.87), the non-vanishing spin connections become

$$\omega^{(0)}{}_{0(1)} = -\frac{C_0}{C_1\, r^2}, \tag{4.101}$$

$$\omega^{(1)}{}_{0(0)} = -\frac{C_0}{C_1\, r^2}, \tag{4.102}$$

which are antisymmetric in indices in a and b. (Please notice that the upper index a has to be lowered for comparison, which gives a sign change for the second connection element.) We have written the Latin indices in parentheses in order to distinguish these numbers from those stemming from Greek indices.

The Hodge duals of the Γ connection are

$$\Lambda^0{}_{23} = -\frac{1}{r}, \tag{4.103}$$

$$\Lambda^0{}_{32} = \frac{1}{r}, \tag{4.104}$$

and are complementary to the Γ's in the lower indices. The non-zero Λ spin connections are

$$\omega_{(\Lambda)}{}^{(0)}{}_{1(0)} = \frac{1}{r}, \tag{4.105}$$

$$\omega_{(\Lambda)}{}^{(0)}{}_{2(3)} = \frac{C_0}{C_3 r^2}, \tag{4.106}$$

$$\omega_{(\Lambda)}{}^{(0)}{}_{3(2)} = -\frac{C_0}{C_2 r^2}. \tag{4.107}$$

It is important to note that the connection $\omega_{(\Lambda)}{}^{(0)}{}_{1(0)}$ has the form that was derived in early papers of the UFT series. In those papers, separate spin connections for Γ and Λ had not yet been defined, and which one was meant depended on the field equation that was being used. We now use the Λ spin connections with the inhomogeneous current (Coulomb and Ampère-Maxwell laws).

The non-vanishing torsion and curvature tensor elements are

$$T^0_{01} = -T^0_{10} = \frac{2}{r}, \tag{4.108}$$

$$R^0{}_{101} = -R^0{}_{110} = \frac{2}{r^2}, \tag{4.109}$$

$$R^1{}_{001} = -R^1{}_{010} = \frac{2C_0^2}{C_1^2 r^4}, \tag{4.110}$$

which are all antisymmetric in the last two indices. The same holds for the torsion and curvature forms:

$$T^{(0)}{}_{01} = -T^{(0)}{}_{10} = \frac{C_0}{r^2}, \tag{4.111}$$

$$R^{(0)}{}_{(1)01} = -R^{(0)}{}_{(1)10} = -\frac{2C_0}{C_1 r^3}, \tag{4.112}$$

$$R^{(1)}{}_{(0)01} = -R^{(1)}{}_{(0)10} = -\frac{2C_0}{C_1 r^3}. \tag{4.113}$$

The final results are obtained by inserting the torsion elements into Eq. (4.65) leading, e.g., to

$$F^{a01} = A^{(0)} T^{a01} = \frac{E^{a1}}{c}, \tag{4.114}$$

from which we immediately get

$$E^{a1} = cA^{(0)} T^{a01}, \tag{4.115}$$

which is the r-component of the fields $\mathbf{E}^a$. The field elements for all other a values are obtained in the same way. This gives, for the electric fields:

$$\mathbf{E}^{(0)} = cA^{(0)} \begin{bmatrix} \frac{C_0}{r^2} \\ 0 \\ 0 \end{bmatrix}, \tag{4.116}$$

$$\mathbf{E}^{(1)} = \mathbf{E}^{(2)} = \mathbf{E}^{(3)} = \mathbf{0}, \tag{4.117}$$

and for the magnetic fields:

$$\mathbf{B}^{(0)} = \mathbf{B}^{(1)} = \mathbf{B}^{(2)} = \mathbf{B}^{(3)} = \mathbf{0}. \tag{4.118}$$

Only the electric 0-component of polarization is not a zero vector, and all polarizations of the magnetic field vanish. This is exactly the classical result

$$\mathbf{E}^{(0)} = \mathbf{E} = \frac{q_e}{4\pi\varepsilon_0 r^2}. \tag{4.119}$$

We can see that – despite the purely classical result – there are non-vanishing spin connections and curvature tensor elements. This shows that ECE theory gives results beyond classical electromagnetism, which is based only on special relativity. ■

Circularly polarized plane wave in a complex basis – the $\mathbf{B}^{(3)}$ field

■ **Example 4.2** In the early 1990s, Myron Evans developed what is known as the B(3) field, a longitudinal field of electrodynamics [26–28] that describes a longitudinal component of electromagnetic waves. He then generalized this theory to O(3) electrodynamics [28, 29], in the late 1990s, and these two theories culminated in ECE theory, in 2003.

We will now present the basics of O(3) electrodynamics and show how circularly polarized plane waves (Figs. 4.1, 4.2) can be attributed to different polarization vectors of the electric and magnetic fields.

The usual orthonormal basis of the three-dimensional cartesian space is described by the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.120}$$

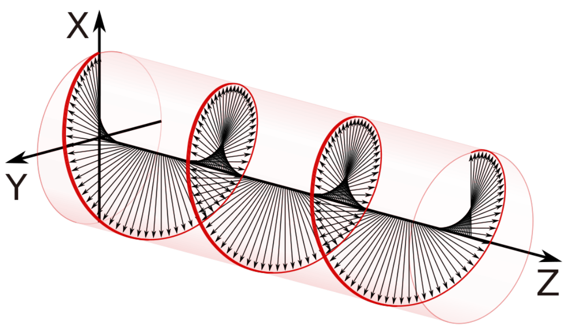


Figure 4.1: Right-polarized wave [180].

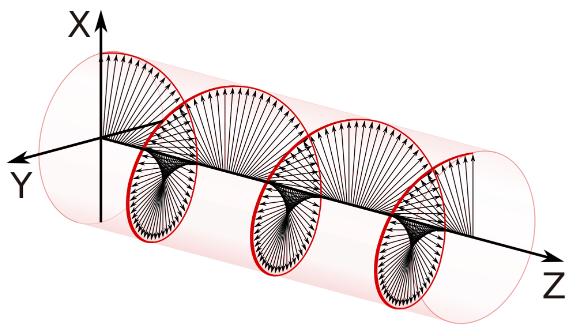


Figure 4.2: Left-polarized wave [181].

These unit vectors satisfy the circular relations

$$\mathbf{i}\times\mathbf{j}=\mathbf{k}, \tag{4.121}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i}, \tag{4.122}$$
$$\mathbf{k}\times\mathbf{i}=\mathbf{j}. \tag{4.123}$$

In O(3) electrodynamics, a complex circular basis in flat space is used, denoted by $\mathbf{q}^{(1)},\mathbf{q}^{(2)},\mathbf{q}^{(3)}$. The transformation equations from the cartesian to the complex circular basis are

$$\mathbf{q}^{(1)}=\frac{1}{\sqrt{2}}(\mathbf{i}-i\mathbf{j}), \tag{4.124}$$
$$\mathbf{q}^{(2)}=\frac{1}{\sqrt{2}}(\mathbf{i}+i\mathbf{j}), \tag{4.125}$$
$$\mathbf{q}^{(3)}=\mathbf{k}, \tag{4.126}$$

where i is the imaginary unit (not to be confused with the unit vector $\mathbf{i}$). These vectors have a spatial circular relation, which is characteristic of O(3) symmetry:

$$\mathbf{q}^{(1)}\times\mathbf{q}^{(2)}=i\mathbf{q}^{(3)*}, \tag{4.127}$$
$$\mathbf{q}^{(2)}\times\mathbf{q}^{(1)}=i\mathbf{q}^{(1)*}, \tag{4.128}$$
$$\mathbf{q}^{(3)}\times\mathbf{q}^{(1)}=i\mathbf{q}^{(2)*}. \tag{4.129}$$

In contrast to (4.121 - 4.123), the cross product of two basis vectors is not the third vector, but rather the conjugate vector, multiplied by the imaginary unit. Eqs. (4.124 - 4.126) are basis transformations; therefore, they can be interpreted directly as a Cartan tetrad $q^a{}_\mu$, where indices a and μ run from 1 to 3. Consequently, we can define vector potentials according to the first ECE axiom (4.11), by multiplying the q's by the factor $A^{(0)}$:

$$\mathbf{A}^{(1)}=A^{(0)}\mathbf{q}^{(1)}, \tag{4.130}$$
$$\mathbf{A}^{(2)}=A^{(0)}\mathbf{q}^{(2)}, \tag{4.131}$$
$$\mathbf{A}^{(3)}=A^{(0)}\mathbf{q}^{(3)}. \tag{4.132}$$

In so doing, we have defined potentials for three polarization directions, where these directions coincide with the axes of the coordinate system. The polarization in the Z direction is constant:

$$\mathbf{A}^{(3)}=A^{(0)}\mathbf{k}. \tag{4.133}$$

In the absence of rotation around the Z axis, we have

$$\nabla\times\mathbf{A}^{(1)}=\nabla\times\mathbf{A}^{(2)}=\mathbf{0}. \tag{4.134}$$

Next, we define a wave of $\mathbf{A}$ vectors rotating in the XY plane:

$$\mathbf{A}^{(1)}=A^{(0)}\mathbf{q}^{(1)}e^{i(\omega t-\kappa Z)}, \tag{4.135}$$
$$\mathbf{A}^{(2)}=A^{(0)}\mathbf{q}^{(2)}e^{-i(\omega t-\kappa Z)}, \tag{4.136}$$
$$\mathbf{A}^{(3)}=A^{(0)}\mathbf{q}^{(3)}, \tag{4.137}$$

where ω is a time frequency, and κ is a wave number (the spatial frequency of a wave, measured in cycles per unit distance) in the Z direction. The first two $\mathbf{A}$ vectors define a left and right rotation around the Z axis. They are related by

$$\mathbf{A}^{(1)}=\mathbf{A}^{(2)*}. \tag{4.138}$$

$\mathbf{A}^{(3)}$ is a constant vector in the Z direction.

The magnetic fields of O(3) electrodynamics are defined by

$$\mathbf{B}^{(1)*} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(2)} \times \mathbf{A}^{(3)}, \tag{4.139}$$

$$\mathbf{B}^{(2)*} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(3)} \times \mathbf{A}^{(1)}, \tag{4.140}$$

$$\mathbf{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}, \tag{4.141}$$

(notice that these are definitions of the conjugated quantities that appear on the left side). These fields obey the *B cyclic theorem*:

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = iA^{(0)}\kappa\,\mathbf{B}^{(3)*}, \tag{4.142}$$

$$\mathbf{B}^{(2)} \times \mathbf{B}^{(3)} = iA^{(0)}\kappa\,\mathbf{B}^{(1)*}, \tag{4.143}$$

$$\mathbf{B}^{(3)} \times \mathbf{B}^{(1)} = iA^{(0)}\kappa\,\mathbf{B}^{(2)*}, \tag{4.144}$$

and this has been proven using computer algebra [131], along with other theorems of O(3) theory that have not been mentioned here. It is also possible to define the first two polarizations in the same way as in conventional electrodynamics:

$$\mathbf{B}^{(1)} = \boldsymbol{\nabla} \times \mathbf{A}^{(1)}, \tag{4.145}$$

$$\mathbf{B}^{(2)} = \boldsymbol{\nabla} \times \mathbf{A}^{(2)}. \tag{4.146}$$

However, then there is no $\mathbf{B}^{(3)}$ field, because $\mathbf{A}^{(3)}$ is constant and therefore

$$\boldsymbol{\nabla} \times \mathbf{A}^{(3)} = \mathbf{0}. \tag{4.147}$$

Consequently, $\mathbf{B}^{(3)}$ can only be defined by Eq. (4.141):

$$\mathbf{B}^{(3)} = i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(1)*} \times \mathbf{A}^{(2)*}. \tag{4.148}$$

The $\mathbf{B}^{(3)}$ field,

$$\mathbf{B}^{(3)} = A^{(0)}\kappa\,\mathbf{k}, \tag{4.149}$$

has been studied in great detail [26,28]. It is a radiated magnetic field or flux density in the direction of wave propagation, and this type of field is not known in ordinary electrodynamics. When the wave hits matter, it creates a magnetization in the propagation direction, which is known as the inverse Faraday effect. There are other spectroscopic effects that can be explained by the $\mathbf{B}^{(3)}$ field, specifically in optical NMR and in laser technology [26].

Interestingly, there is no electrical $\mathbf{E}^{(3)}$ field. Electric field vectors of both left and right circular polarization can be defined by

$$\mathbf{E}^{(1)} = E^{(0)}\frac{1}{\sqrt{2}}(\mathbf{i} + i\mathbf{j})e^{i(\omega t - \kappa Z)}, \tag{4.150}$$

$$\mathbf{E}^{(2)} = E^{(0)}\frac{1}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})e^{-i(\omega t - \kappa Z)}, \tag{4.151}$$

and are perpendicular to the magnetic fields. The $\mathbf{E}^{(3)}$ field would have been the cross product of $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, which would have given

$$\mathbf{E}^{(3)} = \frac{1}{E^{(0)}}\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = -iE^{(0)}\mathbf{k} \tag{4.152}$$

with a suitable constant $E^{(0)}$. This field would be purely imaginary; therefore, no physical $\mathbf{E}^{(3)}$ field can exist (and has never been observed).

There is another interesting relation for circular plane waves. By computer algebra we find that Eqs. (4.139 - 4.141) can be rewritten in the following form:

$$\mathbf{B}^{(1)} = \kappa_1\, \mathbf{A}^{(1)}, \tag{4.153}$$

$$\mathbf{B}^{(2)} = \kappa_2\, \mathbf{A}^{(2)}, \tag{4.154}$$

$$\mathbf{B}^{(3)} = \kappa_3\, \mathbf{A}^{(3)}, \tag{4.155}$$

and then, by comparing these results with (4.145 - 4.147), we arrive at three *Beltrami conditions*:

$$\boldsymbol{\nabla} \times \mathbf{A}^{(1)} = \kappa\, \mathbf{A}^{(1)}, \tag{4.156}$$

$$\boldsymbol{\nabla} \times \mathbf{A}^{(2)} = \kappa\, \mathbf{A}^{(2)}, \tag{4.157}$$

$$\boldsymbol{\nabla} \times \mathbf{A}^{(3)} = 0 \cdot \mathbf{A}^{(3)}. \tag{4.158}$$

Beltrami conditions involve longitudinal waves, where electric and magnetic fields are not perpendicular to each other. These conditions will be discussed in later chapters. All calculations in this example have been verified by computer algebra code [131]. ■

4.3 Wave equation

In Section 2.5.5, we showed that a wave equation can be derived from the tetrad postulate. This wave equation is called the Evans lemma, Eq. (2.258):

$$\square\, q^a{}_\nu + R q^a{}_\nu = 0. \tag{4.159}$$

It is an equation for the tetrad, and contains a scalar curvature R that (according to Eq. (2.257)) is defined by the tetrad itself, the spin connection and the Γ connection terms of Cartan geometry:

$$R = q^\nu{}_a \left(\partial^\mu (\omega^a{}_{\mu b} q^b{}_\nu) - \partial^\mu (\Gamma^\lambda{}_{\mu\nu} q^a{}_\lambda) \right). \tag{4.160}$$

The Evans lemma can be transformed easily into a physical equation by applying the first ECE postulate (4.11), i.e., multiplying the equation by the constant $A^{(0)}$, which gives us the physical units of a potential:

$$\square\, A^a{}_\nu + R A^a{}_\nu = 0. \tag{4.161}$$

In Section 2.5.5 we introduced the d'Alembert operator $\square$, which in cartesian coordinates reads:

$$\square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} - \frac{\partial^2}{\partial Z^2}. \tag{4.162}$$

Eq. (4.161) is different from the wave equation of standard electrodynamics, which is

$$\square\, A^a{}_\nu = 0. \tag{4.163}$$

This equation has no curvature term, so it follows from (4.161), with $R = 0$, that the spacetime is flat, that it is without curvature and torsion. When the potential $A^a{}_\nu$ is derived as a solution of this equation, it is not uniquely determined. $A^a{}_\nu$ may be changed by a tensorial function $\phi^a{}_\nu$, whose derivatives vanish:

$$\square\, \phi^a{}_\nu = 0 \tag{4.164}$$

or, in a particular case,

$$\frac{\partial^2 \phi^a{}_\nu}{\partial t^2} = 0 \quad \text{and} \qquad (4.165)$$
$$\nabla^2 \phi^a{}_\nu = 0,$$

so that

$$\square \ (A^a{}_\nu + \phi^a{}_\nu) = 0. \qquad (4.166)$$

This is an example of a *gauge transformation*, and these transformations are the basis of quantum electrodynamics. When we use the ECE wave equation (4.161) instead, a re-gauging of the vector potential is no longer possible, and therefore quantum electrodynamics is not compatible with the framework of ECE theory. All well-known effects of quantum electrodynamics, for example, the Lamb shift in atomic spectra, can be explained by ECE theory directly, as a consequence of the fact that spacetime is not flat.

As a preview of ECE quantum mechanics, we state that the wave equation (4.161) can be quantized and is the basis of the Fermion equation, which is comparable to the Dirac equation in establishing relativistic quantum mechanics. The Fermion equation is not based on special relativity, but on general relativity, and a spacetime with both curvature and torsion. The non-relativistic Schrödinger equation can be derived as an approximation of the Fermion equation or, alternatively, from Einstein's relation for total energy. Thus, the ECE wave equation provides the foundation for all of ECE quantum mechanics.

The curvature term R in the ECE wave equation describes a coupling between electromagnetism and spacetime, and even electromagnetism and gravitation. Therefore, R may be replaced with a mass term and a dimensional factor. In order to obtain curvature units ($1/\mathrm{m}^2$), we replace R by $(m_0 c/\hbar)^2$, where m_0 may be constant. This ensures that the dimensions are correct, and we can write

$$\square A^a{}_\nu + \left(\frac{m_0 c}{\hbar}\right)^2 A^a{}_\nu = 0. \qquad (4.167)$$

When we apply this equation to electromagnetic waves, we see that photons have a rest mass m_0. Equation (4.167) is identical to the Proca equation [30], which was derived by Alexandru Proca independently, before the advent of ECE theory. In the Standard Model, the Proca equation is directly incompatible with gauge invariance. The gauge principle is not tenable in a unified field theory such as ECE, because the potential in ECE is physically relevant and cannot be "re-gauged". In ECE theory the tetrad postulate is invariant under general coordinate transformation, and this is the principle that governs the potential field. In a sense, the tetrad postulate replaces the gauge theory of the Standard Model.

4.4 Field equations in terms of potentials

The field equations of ECE theory are formally identical to Maxwell's equations, augmented by a polarization index, and are valid in a spacetime of general relativity. The connections of spacetime are not directly visible in this representation of the electric and magnetic force fields. However, we can make the underlying structure evident by replacing the force fields with their potentials. Because the force fields are identical to Cartan torsion, within a factor of proportionality, we use the first Maurer-Cartan structure equation to express the force fields by their potentials and connections. The first Maurer-Cartan structure, Eq. (2.283), is the 2-form:

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b \qquad (4.168)$$

or, written in tensor form:

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu. \tag{4.169}$$

Applying the ECE axioms

$$A^a{}_\mu = A^{(0)} q^a{}_\mu, \tag{4.170}$$
$$F^a{}_{\mu\nu} = A^{(0)} T^a{}_{\mu\nu}, \tag{4.171}$$

then leads to the equation

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + \omega^a{}_{\mu b} A^b{}_\nu - \omega^a{}_{\nu b} A^b{}_\mu. \tag{4.172}$$

It can be seen directly that this is an antisymmetric tensor consisting of the potential A and spin connection terms. In classical electrodynamics, the field tensor $F_{\mu\nu}$ is defined from the potentials by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.173}$$

There is no polarization index and there are no spin connection terms, indicating that this is an equation of Minkowski space without curved spacetime. Eq. (4.172) is a generalized version of this equation that introduces curvature and torsion.

We introduced the vector representations of the ECE field equations in Section 4.2.3. However, it will now be useful to replace the vectors $\mathbf{E}^a$ and $\mathbf{B}^a$ with their potentials according to Eq. (4.172). In classical physics, we have

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{4.174}$$
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{4.175}$$

where ϕ is the electric scalar potential and $\mathbf{A}$ is the magnetic vector potential. The mapping between the elements of F and the electric and magnetic fields is given by Eq. (4.65), and Eq. (4.56) gives an equivalent mapping for the Hodge dual of F. Here we will use Eq. (4.65):

$$\mathbf{E}^a = \begin{bmatrix} E^{a1} \\ E^{a2} \\ E^{a3} \end{bmatrix} = -c \begin{bmatrix} F^{a01} \\ F^{a02} \\ F^{a03} \end{bmatrix}, \tag{4.176}$$
$$\mathbf{B}^a = \begin{bmatrix} B^{a1} \\ B^{a2} \\ B^{a3} \end{bmatrix} = \begin{bmatrix} F^{a32} \\ F^{a13} \\ F^{a21} \end{bmatrix}. \tag{4.177}$$

Then from Eq. (4.172), with $\mu = 0$ and $\nu = 1,2,3$, we get:

$$F^a{}_{01} = \partial_0 A^a{}_1 - \partial_1 A^a{}_0 + \omega^a{}_{0b} A^b{}_1 - \omega^a{}_{1b} A^b{}_0, \tag{4.178}$$
$$F^a{}_{02} = \partial_0 A^a{}_2 - \partial_2 A^a{}_0 + \omega^a{}_{0b} A^b{}_2 - \omega^a{}_{2b} A^b{}_0, \tag{4.179}$$
$$F^a{}_{03} = \partial_0 A^a{}_3 - \partial_3 A^a{}_0 + \omega^a{}_{0b} A^b{}_3 - \omega^a{}_{3b} A^b{}_0. \tag{4.180}$$

We raise the Greek indices in F, A and ω, which results in sign changes for $\nu = 1,2,3$:

$$-F^{a01} = -\partial_0 A^{a1} - \partial_1 A^{a0} - \omega^a{}_{0b} A^{b1} + \omega^{a1}{}_b A^{b0}, \tag{4.181}$$
$$-F^{a02} = -\partial_0 A^{a2} - \partial_2 A^{a0} - \omega^a{}_{0b} A^{b2} + \omega^{a2}{}_b A^{b0}, \tag{4.182}$$
$$-F^{a03} = -\partial_0 A^{a3} - \partial_3 A^{a0} - \omega^a{}_{0b} A^{b3} + \omega^{a3}{}_b A^{b0}. \tag{4.183}$$

Inserting (4.176) gives us

$$\frac{1}{c}E^{a1} = -\partial_0 A^{a1} - \partial_1 A^{a0} - \omega^a{}_{0b} A^{b1} + \omega^{a1}{}_b A^{b0}, \tag{4.184}$$
$$\frac{1}{c}E^{a2} = -\partial_0 A^{a2} - \partial_2 A^{a0} - \omega^a{}_{0b} A^{b2} + \omega^{a2}{}_b A^{b0}, \tag{4.185}$$
$$\frac{1}{c}E^{a3} = -\partial_0 A^{a3} - \partial_3 A^{a0} - \omega^a{}_{0b} A^{b3} + \omega^{a3}{}_b A^{b0}, \tag{4.186}$$

and with

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \quad \text{and} \quad A^{a0} = \frac{\phi^a}{c}, \tag{4.187}$$

we get for the electric field in vector form:

$$\mathbf{E}^a = -\boldsymbol{\nabla}\phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b}\mathbf{A}^b + \boldsymbol{\omega}^a{}_b \phi^b. \tag{4.188}$$

Please notice that $\omega^{a0}{}_b$ is a scalar and $\boldsymbol{\omega}^a{}_b$ is a vector:

$$\boldsymbol{\omega}^a{}_b = \begin{bmatrix} \omega^{a1}{}_b \\ \omega^{a2}{}_b \\ \omega^{a3}{}_b \end{bmatrix}. \tag{4.189}$$

The representation of the magnetic field vector is obtained from Eq. (4.177) by using the corresponding combinations of μ and ν:

$$F^a{}_{32} = \partial_3 A^a{}_2 - \partial_2 A^a{}_3 + \omega^a{}_{3b} A^b{}_2 - \omega^a{}_{2b} A^b{}_3, \tag{4.190}$$
$$F^a{}_{13} = \partial_1 A^a{}_3 - \partial_3 A^a{}_1 + \omega^a{}_{1b} A^b{}_3 - \omega^a{}_{3b} A^b{}_1, \tag{4.191}$$
$$F^a{}_{21} = \partial_2 A^a{}_1 - \partial_1 A^a{}_2 + \omega^a{}_{2b} A^b{}_1 - \omega^a{}_{1b} A^b{}_2. \tag{4.192}$$

Then, raising the Greek indices gives us

$$F^{a32} = -\partial_3 A^{a2} + \partial_2 A^{a3} + \omega^{a3}{}_b A^{b2} - \omega^{a2}{}_b A^{b3}, \tag{4.193}$$
$$F^{a13} = -\partial_1 A^{a3} + \partial_3 A^{a1} + \omega^{a1}{}_b A^{b3} - \omega^{a3}{}_b A^{b1}, \tag{4.194}$$
$$F^{a21} = -\partial_2 A^{a1} + \partial_1 A^{a2} + \omega^{a2}{}_b A^{b1} - \omega^{a1}{}_b A^{b2}. \tag{4.195}$$

This can be written as the magnetic field in vector form:

$$\mathbf{B}^a = \boldsymbol{\nabla} \times \mathbf{A}^a - \boldsymbol{\omega}^a{}_b \times \mathbf{A}^b. \tag{4.196}$$

The first Maurer-Cartan structure equation thus leads to the complete set of field-potential relations in ECE electrodynamics:

$$\boxed{\begin{aligned} \mathbf{E}^a &= -\boldsymbol{\nabla}\phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b}\mathbf{A}^b + \boldsymbol{\omega}^a{}_b \phi^b, && (4.197) \\ \mathbf{B}^a &= \boldsymbol{\nabla} \times \mathbf{A}^a - \boldsymbol{\omega}^a{}_b \times \mathbf{A}^b. && (4.198) \end{aligned}}$$

Inserting these relations into the field equations (4.72 - 4.75) leads to

$$\nabla \cdot \left(\nabla \times \mathbf{A}^a - \omega^a{}_b \times \mathbf{A}^b \right) = -\mu_0 j^{a0}, \tag{4.199}$$

$$\frac{\partial \left(\nabla \times \mathbf{A}^a - \omega^a{}_b \times \mathbf{A}^b \right)}{\partial t} + \nabla \times \left(-\nabla \phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b} \mathbf{A}^b + \omega^a{}_b \phi^b \right) = c\,\mu_0 \mathbf{j}^a, \tag{4.200}$$

$$\nabla \cdot \left(-\nabla \phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b} \mathbf{A}^b + \omega^a{}_b \phi^b \right) = \frac{\rho^a}{\varepsilon_0}, \tag{4.201}$$

$$-\frac{1}{c^2} \frac{\partial \left(-\nabla \phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b} \mathbf{A}^b + \omega^a{}_b \phi^b \right)}{\partial t} + \nabla \times \left(\nabla \times \mathbf{A}^a - \omega^a{}_b \times \mathbf{A}^b \right) = \mu_0 \mathbf{J}^a. \tag{4.202}$$

These equations can be simplified by using the theorems of vector algebra to give:

$$\nabla \cdot \left(\omega^a{}_b \times \mathbf{A}^b \right) = \mu_0 j^{a0}, \tag{4.203}$$

$$-c\nabla \times \left(\omega^a{}_{0b} \mathbf{A}^b \right) + \nabla \times \left(\omega^a{}_b \phi^b \right) - \frac{\partial \left(\omega^a{}_b \times \mathbf{A}^b \right)}{\partial t} = c\,\mu_0 \mathbf{j}^a, \tag{4.204}$$

$$\nabla \cdot \frac{\partial \mathbf{A}^a}{\partial t} + \nabla^2 \phi^a + c\nabla \cdot \left(\omega^a{}_{0b} \mathbf{A}^b \right) - \nabla \cdot \left(\omega^a{}_b \phi^b \right) = -\frac{\rho^a}{\varepsilon_0}, \tag{4.205}$$

$$\nabla \left(\nabla \cdot \mathbf{A}^a \right) - \nabla^2 \mathbf{A}^a - \nabla \times \left(\omega^a{}_b \times \mathbf{A}^b \right) + \frac{1}{c^2} \left(\frac{\partial^2 \mathbf{A}^a}{\partial t^2} + c\frac{\partial \left(\omega^a{}_{0b} \mathbf{A}^b \right)}{\partial t} + \nabla \frac{\partial \phi^a}{\partial t} - \frac{\partial \left(\omega^a{}_b \phi^b \right)}{\partial t} \right) = \mu_0 \mathbf{J}^a. \tag{4.206}$$

These are the field equations in potential form. They are much more complicated than those written in terms of force fields. In the standard case of vanishing magnetic monopoles, we have $j^{a0} = 0$ and $\mathbf{j}^a = \mathbf{0}$, as usual. If there are no spin connections, the first two equations have zero terms on their left sides. This indicates that non-vanishing magnetic monopoles are only possible in a spacetime of general relativity.

If no polarization is present, we can omit the corresponding Latin indices. In this case, we only have one scalar and one vector potential, and one scalar and one vector spin connection:

$$\phi^a \to \phi, \tag{4.207}$$

$$\mathbf{A}^a \to \mathbf{A}, \tag{4.208}$$

$$\omega^a{}_{0b} \to \omega_0, \tag{4.209}$$

$$\omega^a{}_b \to \omega. \tag{4.210}$$

Eqs. (4.197 - 4.98) then simplify to

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} - c\omega_0 \mathbf{A} + \omega \phi, \tag{4.211}$$

$$\mathbf{B} = \nabla \times \mathbf{A} - \omega \times \mathbf{A}. \tag{4.212}$$

Compared to their form in Maxwell-Heaviside theory (Eqs. (4.174 - 4.175)), both the **E** and **B** fields contain additional spin connection terms.

When no polarization is present, the field equations (4.203 - 4.206) simplify to

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\omega}\times\mathbf{A}) = \mu_0 j^0, \tag{4.213}$$

$$-c\boldsymbol{\nabla}\times(\omega_0\mathbf{A})+\boldsymbol{\nabla}\times(\boldsymbol{\omega}\phi)-\frac{\partial(\boldsymbol{\omega}\times\mathbf{A})}{\partial t} = c\,\mu_0\,\mathbf{j}, \tag{4.214}$$

$$\boldsymbol{\nabla}\cdot\frac{\partial\mathbf{A}}{\partial t}+\Delta\phi+c\boldsymbol{\nabla}\cdot(\omega_0\mathbf{A})-\boldsymbol{\nabla}\cdot(\boldsymbol{\omega}\phi) = -\frac{\rho}{\varepsilon_0}, \tag{4.215}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})-\Delta\mathbf{A}-\boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}}{\partial t^2}+c\frac{\partial(\omega_0\mathbf{A})}{\partial t}+\boldsymbol{\nabla}\frac{\partial\phi}{\partial t}-\frac{\partial(\boldsymbol{\omega}\phi)}{\partial t}\right) = \mu_0\,\mathbf{J}. \tag{4.216}$$

These are eight component equations for eight potential and spin connection variables. Formally, this equation system is uniquely defined, but the Gauss law is not independent from the Faraday law, and the Coulomb law is not independent from the Ampère-Maxwell law. This has to be taken into account when solving the equation system. The solutions become unique (the equations become independent) when the charge density and the current density are chosen in an unrelated way [34].

5. Advanced properties of electrodynamics

In this chapter, we will complete our discussion of ECE electrodynamics by introducing special features and showing, through detailed examples, that they can be derived directly and easily with ECE theory, but not at all (or only with difficulty and inconsistencies) using standard physics.

5.1 Antisymmetry laws

As shown in Section 2.5.4, the tetrad postulate can be written in the form of Eq. (2.236):

$$q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} = \partial_\mu q^a{}_\lambda + q^b{}_\lambda \omega^a{}_{\mu b}. \tag{5.1}$$

We can define a mixed-index Γ connection by

$$\Gamma^a{}_{\mu\lambda} := q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} = \partial_\mu q^a{}_\lambda + q^b{}_\lambda \omega^a{}_{\mu b}, \tag{5.2}$$

so that the antisymmetric Cartan torsion can be written (with index renaming) as

$$T^a{}_{\mu\nu} = \Gamma^a{}_{\mu\nu} - \Gamma^a{}_{\nu\mu}. \tag{5.3}$$

In Chapter 2 we found that the Γ connection may contain non-vanishing elements for $\mu = \nu$, but only the antisymmetric parts for $\mu \neq \nu$ are relevant. For a priori antisymmetric non-diagonal elements of Γ we have

$$\Gamma^a{}_{\mu\nu} = -\Gamma^a{}_{\nu\mu}. \tag{5.4}$$

Then, for the torsion form (5.3), it follows that

$$T^a{}_{\mu\nu} = 2\,\Gamma^a{}_{\mu\nu} = 2\left(\partial_\mu q^a{}_\nu + q^b{}_\nu \omega^a{}_{\mu b}\right). \tag{5.5}$$

On the other hand, torsion can also be written according to Eq. (4.169) as

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu. \tag{5.6}$$

Equating both expressions for $T^a{}_{\mu\nu}$ then gives the relation

$$2\left(\partial_\mu q^a{}_\nu + q^b{}_\nu \omega^a{}_{\mu b}\right) = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu, \tag{5.7}$$

which can be rearranged to give

$$\boxed{\partial_\mu q^a{}_\nu + \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu + \omega^a{}_{\nu b} q^b{}_\mu = 0.} \tag{5.8}$$

This is the *antisymmetry condition* of ECE theory.

Next, we will discuss how this impacts the vector notation of **E** and **B** fields. Applying the ECE axioms (4.170 - 4.171) gives

$$\partial_\mu A^a{}_\nu + \partial_\nu A^a{}_\mu + \omega^a{}_{\mu b} A^b{}_\nu + \omega^a{}_{\nu b} A^b{}_\mu = 0. \tag{5.9}$$

For $\mu = 0, \nu = 1,2,3$ we obtain

$$\partial_0 A^a{}_1 + \partial_1 A^a{}_0 + \omega^a{}_{0b} A^b{}_1 + \omega^a{}_{1b} A^b{}_0 = 0, \tag{5.10}$$
$$\partial_0 A^a{}_2 + \partial_2 A^a{}_0 + \omega^a{}_{0b} A^b{}_2 + \omega^a{}_{2b} A^b{}_0 = 0, \tag{5.11}$$
$$\partial_0 A^a{}_3 + \partial_3 A^a{}_0 + \omega^a{}_{0b} A^b{}_3 + \omega^a{}_{3b} A^b{}_0 = 0. \tag{5.12}$$

The Greek indices of A and ω can be raised with sign change for $\nu = 1,2,3$. In vector notation this gives

$$-\frac{1}{c}\frac{\partial \mathbf{A}^a}{\partial t} + \boldsymbol{\nabla} A^a{}_0 - \omega^a{}_{0b}\mathbf{A}^b - \boldsymbol{\omega}^a{}_b A^b{}_0 = \mathbf{0} \tag{5.13}$$

or, with $A^a{}_0 = \phi^a / c$,

$$\boxed{-\frac{\partial \mathbf{A}^a}{\partial t} + \boldsymbol{\nabla}\phi^a - c\,\omega^a{}_{0b}\mathbf{A}^b - \boldsymbol{\omega}^a{}_b \phi^b = \mathbf{0}.} \tag{5.14}$$

These are the *electric antisymmetry conditions*, because the terms of the ECE electric field appear.

For $\mu \neq 0$ we obtain from (5.9):

$$\partial_3 A^a{}_2 + \partial_2 A^a{}_3 + \omega^a{}_{3b} A^b{}_2 + \omega^a{}_{2b} A^b{}_3 = 0, \tag{5.15}$$
$$\partial_1 A^a{}_3 + \partial_3 A^a{}_1 + \omega^a{}_{1b} A^b{}_3 + \omega^a{}_{3b} A^b{}_1 = 0, \tag{5.16}$$
$$\partial_2 A^a{}_1 + \partial_1 A^a{}_2 + \omega^a{}_{2b} A^b{}_1 + \omega^a{}_{1b} A^b{}_2 = 0, \tag{5.17}$$

and with indices raised:

$$-\partial_3 A^{a2} - \partial_2 A^{a3} + \omega^{a3}{}_b A^{b2} + \omega^{a2}{}_b A^{b3} = 0, \tag{5.18}$$
$$-\partial_1 A^{a3} - \partial_3 A^{a1} + \omega^{a1}{}_b A^{b3} + \omega^{a3}{}_b A^{b1} = 0, \tag{5.19}$$
$$-\partial_2 A^{a1} - \partial_1 A^{a2} + \omega^{a2}{}_b A^{b1} + \omega^{a1}{}_b A^{b2} = 0. \tag{5.20}$$

These equations are called the *magnetic antisymmetry conditions*, because they relate to the magnetic vector potential $\mathbf{A}^a$. These equations have a permutational structure and cannot be written in vector form.

The antisymmetry conditions are constraints for the fields $\mathbf{E}^a$ and $\mathbf{B}^a$. Therefore, Eqs. (4.188) and (4.196) can be reformulated. First, we will use Eq. (5.5) directly. The ECE axioms allow us to write

$$F^a{}_{\mu\nu} = 2A^{(0)}\,\Gamma^a{}_{\mu\nu} = 2A^{(0)}\left(\partial_\mu q^a{}_\nu + q^b{}_\nu \omega^a{}_{\mu b}\right) \tag{5.21}$$
$$= 2\left(\partial_\mu A^a{}_\nu + \omega^a{}_{\mu b} A^b{}_\nu\right).$$

For $\mu = 0$, we obtain the electric field:

$$\mathbf{E}^a = -\begin{bmatrix} F^{a01} \\ F^{a02} \\ F^{a03} \end{bmatrix} = \begin{bmatrix} F^a{}_{01} \\ F^a{}_{02} \\ F^a{}_{03} \end{bmatrix} = 2c \begin{bmatrix} \partial_0 A^a{}_1 + \omega^a{}_{0b} A^b{}_1 \\ \partial_0 A^a{}_2 + \omega^a{}_{0b} A^b{}_2 \\ \partial_0 A^a{}_3 + \omega^a{}_{0b} A^b{}_3 \end{bmatrix} = -2c \begin{bmatrix} \partial_0 A^{a1} + \omega^a{}_{0b} A^{b1} \\ \partial_0 A^{a2} + \omega^a{}_{0b} A^{b2} \\ \partial_0 A^{a3} + \omega^a{}_{0b} A^{b3} \end{bmatrix} \tag{5.22}$$
$$= -2\left(\frac{\partial \mathbf{A}^a}{\partial t} + c\omega^a{}_{0b}\mathbf{A}^b.\right)$$

For the magnetic field, we obtain

$$\mathbf{B}^a = \begin{bmatrix} F^{a32} \\ F^{a13} \\ F^{a21} \end{bmatrix} = \begin{bmatrix} F^a{}_{32} \\ F^a{}_{13} \\ F^a{}_{21} \end{bmatrix} = 2 \begin{bmatrix} \partial_3 A^a{}_2 + \omega^a{}_{2b} A^b{}_3 \\ \partial_1 A^a{}_3 + \omega^a{}_{3b} A^b{}_1 \\ \partial_2 A^a{}_1 + \omega^a{}_{1b} A^b{}_2 \end{bmatrix} \tag{5.23}$$
$$= 2 \begin{bmatrix} -\partial_3 A^{a2} + \omega^{a2}{}_b A^{b3} \\ -\partial_1 A^{a3} + \omega^{a3}{}_b A^{b1} \\ -\partial_2 A^{a1} + \omega^{a1}{}_b A^{b2} \end{bmatrix}.$$

This equation cannot be written in the form of vector operators.

The electric antisymmetry condition (5.14) can be used to replace the two terms containing the $\mathbf{A}^a$ field with terms of the potential ϕ^a. Inserting them into Eq. (5.22) gives us two formulations for the electric field vector:

$$\mathbf{E}^a = -2\left(\frac{\partial \mathbf{A}^a}{\partial t} + c\omega^a{}_{0b}\mathbf{A}^b\right) = -2\left(\boldsymbol{\nabla}\phi^a - \omega^a{}_b\phi^b\right). \tag{5.24}$$

This is a remarkable result. The electric field is defined by either the vector potential or the scalar potential, in combination with the scalar and vector spin connections. There is no counterpart in classical electrodynamics. If we omitted the spin connections, we would have $\frac{\partial \mathbf{A}^a}{\partial t} = \boldsymbol{\nabla}\phi^a$, which is not generally true in Maxwellian electrodynamics.

Finally, we can also rewrite the ECE magnetic field (5.23) by using the magnetic antisymmetry conditions (5.18 - 5.20):

$$\mathbf{B}^a = 2 \begin{bmatrix} -\partial_3 A^{a2} + \omega^{a2}{}_b A^{b3} \\ -\partial_1 A^{a3} + \omega^{a3}{}_b A^{b1} \\ -\partial_2 A^{a1} + \omega^{a1}{}_b A^{b2} \end{bmatrix} = 2 \begin{bmatrix} \partial_2 A^{a3} - \omega^{a3}{}_b A^{b2} \\ \partial_3 A^{a1} - \omega^{a1}{}_b A^{b3} \\ \partial_1 A^{a2} - \omega^{a2}{}_b A^{b1} \end{bmatrix}. \tag{5.25}$$

This is simply the application of an antisymmetry operation. The factor of 2 appears in Eqs. (5.24) and (5.25) for the "missing terms" when compared with the original definitions (4.197, 4.198).

■ **Example 5.1** In this example we show that classical electrodynamics, which uses U(1) symmetry, is not compatible with the antisymmetry laws of ECE theory. The antisymmetric field tensor is defined in U(1) symmetry [31–34] by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{5.26}$$

and the antisymmetry of this definition requires that

$$\partial_\mu A_\nu = -\partial_\nu A_\mu. \tag{5.27}$$

There is no polarization index and there are no spin connection terms. The electric and magnetic field vectors, in terms of potentials, have the well-known forms:

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{5.28}$$
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{5.29}$$

From the antisymmetry law (5.27), it follows that

$$\nabla\phi = \frac{\partial \mathbf{A}}{\partial t} \tag{5.30}$$

and then, because the curl of a gradient field vanishes:

$$\nabla \times \nabla\phi = \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \mathbf{0}. \tag{5.31}$$

Therefore:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \tag{5.32}$$

From Eq. (5.31) it follows that

$$\nabla \times \mathbf{E} = \mathbf{0}. \tag{5.33}$$

On the other hand, the Faraday law is

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \tag{5.34}$$

If **E** were a time-dependent field, we would have $\frac{\partial \mathbf{B}}{\partial t} \neq \mathbf{0}$; therefore, **E** must be a static field. From the antisymmetry equation (5.30) it follows that

$$\nabla\phi = \frac{\partial \mathbf{A}}{\partial t} = \mathbf{0} \tag{5.35}$$

and so, especially for a static electric field,

$$\mathbf{E} = -\nabla\phi = \mathbf{0}. \tag{5.36}$$

In standard theory it is assumed that $\mathbf{A} = \mathbf{0}$ for static electric fields; consequently, Eq. (5.32) is reduced to

$$\mathbf{B} = \mathbf{0}. \tag{5.37}$$

This has severe consequences that are described in [32] as follows:

The catastrophic result is obtained that the [static] *E and B fields vanish on the U(1) level. All attempts at constructing a unified field theory based on a U(1) sector symmetry are incorrect fundamentally. Even worse for the standard physics is that the method introduced by Heaviside of expressing electric and magnetic fields through* [Eqs. (5.28) and (5.29)] *must be abandoned, so all of twentieth century gauge theory is proven to be empty dogma. This conclusion reinforces many other ways of showing that a U(1) gauge theory of electromagnetism is incorrect and that gauge freedom in the natural sciences is an illusion.*

Here no gauge freedom means that the potential cannot be shifted arbitrarily, because it has a physical meaning. ■

5.2 Polarization and Magnetization

5.2.1 Derivation from standard theory

Standard electrodynamics theory has been extended to media that are polarizable by electric fields and magnetizable by magnetic fields. These material properties evoke additional fields in the media, polarization **P** and magnetization **M**. The resulting total electric field is the dielectric displacement

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}. \tag{5.38}$$

For magnetic materials, the induction is the sum of the magnetic field **H** and magnetization **M**:

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}). \tag{5.39}$$

ε_0 is the vacuum permittivity and μ_0 the vacuum permeability. They are related by the velocity of light in vacuo c:

$$\varepsilon_0\mu_0 = \frac{1}{c^2}. \tag{5.40}$$

In the case of isotropic materials with linear polarization/magnetization properties, the material fields depend linearly on the electric and magnetic fields and can be written as

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}, \tag{5.41}$$

$$\mathbf{B} = \mu_0\mu_r\mathbf{H}, \tag{5.42}$$

where ε_r is the relative permittivity and μ_r is the relative permeability. In vacuo:

$$\varepsilon_r = 1, \quad \mu_r = 1. \tag{5.43}$$

The material equations (5.41, 5.42) can be generalized to ECE equations in a spacetime of general relativity, as we have done for the **E** and **B** fields. Since these are linear relations, the displacement and magnetic field can be augmented by an ECE polarization[1] index a:

$$\mathbf{D}^a = \varepsilon_0\varepsilon_r\mathbf{E}^a, \tag{5.44}$$

$$\mathbf{B}^a = \mu_0\mu_r\mathbf{H}^a. \tag{5.45}$$

The Faraday law in vacuo,

$$\frac{\partial \mathbf{B}^a}{\partial t} + \boldsymbol{\nabla}\times\mathbf{E}^a = \mathbf{0}, \tag{5.46}$$

can be rewritten with the aid of (5.44, 5.45) and with $\varepsilon_r = 1, \mu_r = 1$, as

$$\frac{1}{c^2}\frac{\partial \mathbf{H}^a}{\partial t} + \boldsymbol{\nabla}\times\mathbf{D}^a = \mathbf{0}. \tag{5.47}$$

In matter, the **H** and **D** fields are changed according to Eqs. (5.38, 5.39). Using the simplified relations (5.41, 5.42), we can express these fields by the vacuum fields **E** and **B** (but we would have to use different variable names to be fully correct). By introducing the a index as before, we obtain

$$\frac{1}{\mu_r}\frac{\partial \mathbf{B}^a}{\partial t} + \varepsilon_r\boldsymbol{\nabla}\times\mathbf{E}^a = \mathbf{0}, \tag{5.48}$$

which is an alternative version of the Faraday law in matter. In the same way, for the Ampère-Maxwell law we obtain

$$-c^2\frac{\partial \mathbf{D}^a}{\partial t} + \boldsymbol{\nabla}\times\mathbf{H}^a = \mathbf{J}^a \tag{5.49}$$

and

$$-\varepsilon_r\frac{\partial \mathbf{E}^a}{\partial t} + \frac{1}{\mu_r}\boldsymbol{\nabla}\times\mathbf{B}^a = \mu_0\mathbf{J}^a, \tag{5.50}$$

[1] Please note that this "polarization" is a spacetime property and does not have anything to do with dielectric polarization.

where $\mathbf{J}^a$ is a "free" external current, independent of polarization and magnetization. The Gauss law remains as is, but the Coulomb law becomes

$$\boldsymbol{\nabla} \cdot \mathbf{E}^a = \frac{\rho^a}{\varepsilon_0 \varepsilon_r}. \tag{5.51}$$

Ultimately, we arrive at the ECE field equations for polarizable and magnetizable materials, in vector form:

$$\boxed{\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B}^a &= 0, \\ \frac{1}{\mu_r}\frac{\partial \mathbf{B}^a}{\partial t} + \varepsilon_r \boldsymbol{\nabla} \times \mathbf{E}^a &= \mathbf{0}, \\ \boldsymbol{\nabla} \cdot \mathbf{E}^a &= \frac{\rho^a}{\varepsilon_0 \varepsilon_r}, \\ -\varepsilon_r \frac{\partial \mathbf{E}^a}{\partial t} + \frac{1}{\mu_r} \boldsymbol{\nabla} \times \mathbf{B}^a &= \mu_0 \mathbf{J}^a. \end{aligned}} \qquad \begin{aligned} &(5.52)\\ &(5.53)\\ &(5.54)\\ &(5.55)\end{aligned}$$

The refractive index n is defined in standard dielectric theory by

$$n^2 := \varepsilon_r \mu_r. \tag{5.56}$$

Inserting this into the Faraday law (5.53) gives

$$\frac{\partial \mathbf{B}^a}{\partial t} + n^2 \, \boldsymbol{\nabla} \times \mathbf{E}^a = \mathbf{0}, \tag{5.57}$$

which is a law of optics. Therefore, optical properties can also be described by the ECE polarization and magnetization laws.

5.2.2 Derivation from the ECE homogeneous current

Instead of assuming an isotropic, linear medium, we can base our derivation on the more general laws for polarization and magnetization (5.38) and (5.39) directly. Since these are vector laws, we can transfer the ECE polarization index a to $\mathbf{P}$ and $\mathbf{M}$:

$$\mathbf{D}^a = \varepsilon_0 \mathbf{E}^a + \mathbf{P}^a, \tag{5.58}$$

$$\mathbf{H}^a = \frac{1}{\mu_0} \mathbf{B}^a - \mathbf{M}^a. \tag{5.59}$$

We can incorporate polarization and magnetization into the Faraday law in vacuo, Eq. (5.47), by inserting $\mathbf{D}^a$ and $\mathbf{H}^a$:

$$\frac{1}{c^2} \frac{\partial (\frac{1}{\mu_0} \mathbf{B}^a - \mathbf{M}^a)}{\partial t} + \boldsymbol{\nabla} \times (\varepsilon_0 \mathbf{E}^a + \mathbf{P}^a) = \mathbf{0}. \tag{5.60}$$

Rearranging the terms gives

$$\frac{\partial \mathbf{B}^a}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E}^a = \mu_0 \left(\frac{\partial \mathbf{M}^a}{\partial t} - c^2 \boldsymbol{\nabla} \times \mathbf{P}^a \right). \tag{5.61}$$

This is the Faraday law of ECE theory, with homogeneous current $\mathbf{j}^a$:

$$\frac{\partial \mathbf{B}^a}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E}^a = \mu_0 \mathbf{j}^a \tag{5.62}$$

with

$$\mathbf{j}^a = \frac{\partial \mathbf{M}^a}{\partial t} - c^2 \boldsymbol{\nabla} \times \mathbf{P}^a. \tag{5.63}$$

In this derivation it can be seen clearly that the homogeneous current is equivalent to a spacetime with polarization and magnetization. The effect is like a current of magnetic charges, which may be observable only in cosmic dimensions, where it can be amplified by the path length to measurable levels.

■ **Example 5.2** We show that the cosmological red shift can be described by optical properties of spacetime. We cite from [35]:

[The homogeneous current (5.63) may appear in cosmic dimensions] *and is the mechanism responsible for the interaction of gravitation with the light beam as the latter travels from source to telescope, a distance Z. Over this immense distance it is certain that the light beam encounters myriad species of gravitational field before reaching the telescope and the observer. However weak these fields may be in inter-stellar and inter-galactic ECE spacetime, the enormous path length Z amplifies the current* $\mathbf{j}^a$ *to measurable levels, and appears in the telescope as a red shift. This inference is analogous to the well-known fact that the absorption coefficient in spectroscopy depends on the path length – the greater the path length the greater the absorption of the light beam and the weaker the signal at the detector. Therefore, what is always observed in astronomy is the effect of gravitation on light through the current of* [Eq. (5.63)] *– in general an absorption (or dielectric loss) accompanied by a dispersion (a change in the refractive index).*

It is also well known in spectroscopy that the more dilute the sample the sharper are the spectral features (the effect of collisional broadening is decreased by dilution). Since inter-stellar and inter-galactic spacetime is very tenuous (or dilute), the stars and galaxies appear sharply defined. This does not mean at all that the spacetime is empty or void as in Big Bang theory. The empty inter-stellar and inter-galactic spacetime of Big Bang is defined by Einstein-Hilbert theory alone, without any classical consideration of the classical effect of gravitation on a light beam. The red shifts are defined in Big Bang by a particular solution to the Einstein-Hilbert field equations using a given metric. No account is taken of the homogeneous current $\mathbf{j}^a$ *and so the effect of gravitation on light is not considered classically. These are major omissions, leading to the apparent conclusion that the universe is expanding – simply because the metric demands this conclusion. This is, however, a circular argument – the conclusion (expanding metric deduced) is programmed in at the beginning (expanding metric assumed).*

We emphasize that this explanation strongly supports the tired light theory [36]. The common argument against this theory is that light is dispersed by myriads of collisions with particles of the interstellar medium or even the quantum vacuum. This should lead to very diffuse images in telescopes. However, according to the explanation above, dispersion does not appear because the interstellar medium is very dilute. The red shift is fully explainable on a macroscopic level. The Faraday law can be written for a permeability and permittivity of spacetime, which is space- (and possibly time-) dependent, in the form:

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{B}^a}{\mu_r}\right) + \boldsymbol{\nabla} \times (\varepsilon_r \mathbf{E}^a) = \mathbf{0}. \tag{5.64}$$

This shows how regions with varying μ_r and ε_r alter the electromagnetic properties of light. The red shift can be explained in the following way: assume an electromagnetic plane wave, given in cartesian coordinates (with basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$) by

$$\mathbf{E}^a = \mathbf{i} E_0 e^{i\phi}, \quad \mathbf{B}^a = \mathbf{j} B_0 e^{i\phi}, \tag{5.65}$$

with phase factor

$$\phi = \omega t - \kappa Z. \tag{5.66}$$

This is a wave propagating in the **k** direction with time frequency ω and wave number $\kappa_Z = \kappa$. We insert this into the Faraday equation of free space (5.46):

$$\frac{\partial \mathbf{B}^a}{\partial t} + \nabla \times \mathbf{E}^a = \mathbf{0}, \tag{5.67}$$

and obtain for the single terms:

$$\frac{\partial \mathbf{B}^a}{\partial t} = i\omega \mathbf{j} B_0 e^{i\phi}, \tag{5.68}$$

$$\nabla \times \mathbf{E}^a = -i\kappa \mathbf{j} E_0 e^{i\phi}, \tag{5.69}$$

(see computer algebra code [132] for details). Therefore, from the Faraday law we then obtain

$$\omega B_0 - \kappa E_0 = 0. \tag{5.70}$$

In free space, without dispersion,

$$\frac{\omega}{\kappa} = c, \tag{5.71}$$

and from Section 4.2.3 (the definition of the electromagnetic field tensor) we know that

$$\frac{E_0}{B_0} = c. \tag{5.72}$$

Therefore, the left side of (5.67) gives $1 - 1 = 0$, and the Faraday equation is satisfied. Now we want to know how we have to modify the definitions of the electric and magnetic fields to satisfy the Faraday equation for dielectric space (5.53):

$$\frac{1}{\mu_r} \frac{\partial \mathbf{B}^a}{\partial t} + \varepsilon_r \nabla \times \mathbf{E}^a = \mathbf{0}. \tag{5.73}$$

Inserting the fields (5.65) leads to

$$\frac{\omega}{\mu_r} B_0 - \varepsilon_r \kappa E_0 = 0. \tag{5.74}$$

We obtain Eq. (5.73) by changing the definition of the phase factor in the following way:

$$\omega \rightarrow \frac{\omega}{\mu_r}, \tag{5.75}$$

$$\kappa \rightarrow \varepsilon_r \kappa, \tag{5.76}$$

which leads to

$$\phi = \frac{\omega}{\mu_r} t - \varepsilon_r \kappa Z \tag{5.77}$$

(see computer algebra code [132]). The frequency is lowered by a factor of $1/\mu_r$ with $\mu_r > 1$, and this is the cosmological red shift. As explained above, this is an optical effect that has nothing to do with an expanding universe.

From Eq. (5.74) it follows that

$$\frac{\omega}{\kappa} - \varepsilon_r \mu_r c = 0 \tag{5.78}$$

or

$$\frac{\omega}{\kappa} = n^2 c, \tag{5.79}$$

where

$$n^2 = \varepsilon_r \mu_r \tag{5.80}$$

is the optical refraction index. In optics, n can be complex valued,

$$n = n' + i\, n'' \tag{5.81}$$

with real part n', and imaginary part n'', which describes absorption effects. The frequency value in Eq. (5.79) then becomes complex:

$$\omega = n^2 \omega_0, \tag{5.82}$$

where $\omega_0 = \kappa c$ is the frequency of the wave in vacuo. The frequency part of the phase factor becomes

$$e^{i\omega t} = e^{i\, n^2 \omega_0\, t} = e^{i(\omega_r + i\omega_i)t} \tag{5.83}$$

with real and imaginary frequency parts

$$\omega_r = (n'^2 - n''^2)\omega_0, \quad \omega_i = 2n'n''\omega_0. \tag{5.84}$$

This gives two phase factors:

$$e^{i\omega t} = e^{i\,(n'^2 - n''^2)\omega_0\, t}\; e^{-2n'n''\omega_0\, t}. \tag{5.85}$$

The first factor describes a frequency reduction, and the second factor describes an exponential damping of the wave. Many more details are given in [35]. This gives us a light wave that transfers energy to spacetime, resulting in a lowering of the frequency, which is a red shift effect, again. ∎

5.3 Conservation theorems

5.3.1 Poynting theorem

In ECE theory, the Poynting theorem can be developed in the same way as it is in classical electrodynamics. In addition, a coupling between electromagnetism and gravitation can be included, which is more complete than in the classical theory, because gravitation is described much more completely, compared to classical mechanics [37]. In this discussion we will restrict ourselves to the basic features of ECE electrodynamics.

The total rate P of work in a volume V is given by

$$P = \int_V \mathbf{J} \cdot \mathbf{E}\, d^3x. \tag{5.86}$$

Since in ECE theory all electromagnetic quantities have a polarization index a, we can write this directly as

$$P^a = \int_V \mathbf{J}^a \cdot \mathbf{E}^a\, d^3x. \tag{5.87}$$

All subsequent derivations follow the same path as in standard textbooks [38], except that there is an additional polarization index for all quantities. The energy density of the force fields in materials is

$$u^a = \frac{1}{2} (\mathbf{E}^a \cdot \mathbf{D}^a + \mathbf{B}^a \cdot \mathbf{H}^a) . \tag{5.88}$$

By substituting terms using the field equations, we obtain the following from Eq. (5.86):

$$\boxed{\frac{\partial u^a}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S}^a = -\mathbf{J}^a \cdot \mathbf{E}^a.} \tag{5.89}$$

This is the *Poynting theorem*, in which the *Poynting vector* $\mathbf{S}^a$ is defined by

$$\boxed{\mathbf{S}^a := \mathbf{E}^a \times \mathbf{H}^a.} \tag{5.90}$$

The Poynting vector has the dimensions of energy/(area·time) and describes the energy flow of the fields. The term $\mathbf{J}^a \cdot \mathbf{E}^a$ is the energy density originating in charges moving in an electric field. Magnetic fields are irrelevant since the charges move perpendicularly to the magnetic field due to the Lorentz force. The Poynting vector describes the energy flow and is proportional to the electromagnetic momentum.

In standard theory, only the energy density and energy flow that originate from the force fields are considered. In ECE theory, the potential also is physical. Spacetime itself can be considered to be a background potential. Therefore, a potential without fields is a kind of flux field and has a field energy. This type of potential is not taken into account in the Poynting theorem. There is no classical counterpart for the background potential. There are some possible approaches in [39], for example, for the energy densities contributed by the vector and scalar potentials of ECE theory:

$$u_{A_{ECE}}(\mathbf{r},t) = \frac{1}{2\mu_0} \sum_i \left(\frac{1}{c^2} |\omega_0 A_i|^2 + |\omega_i A_i|^2 \right), \tag{5.91}$$

$$u_{\Phi_{ECE}}(\mathbf{r},t) = \frac{1}{2} \varepsilon_0 \left(\frac{1}{c^2} (\omega_0 \Phi)^2 + \sum_i |\omega_i \Phi|^2 \right). \tag{5.92}$$

The i index numbers the components of the vectors $\mathbf{A}$ and $\boldsymbol{\omega}$.

5.3.2 Continuity equation

In electrodynamics, charges and current densities are conserved. In ECE theory, both are geometrical quantities, which are subject to change due to the structure of spacetime. Nevertheless, they are conserved as in classical theory, indicating that the ECE approach is in agreement with essential physics concepts.

The second field equation (4.48) reads:

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu}, \tag{5.93}$$

where $J^{a\nu}$ is the ECE 4-current density as given by Eq. (4.50). According to (4.172), the electromagnetic field tensor in contravariant form is

$$F^{a\mu\nu} = \partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + \omega^{a\mu}{}_b A^{b\nu} - \omega^{a\nu}{}_b A^{b\mu}. \tag{5.94}$$

The derivative in (5.93) can therefore be written as

$$\partial_\mu F^{a\mu\nu} = \partial_\mu \partial^\mu A^{a\nu} - \partial_\mu \partial^\nu A^{a\mu} + \partial_\mu (\omega^{a\mu}{}_b A^{b\nu}) - \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}). \tag{5.95}$$

We apply an additional derivative ∂_ν to both sides of (5.93). The left side then becomes

$$\partial_\nu \partial_\mu F^{a\mu\nu} = \partial_\nu \partial_\mu \partial^\mu A^{a\nu} - \partial_\nu \partial_\mu \partial^\nu A^{a\mu} + \partial_\nu \partial_\mu (\omega^{a\mu}{}_b A^{b\nu}) - \partial_\nu \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}). \tag{5.96}$$

μ and ν are dummy indices, and thus they can be renamed (by interchanging their names). Partial derivatives can also be commuted:

$$\partial_\nu \partial_\mu F^{a\mu\nu} = \partial_\nu (\partial_\mu \partial^\mu) A^{a\nu} - \partial_\nu (\partial_\mu \partial^\mu) A^{a\nu} + \partial_\nu \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}) - \partial_\nu \partial_\mu (\omega^{a\nu}{}_b A^{b\mu}). \tag{5.97}$$

The terms on the right side of the last equation cancel out, resulting in

$$\partial_\nu \partial_\mu F^{a\mu\nu} = 0. \tag{5.98}$$

Inserting this into Eq. (5.93) gives the *continuity equation* in generally covariant form:

$$\boxed{\partial_\nu J^{a\nu} = 0,} \tag{5.99}$$

which is the 4-divergence of the 4-current density. By applying Eq. (4.53), $J^{a0} = c\rho^a$, it can be written in vector form:

$$\boxed{\frac{\partial \rho^a}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}^a = 0.} \tag{5.100}$$

This form of the continuity equation is identical to that of standard electrodynamics, but holds in a spacetime with curvature and torsion, thus expanding the range of validity significantly.

5.4 Examples of ECE electrodynamics

5.4.1 Gravity-induced polarization changes

■ **Example 5.3** As shown in Example 5.2, the electromagnetic fields of spacetime have optical properties that enable magnetization and polarization. In this example, we explore these optical properties with respect to polarization changes that are induced by gravity. Assume that a circularly polarized electromagnetic wave travels through space in the Z direction. We assume only one polarization of the a index, and therefore the index can be omitted. According to Eqs. (4.150, 4.151) of Example 4.2, the electric and magnetic field (induction) of the wave is then:

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}} (\mathbf{i} + i\mathbf{j}) e^{i\phi}, \tag{5.101}$$

$$\mathbf{B} = \frac{B^{(0)}}{\sqrt{2}} (\mathbf{i} - i\mathbf{j}) e^{-i\phi}, \tag{5.102}$$

with phase factor

$$\phi = \omega t - \kappa Z. \tag{5.103}$$

In an optically active region of spacetime, with $\mu_r \neq 1$ and $\varepsilon_r \neq 1$, the phase is changed to

$$\phi_1 = \frac{\omega}{\mu_r} t - \varepsilon_r \kappa Z. \tag{5.104}$$

After we apply the definition of the refraction index,

$$n^2 = \mu_r \varepsilon_r, \tag{5.105}$$

the Faraday law in media, Eq. (5.73), then reads:

$$\frac{1}{n}\frac{\partial \mathbf{B}^a}{\partial t} + n\boldsymbol{\nabla}\times\mathbf{E}^a = \mathbf{0}. \tag{5.106}$$

The force fields are changed in the following way:

$$\mathbf{E} \rightarrow n\mathbf{E}, \qquad \mathbf{B} \rightarrow \frac{1}{n}\mathbf{B}. \tag{5.107}$$

The real and physical part of Eq. (5.101) in vacuo is

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}\left(\mathbf{i}\cos(\phi) + \mathbf{j}\sin(\phi)\right) \tag{5.108}$$

(see computer algebra code [133]). In an optically active spacetime, the phase factor ϕ is changed to ϕ_1, as described above, and then:

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}\left(\mathbf{i}\cos(\phi_1) + \mathbf{j}\sin(\phi_1)\right). \tag{5.109}$$

Since $\mathbf{E}$ depends on the phase factor in a nonlinear way, the ratio between the X and Y components changes. If

$$\cos(\phi_1) = a\cos(\phi), \tag{5.110}$$
$$\sin(\phi_1) = b\sin(\phi), \tag{5.111}$$

then

$$\mathbf{E} = \frac{E^{(0)}}{\sqrt{2}}\left(a\,\mathbf{i}\cos(\phi) + b\,\mathbf{j}\sin(\phi)\right). \tag{5.112}$$

This is an elliptically polarized wave. For example, for $\phi = 45°$, $\phi_1 = 60°$, we obtain the values $a = 0.707$, $b = 1.225$ (see computer algebra code [133]).

It has been shown that changes in the optical properties of spacetime, due to matter, cause a change of polarization in light passing through a region where these properties are affected by gravitation. Such polarization changes from a white dwarf have been reported by Preuss et al. [40]. Details are discussed in [41]. The ECE theory in this example describes the change of polarization qualitatively and straightforwardly, and a quantitative description could be developed for given parameter functions $\mu_r(\mathbf{r})$ and $\varepsilon_r(\mathbf{r})$ or, alternatively, for given curvature/torsion parameters of the homogeneous current in the respective region. This capability is not present in Einsteinian general relativity. ■

5.4.2 Effects of spacetime properties on optics and spectroscopy

■ **Example 5.4** We show that the Sagnac effect is a consequence of the rotation of spacetime [42]. Consider the rotation of a beam of light of any polarization around a circle in the XY plane, at an angular frequency ω_1 that is to be determined. The rotation is a rotation of spacetime and is described by the rotating tetrad field vector

$$\mathbf{q}^{(1)} = \frac{1}{\sqrt{2}}\left(\mathbf{i} - i\mathbf{j}\right)e^{i\omega_1 t}, \tag{5.113}$$

i.e., rotation around the rim of the circular platform of the static Sagnac interferometer with the beam of light, see Fig. 5.1. The ECE ansatz converts the geometry into physics as follows:

$$\mathbf{A}^{(1)} = A^{(0)}\mathbf{q}^{(1)}. \tag{5.114}$$

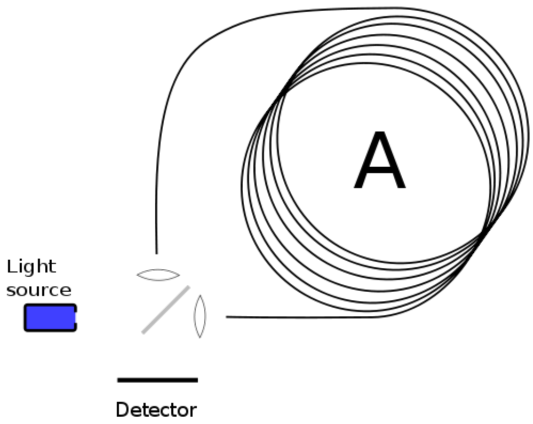


Figure 5.1: Sagnac interferometer [182].

This equation describes a vector potential field rotating around the rim of the circular Sagnac platform at rest. Rotation to the left is described by:

$$\mathbf{A}_L^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\mathbf{i} - i\mathbf{j})\, e^{i\omega_1 t}, \tag{5.115}$$

and rotation to the right is described by:

$$\mathbf{A}_R^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\mathbf{i} + i\mathbf{j})\, e^{i\omega_1 t}. \tag{5.116}$$

This can be seen by computing the real, physical parts:

$$Re(\mathbf{A}_L^{(1)}) = \frac{A^{(0)}}{\sqrt{2}} (\mathbf{i}\cos(\omega_1 t) + \mathbf{j}\sin(\omega_1 t)\,, \tag{5.117}$$

$$Re(\mathbf{A}_R^{(1)}) = \frac{A^{(0)}}{\sqrt{2}} (\mathbf{i}\cos(\omega_1 t) - \mathbf{j}\sin(\omega_1 t)\,, \tag{5.118}$$

which are circular motions in the left and right directions (see computer algebra code [134]).

When the platform is at rest, a beam going around left-wise takes the same time to reach its starting point on the circle as a beam going around right-wise. The time delay between the two beams is

$$\Delta t = 2\pi \left(\frac{1}{\omega_1} - \frac{1}{\omega_1} \right) = 0. \tag{5.119}$$

Note carefully that Eqs. (5.115) and (5.116) do not exist in special relativity because electromagnetism is thought of as an entity superimposed on a passive or static frame, which never rotates. Now consider the beam (5.115) rotating left-wise and spin the platform left-wise at an angular frequency Ω. The result is an increase in the angular frequency of the rotating tetrad (because the spacetime is spinning more quickly):

$$\omega_1 \to \omega_1 + \Omega. \tag{5.120}$$

Similarly, consider the beam (5.115) rotating left-wise and spin the platform right-wise at the same angular frequency Ω. The result is a decrease in the angular frequency of the rotating tetrad (because the spacetime is spinning more slowly):

$$\omega_1 \to \omega_1 - \Omega. \tag{5.121}$$

The time delay between a beam circling left-wise *with* the platform and a beam circling left-wise *against* the platform is

$$\Delta t = 2\pi \left(\frac{1}{\omega_1 - \Omega} - \frac{1}{\omega_1 + \Omega} \right) = \frac{4\pi\Omega}{\omega_1^2 - \Omega^2}, \tag{5.122}$$

and this is called the *Sagnac effect*. Winding more turns of fiber on the interferometer, as indicated in Fig. 5.1, increases the time difference as a multiple of the number of windings.

In order to calculate the angular frequency ω_1 of the rotating light, we start with the fact that the time it takes for light to traverse an infinitesimal length element dl is

$$dt = \frac{dl}{c}. \tag{5.123}$$

The apparatus rotates during this time by an angle Ωdt. The radius of the interferometer is r, and the tangential velocity v of mechanical rotation at radius r is $v = \Omega r$ (when $v << c$). The amount of increase or decrease in the path length of the beam, in a tangential direction, is

$$dx = \Omega\, r\, dt = \frac{\Omega\, r}{c} dl. \tag{5.124}$$

For a complete rotation we obtain

$$x = \oint dx = \oint \frac{\Omega\, r}{c} dl = \frac{\Omega}{c} \cdot 2\pi r \cdot r = \frac{\Omega}{c} \cdot 2A, \tag{5.125}$$

where $A = \pi r^2$ is the area enclosed by a circular beam. The difference between the paths of the two circulating light beams is $2x$; therefore, from (5.123):

$$\Delta t = \frac{2x}{c} = \frac{4\Omega}{c^2} A, \tag{5.126}$$

and equating this with (5.122) gives us

$$\frac{4\Omega}{c^2} \pi r^2 = \frac{4\pi\Omega}{\omega_1^2 - \Omega^2}. \tag{5.127}$$

For $\Omega << \omega_1$ we obtain (see computer algebra code [134]):

$$\omega_1 = \frac{c}{r} = c\kappa \tag{5.128}$$

with a wave number $\kappa = 1/r$. This is the angular frequency of the rotating tetrad, or rotating spacetime. ■

The Sagnac effect is an example of a geometrical phase effect, which is also called the *Berry phase*. In quantum physics, the phase of the wave function changes when a quantum mechanical object is moved on different paths from point A to point B, or from point B back to point A. Prominent examples are the Aharonov-Bohm effect (discussed in Vol. 2 of this book) and the Tomita-Chiao effect (which has also been explained by ECE theory [43]). The essential point of the Tomita-Chiao effect is that light in an optical fiber changes its phase, depending on whether the fiber is laid straight or in a curved way. This is not due to differences in refraction when the fiber is bent or wound around a cylinder. In standard physics, the Berry phase is explained by complicated quantum mechanical methods, but ECE theory is able to give an explanation on the classical level, as it did for the Sagnac interferometer.

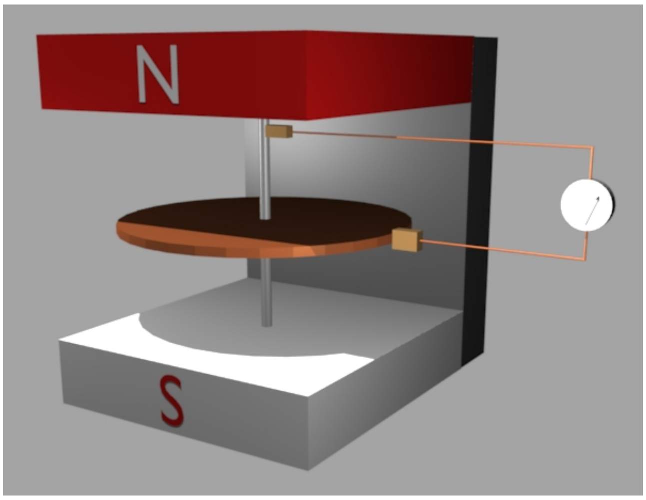


Figure 5.2: Principal setup of the homopolar generator (Faraday disk) [183].

5.4.3 Homopolar generator

■ **Example 5.5** The *homopolar generator* or *Faraday disk* was the first electric generator, and was invented by Michael Faraday in 1831. The original experiment by Faraday was recorded in his diary on December 26, 1831, and consisted of a disc placed on top of a permanent magnet and separated from the magnet by paper. The assembly was suspended by a string and the complete assembly rotated. An electro-motive force (electric field) was observed between the center of the disc and the outer edge of the disc, but this force vanished when the mechanical torsion (rotation) was absent. Alternatively, the disc can be rotated between the poles of a spatially fixed permanent magnet (see Fig. 5.2).

The Faraday law of induction of the Standard Model (special relativistic electrodynamics) was later used to describe the induction seen when a magnet is translated with respect to a stationary induction loop. However, this law does not cover the Faraday disk generator, in which the magnet is stationary. In standard electrodynamics, the Faraday disk is explained by the Lorentz force law, which is the translation law of charges moving in a magnetic field. Since the Lorentz force law is not part of the Maxwell-Heaviside equations, it is sometimes stated that the homopolar generator is not explainable by standard electrodynamics, although this point of view is more or less arbitrary.

In standard theory, any field is considered to be an entity distinct from the passive frame, especially if the field is moving or spinning. When the Faraday disc is described by ECE theory, the frame itself is considered to be spinning. This can be described by the circular complex basis as shown, for example, in Eq. (5.113). The two transverse basis vectors $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ can be described by

$$\mathbf{q}^{(1)} = \mathbf{q}^{(2)*} = \frac{1}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})\, e^{i\Omega t}. \tag{5.129}$$

Ω is the frequency at which the disc is mechanically spun. According to the Evans ansatz (5.114), this generates the vector potentials

$$\mathbf{A}^{(1)} = A^{(0)}\mathbf{q}^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})\, e^{i\Omega t}, \tag{5.130}$$

$$\mathbf{A}^{(2)} = \mathbf{A}^{(1)*} = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i} + i\mathbf{j})\, e^{-i\Omega t}. \tag{5.131}$$

Both vector potentials have the same real, physical part:

$$Re(\mathbf{A}^{(1)}) = Re(\mathbf{A}^{(2)}) = \frac{A^{(0)}}{\sqrt{2}}\left(\mathbf{i}\cos(\Omega t) + \mathbf{j}\sin(\Omega t)\right). \tag{5.132}$$

According to Eq. (4.197), the ECE electric field is

$$\mathbf{E}^a = -\boldsymbol{\nabla}\phi^a - \frac{\partial \mathbf{A}^a}{\partial t} - c\omega^a{}_{0b}\mathbf{A}^b + \omega^a{}_b\phi^b. \tag{5.133}$$

Since the electric potential ϕ and the spin connections are zero (there is no a priori given electric structure), the electric field evoked by mechanical rotation is

$$\mathbf{E}^a = -\frac{\partial \mathbf{A}^a}{\partial t}. \tag{5.134}$$

With (5.130, 5.131), this leads to the electric fields

$$\mathbf{E}^{(1)} = \frac{A^{(0)}\Omega}{\sqrt{2}}\left(-i\mathbf{i} - \mathbf{j}\right)e^{i\Omega t}, \tag{5.135}$$

$$\mathbf{E}^{(2)} = \frac{A^{(0)}\Omega}{\sqrt{2}}\left(i\mathbf{i} - \mathbf{j}\right)e^{-i\Omega t}, \tag{5.136}$$

whose real part is

$$\mathbf{E} = Re(\mathbf{E}^{(1)}) = Re(\mathbf{E}^{(2)}) = \frac{A^{(0)}\Omega}{\sqrt{2}}\left(\mathbf{i}\sin(\Omega t) - \mathbf{j}\cos(\Omega t)\right) \tag{5.137}$$

(see computer algebra code [135]). This electric field (with strength in volts per meter) spins around the rim of the rotating disk. As observed experimentally, it is proportional to the product of $A^{(0)}$ and Ω, and the factor $A^{(0)}$ stems from the permanent magnet. An electromotive force is set up between the center of the disk and its rim, as first observed by Faraday, and this quantity is measured by a voltmeter at rest with respect to the spinning disk.

The frequency Ω of mechanical rotation can be considered to be a spin connection. Then Eq. (5.133) can be written as

$$\mathbf{E}^a = -\frac{\partial \mathbf{A}^a}{\partial t} - \Omega\mathbf{A}^a. \tag{5.138}$$

The real part of $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ contains sin and cos terms in both the X and Y components, representing a phase shift compared to (5.137), but its graph is equivalent to (5.137) (see computer algebra code [135]). In [44, 45] the spin connection was defined with a complex phase factor:

$$\mathbf{E}^a = -\frac{\partial \mathbf{A}^a}{\partial t} - i\Omega\mathbf{A}^a. \tag{5.139}$$

This gives the simpler result:

$$\mathbf{E} = Re(\mathbf{E}^{(1)}) = 2\frac{A^{(0)}\Omega}{\sqrt{2}}\left(\mathbf{i}\sin(\Omega t) - \mathbf{j}\cos(\Omega t)\right), \tag{5.140}$$

which is – except for a constant factor – identical to (5.137). ∎

ECE theory not only gives a consistent description of the Faraday disk through the field equations, but also allows for resonance enhancements of the induced voltage. This is reported in detail in [45].

5.4.4 Spin connection resonance

We will describe typical resonance effects in the next two examples.

■ **Example 5.6** We now consider the resonant Coulomb law. One of the most important consequences of general relativity applied to electrodynamics is that the spin connection enters the relation between the field and potential, as described in Section 4.4. The equations of electrodynamics, written in terms of the potential, can be reduced to the form of Euler-Bernoulli resonance equations. The method is most simply illustrated by considering the vector form of the Coulomb law deduced in Section 4.2.3:

$$\boldsymbol{\nabla}\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0}, \tag{5.141}$$

where we have written the fields without a polarization index. Assuming the absence of a vector potential (absence of a magnetic field), the electric field in the Standard Model is

$$\mathbf{E}=-\boldsymbol{\nabla}\phi, \tag{5.142}$$

where ϕ is the electric potential. Under the same assumption, the electric field in ECE theory, according to Eq. (4.197), is

$$\mathbf{E}=-\boldsymbol{\nabla}\phi+\boldsymbol{\omega}\phi, \tag{5.143}$$

where ω is the vector spin connection. Therefore, Eq. (5.141) takes on the form

$$\nabla^2\phi-\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\phi-(\boldsymbol{\nabla}\cdot\boldsymbol{\omega})\,\phi=-\frac{\rho}{\varepsilon_0}. \tag{5.144}$$

The equivalent equation in the Standard Model is the Poisson equation, which is a limit of Eq. (5.144) when the spin connection is zero. The Poisson equation does not give resonant solutions. However, Eq. (5.144) has resonant solutions of Euler-Bernoulli type, as can be seen in the following discussion. Restricting consideration to one cartesian coordinate reduces the dependencies to $\phi(X)$, and the spin connection to only an X component, $\omega_X(X)$. Then Eq. (5.144) reads:

$$\frac{d^2\phi}{dX^2}-\omega_X\frac{d\phi}{dX}-\frac{d\omega_X}{dX}\phi=-\frac{\rho}{\varepsilon_0}. \tag{5.145}$$

This equation has the structure of a damped Euler-Bernoulli resonance of the form

$$\frac{d^2\phi}{dx^2}+\alpha\frac{d\phi}{dx}+\kappa_0^2\phi=F_0\cos(\kappa x), \tag{5.146}$$

if we assume that $\omega_X<0$. However, as we will see below, this is not an actual restriction. Here, κ_0 is the spatial eigenfrequency, measured in 1/m, and α is the damping constant. On the right side, there is a periodic driving force with spatial frequency (wave number) κ. The particular solution of this differential equation is

$$\phi=F_0\frac{\alpha\kappa\sin(\kappa x)+\left(\kappa_0{}^2-\kappa^2\right)\cos(\kappa x)}{(\kappa_0^2-\kappa^2)^2+\alpha^2\kappa^2}. \tag{5.147}$$

For vanishing damping, we have

$$\phi\rightarrow F_0\frac{\cos(\kappa x)}{\kappa_0^2-\kappa^2}. \tag{5.148}$$

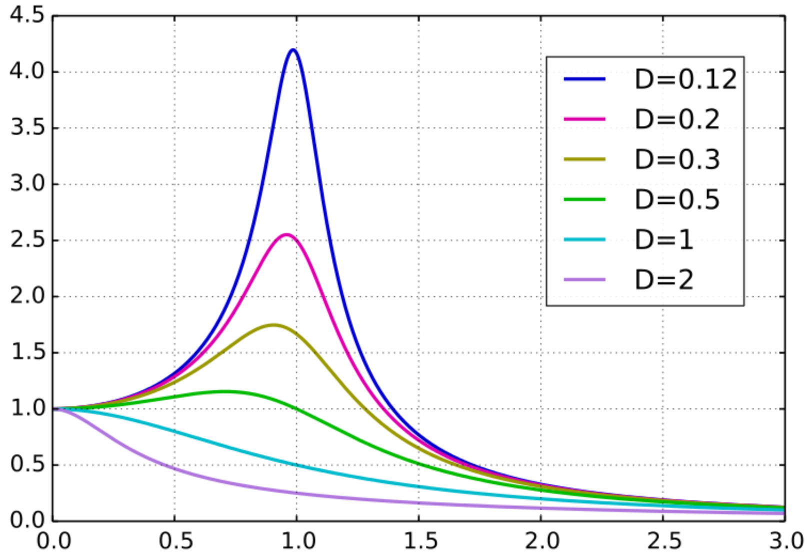


Figure 5.3: Y axis: steady-state amplitude $\phi/\phi_{\text{static}}$ of a damped driven oscillator with different damping constants $D = \alpha/2$. X axis: frequency ratio κ/κ_0 [184].

For $\kappa \to \kappa_0$ the amplitude of $\phi(x)$ approaches infinity. In the case of damping, the amplitude at the resonance point remains finite (see Fig. 5.3).

By comparing Eqs. (5.145) and (5.146), we see that the Coulomb equation (5.144) has no constant coefficients and thus is not an original form of the Euler-Bernoulli resonance. Therefore, we can expect that the solutions may differ significantly from those of the original Euler-Bernoulli equation. To investigate this, we consider an example in spherical polar coordinates. We assume that the potential and the spin connection depend only on the radial coordinate r. For the radial (and only) component of the spin connection, we assume that

$$\omega_r = \frac{1}{r}. \tag{5.149}$$

The differential operators in (5.144) then take the form

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r}, \tag{5.150}$$

$$\boldsymbol{\nabla} \phi = \frac{\partial \phi}{\partial r} \cdot \mathbf{e}_r, \tag{5.151}$$

$$\boldsymbol{\nabla} \cdot (\omega_r\, \mathbf{e}_r) = -\frac{1}{r^2}. \tag{5.152}$$

Then Eq. (5.144), after the right side has been replaced with an oscillatory driving term, reads:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2} = F_0 \sin(\kappa r). \tag{5.153}$$

This equation can be solved analytically (see computer algebra code [136]). The solution contains an expression, $-\cos(\kappa r)/r$, leading to the limit $\phi(r) \to -\infty$ in the case where $r \to 0$, as graphed in Fig. 5.4. Other model examples for $\boldsymbol{\omega}$ are listed in the code. Although the Coulomb law with the spin connection term resembles a resonance equation with damping, there is no damping for $r \to 0$, because of the non-constant coefficients in the equation. ■

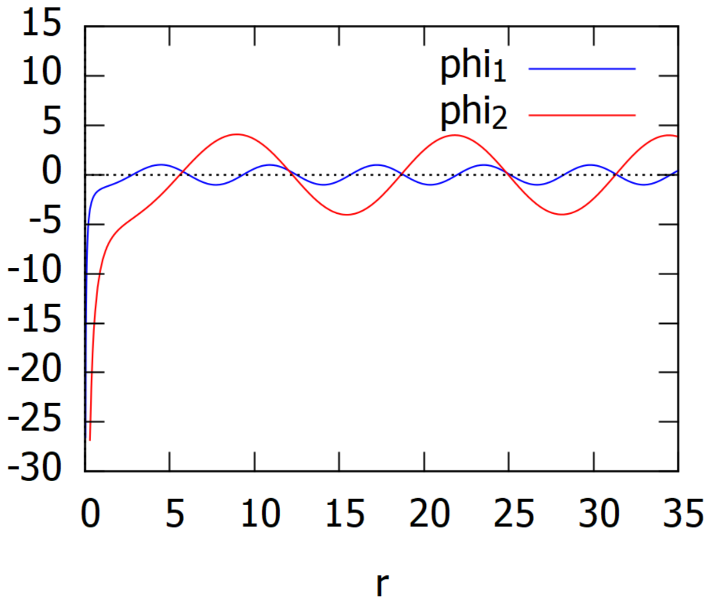


Figure 5.4: Solution of Eq. (5.153), $\kappa = 1$ and $\kappa = 0.5$, other constants normalized.

During the development of ECE theory, the spin connection was incorporated into the Coulomb law [45], which is the basic law used in quantum chemistry. In [45], the electric field was defined in terms of a scalar potential and a spin connection vector. The resulting equation transformed the Poisson equation into a linear inhomogeneous differential equation. This equation was then applied to the hydrogen (H) atom, which serves as a model system for the huge subject area of atomic, molecular and solid-state physics. (The most used method for computation of electronic properties of solids is Density Functional Theory.) We will come back to this in the quantum physics part (Vol. 2) of this book.

ECE theory has also been used to explain and design circuits that use spin connection resonance to take power from spacetime, notably in Papers 63 and 94 [46,47]. In Paper 63, the spin connection was incorporated into the Coulomb law, and through the use of an Euler transform method, the resulting equation in the scalar potential was shown to have resonance solutions. In Paper 94, this method was extended and applied systematically to the Bedini machine, which was shown to have a basis for producing energy from spacetime, although nobody has succeeded in achieving this to date. In addition, spacetime effects in transformers have been found by Osamu Ide and successfully explained by ECE theory [48].

■ **Example 5.7** As another important example, we consider resonant forms of the Ampère-Maxwell law. In potential representation (see Eq. (4.206)) it reads:

$$\begin{aligned} &\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A}^a\right)-\boldsymbol{\nabla}^2\mathbf{A}^a-\boldsymbol{\nabla}\times\left(\boldsymbol{\omega}^a{}_b\times\mathbf{A}^b\right)\\ &+\frac{1}{c^2}\left(\frac{\partial^2\mathbf{A}^a}{\partial t^2}+c\frac{\partial\left(\omega^a{}_{0b}\mathbf{A}^b\right)}{\partial t}+\boldsymbol{\nabla}\frac{\partial\phi^a}{\partial t}-\frac{\partial\left(\boldsymbol{\omega}^a{}_b\phi^b\right)}{\partial t}\right)=\mu_0\mathbf{J}^a. \end{aligned} \tag{5.154}$$

$\mathbf{J}^a$ is a current that may have a polarization dependence. Assuming the simple case – that there is no scalar potential, and that the vector potential is independent of space location and has only a pure time dependence – gives us the equation

$$\frac{\partial^2\mathbf{A}^a}{\partial t^2}+c\frac{\partial\left(\omega^a{}_{0b}\mathbf{A}^b\right)}{\partial t}=\frac{1}{\varepsilon_0}\mathbf{J}^a. \tag{5.155}$$

By restricting this equation to one polarization index, we get

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} + c\frac{\partial\,(\omega_0 \mathbf{A})}{\partial t} = \frac{1}{\varepsilon_0}\mathbf{J}. \tag{5.156}$$

This equation is formally identical to (5.145), except that it is a vector equation and the (only) coordinate is the time coordinate. Here, the spin connection is the scalar spin connection ω_0 with units of 1/m. We replace it by a time frequency, subsuming the factor c:

$$\omega_t = c\ \omega_0, \tag{5.157}$$

so that (5.156) can be written as

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} + \omega_t \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \omega_t}{\partial t}\mathbf{A} = \frac{1}{\varepsilon_0}\mathbf{J}, \tag{5.158}$$

analogously to (5.145). Therefore, the existence of the time spin-connection makes the Ampère-Maxwell law a resonance equation in the same way as was described for the Coulomb law in the preceding example.

Another resonance is possible when there is no scalar potential and we assume that the vector potential is only space-dependent, for example, in magnetic structures. If **A** is divergence-free, we obtain from Eq. (5.154) (again for one direction of polarization):

$$-\boldsymbol{\nabla}^2\mathbf{A} - \boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\mathbf{A}) = \mu_0\,\mathbf{J}, \tag{5.159}$$

where the vector spin connection $\boldsymbol{\omega}$ appears again. Using the vector identity

$$\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} \tag{5.160}$$

and the fact that **A** is divergence-free, we obtain

$$\boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\mathbf{A}) = -\mathbf{A}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{A} \tag{5.161}$$

so that Eq. (5.159) becomes

$$\nabla^2\mathbf{A} + \mathbf{A}(\boldsymbol{\nabla}\cdot\boldsymbol{\omega}) - (\mathbf{A}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{A} = -\mu_0\,\mathbf{J}. \tag{5.162}$$

Because this equation contains derivatives of **A** of zeroth, first and second order, resonances are possible for this special form of the Ampère-Maxwell law. To show this, we first define a special system, where **A** is restricted to two dimensions and the spin connection is perpendicular to the plane of **A**. In cartesian coordinates, we then have

$$\mathbf{A} = \begin{bmatrix} A_X \\ A_Y \\ 0 \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega_Z \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} J_X \\ J_Y \\ J_Z \end{bmatrix}, \tag{5.163}$$

where all variables depend on coordinates X and Y. As shown in computer algebra code [137], it follows that

$$\nabla^2\mathbf{A} = \begin{bmatrix} \frac{\partial^2 A_X}{\partial X^2} + \frac{\partial^2 A_X}{\partial Y^2} \\ \frac{\partial^2 A_Y}{\partial X^2} + \frac{\partial^2 A_Y}{\partial Y^2} \\ 0 \end{bmatrix}, \tag{5.164}$$

$$\boldsymbol{\omega}\times\mathbf{A} = \begin{bmatrix} -A_Y\,\omega_Z \\ A_X\,\omega_Z \\ 0 \end{bmatrix}, \tag{5.165}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\mathbf{A}) = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial \omega_Z}{\partial X}A_X + \frac{\partial \omega_Z}{\partial Y}A_Y + \frac{\partial A_X}{\partial X}\,\omega_Z + \frac{\partial A_Y}{\partial Y}\,\omega_Z \end{bmatrix}. \tag{5.166}$$

Inserting this into Eq. (5.159) leads to three component-equations:

$$\frac{\partial^2 A_X}{\partial X^2} + \frac{\partial^2 A_X}{\partial Y^2} = -\mu_0 J_X, \tag{5.167}$$

$$\frac{\partial^2 A_Y}{\partial X^2} + \frac{\partial^2 A_Y}{\partial Y^2} = -\mu_0 J_Y, \tag{5.168}$$

$$\frac{\partial \omega_Z}{\partial X} A_X + \frac{\partial \omega_Z}{\partial Y} A_Y + \frac{\partial A_X}{\partial X} \omega_Z + \frac{\partial A_Y}{\partial Y} \omega_Z = -\mu_0 J_Z. \tag{5.169}$$

The first two equations decouple from the third, which is of first order in derivatives only. In order to get a sense of how resonances can occur, we simplify this equation set further, so that only one variable, $A_X(X)$, is left:

$$\mathbf{A} = \begin{bmatrix} A_X \\ 0 \\ 0 \end{bmatrix}, \tag{5.170}$$

where $\boldsymbol{\omega}$ and $\mathbf{J}$ remain as in (5.163), but depend only on the X variable. Then the equation set (5.167 - 5.169) simplifies to

$$\frac{\partial^2 A_X}{\partial X^2} = -\mu_0 J_X, \tag{5.171}$$

$$0 = -\mu_0 J_Y, \tag{5.172}$$

$$\frac{\partial \omega_Z}{\partial X} A_X + \frac{\partial A_X}{\partial X} \omega_Z = -\mu_0 J_Z. \tag{5.173}$$

From Eq. (5.172) we get $J_Y = 0$ as a constraint. Eqs. (5.171) and (5.173) are not compatible any more, because they have different solutions for an arbitrary ω_Z [2]. However, if we add these two equations together, we get an analytically solvable equation that combines the properties of both equations:

$$\frac{\partial^2 A_X}{\partial X^2} + \frac{\partial A_X}{\partial X} \omega_Z + \frac{\partial \omega_Z}{\partial X} A_X = -\mu_0 (J_X + J_Z). \tag{5.174}$$

(We will see that this is a meaningful operation, below.) This is a resonance equation with non-constant coefficients, as were Eqs. (5.145) and (5.158). As illustrations, we present a few solutions of this equation in cartesian coordinates. In Table 5.1 we show four solutions of Eq. (5.174) for given combinations of current density $\mathbf{J}$ and spin connection ω_Z. They are graphed in Fig. 5.5. All solutions have divergence points at $X \to 0$, $X \to \pm\infty$, or elsewhere. Eq. (5.171) has an oscillatory solution as expected, but Eq. (5.173), although only of first order, has diverging solutions (Solutions 5-7, in Table 5.1 and Fig. 5.6). Therefore, the combined equation (5.174) is an approximation to a complete calculation where all components of $\mathbf{A}$ and $\boldsymbol{\omega}$ are present, as in Eq. (5.159). ■

It is known from the work of Tesla, for example, that strong resonances in electric power can be obtained with a suitable apparatus, and such resonances cannot be explained using the Standard Model. One consistent explanation of Tesla's well-known results is achieved by incorporating the spin connection into classical electrodynamics.

[2] It is possible to consider both A_X and ω_Z as variables and solve Eqs. (5.171, 5.173) as two equations with two unknowns. The first equation is decoupled from the second; therefore, the solution A_X of the first equation can be inserted into the second equation, and ω_X can be determined, giving resonance solutions for ω_X (see computer algebra code [137]).

Equation	Fig. ref.	$J_X,\ J_Z$	ω_Z	Solution
(5.174)	Solution 1	J_0	$1/X$	$\frac{2}{3}J_0\mu_0 X^2$
	Solution 2	J_0/X	$1/X$	$-J_0\mu_0(X\log(X)+\frac{1}{2}X)$
	Solution 3	$J_0\sin(aX)$	$1/X$	$2\frac{J_0\mu_0}{a^3}(\sin(aX)+\frac{\cos(aX)}{X})$
	Solution 4	J_0X^2	$1/X$	$\frac{2}{15}J_0\mu_0X^4$
(5.171)	Solution 5	$J_0\cos(\kappa_0 X)$		$\frac{\mu_0}{\kappa_0^2}\cos(\kappa_0 X)$
(5.173)	Solution 6	$J_0\cos(\kappa_0 X)$	$\kappa_0\cos(\kappa X)$	$-X\frac{\mu_0}{\kappa_0^2}\frac{\sin(\kappa_0 X)}{\cos(\kappa X)}$
	Solution 7	$J_0\cos(\kappa_0 X)$	$1/X$	$-X\frac{\mu_0}{\kappa_0}\sin(\kappa_0 X)$

Table 5.1: Solutions of model resonance equations.

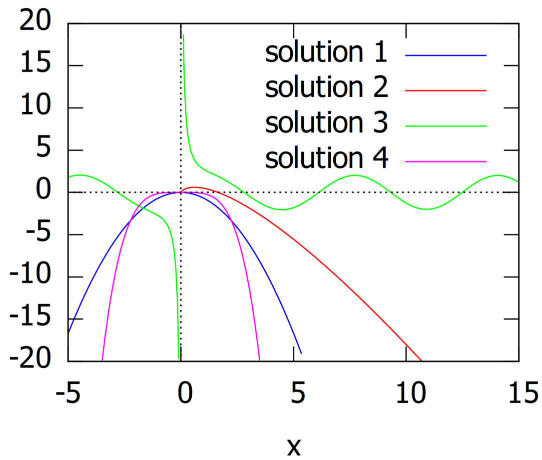


Figure 5.5: Solutions of Eq. (5.174), all constants normalized.

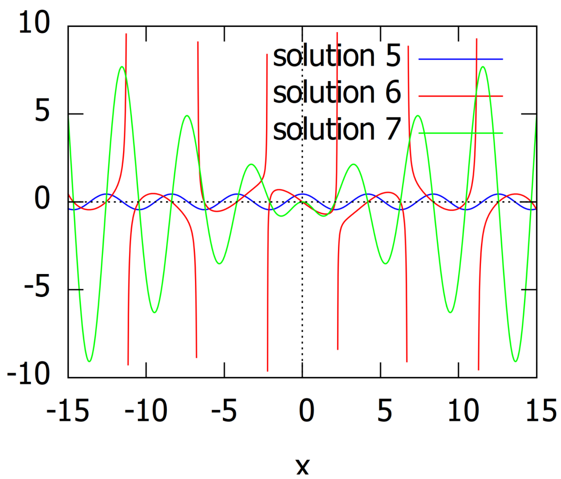


Figure 5.6: Solutions of Eqs. (5.171) and (5.173) with $\kappa_0 = 1.5$ and $\kappa = 0.7$, other constants normalized.

6. ECE2 theory

In the phase of ECE theory that was developed in the preceding chapters, the field equations of electromagnetism were derived from the Cartan-Bianchi and Cartan-Evans identities. In this chapter, we will explore the next phase, in which the Jacobi-Cartan-Evans identity (see Section 3.4.4) will be used to define a new type of curvature, which will be transformed directly into fields using a new ECE hypothesis. This identity gives field equations for the four fundamental fields: electromagnetism, gravitation, and weak and strong nuclear. To illustrate this process, the field equations for electromagnetism will be derived, and it will be shown that the Jacobi-Cartan-Evans identity produces the Maxwell-Heaviside field equations in a space with non-zero torsion and curvature, and with geometrically well-defined magnetic and electric charge-current densities [49].

In this new phase of ECE theory there are no indices of tangent space in the field equations, so for electromagnetism, for example, their format is the same as that of the Maxwell-Heaviside equations. The field-potential relation can be defined without using the spin connections explicitly. However, because Cartan geometry is used, the magnetic and electric charge-current densities are defined geometrically, and the equations are those of a generally covariant unified field theory (ECE theory), and not special relativity. The field equations of gravitation and of the weak and strong nuclear fields have precisely the same format as the field equations of electromagnetism, and specialized field equations for the interaction of the fundamental fields can be developed.

6.1 Curvature-based field equations

6.1.1 Development from the Jacobi-Cartan-Evans identity

To develop the field equations in ECE2 theory, we go back to Cartan geometry as introduced in Chapter 3. The Jacobi-Cartan-Evans identity corrects the original 1902 second Bianchi identity with respect to torsion, and was inferred in Paper 313 [25]. It was written out in Eq. (3.97) with curvature and torsion tensors R and T for the base manifold, i.e., with Greek indices only. As often demonstrated in Chapter 3, the κ index of the curvature tensor in Eq. (3.97) can be replaced with the a index of tangent space:

$$D_\rho R^a{}_{\lambda\mu\nu} + D_\nu R^a{}_{\lambda\rho\mu} + D_\mu R^a{}_{\lambda\nu\rho} = T^\alpha{}_{\mu\nu}\, R^a{}_{\lambda\rho\alpha} + T^\alpha{}_{\rho\mu}\, R^a{}_{\lambda\nu\alpha} + T^\alpha{}_{\nu\rho}\, R^a{}_{\lambda\mu\alpha}, \tag{6.1}$$

where $R^a{}_{\lambda\mu\nu}$ is a mixed-index tensor [49]. This is a cyclic sum of covariant derivatives of curvature tensors. In a space of four dimensions, a second form of the Jacobi-Cartan-Evans identity can be written with Hodge-dual tensors:

$$D_\rho \widetilde{R}^a{}_{\lambda\mu\nu} + D_\nu \widetilde{R}^a{}_{\lambda\rho\mu} + D_\mu \widetilde{R}^a{}_{\lambda\nu\rho} = \widetilde{T}^\alpha{}_{\mu\nu}\, R^a{}_{\lambda\rho\alpha} + \widetilde{T}^\alpha{}_{\rho\mu}\, R^a{}_{\lambda\nu\alpha} + \widetilde{T}^\alpha{}_{\nu\rho}\, R^a{}_{\lambda\mu\alpha}. \tag{6.2}$$

Using the methods of Section 3.3.1, the last two indices of the tensors R and T can be raised by Hodge-dual operations, giving sums of the form

$$D_\mu \widetilde{R}^a{}_\lambda{}^{\mu\nu} = \widetilde{T}^{\alpha\mu\nu} R^a{}_{\lambda\mu\alpha}, \tag{6.3}$$

$$D_\mu R^a{}_\lambda{}^{\mu\nu} = T^{\alpha\mu\nu} R^a{}_{\lambda\mu\alpha}. \tag{6.4}$$

Now we define a new curvature tensor $R^{\mu\nu}$ by

$$R^{\mu\nu} := q^\lambda{}_a R^a{}_\lambda{}^{\mu\nu} \tag{6.5}$$

and its Hodge-dual by

$$\widetilde{R}^{\mu\nu} := q^\lambda{}_a \widetilde{R}^a{}_\lambda{}^{\mu\nu}. \tag{6.6}$$

These fundamentally new curvature definitions were designed to produce curvature tensors in the base manifold, while eliminating the need for the indices of tangent space.

Using the tetrad postulate,

$$D_\mu q^a{}_\lambda = 0, \tag{6.7}$$

it follows for the left side of Eq. (6.3) that

$$D_\mu \widetilde{R}^a{}_\lambda{}^{\mu\nu} = D_\mu (q^a{}_\lambda \widetilde{R}^{\mu\nu}) = (D_\mu q^a{}_\lambda) \widetilde{R}^{\mu\nu} + q^a{}_\lambda D_\mu \widetilde{R}^{\mu\nu} = q^a{}_\lambda D_\mu \widetilde{R}^{\mu\nu}. \tag{6.8}$$

Inserting this into (6.3) and multiplying by $q^\lambda{}_a$, and using the covariant version of (6.5), gives us

$$D_\mu \widetilde{R}^{\mu\nu} = \widetilde{T}^{\alpha\mu\nu} R_{\mu\alpha}. \tag{6.9}$$

By following the same procedure for (6.4), we obtain

$$D_\mu R^{\mu\nu} = T^{\alpha\mu\nu} R_{\mu\alpha}. \tag{6.10}$$

From the definition of the covariant derivative (2.130) of a rank-2 tensor, it follows for both equations that

$$\partial_\mu \widetilde{R}^{\mu\nu} + \Gamma^\mu{}_{\mu\lambda} \widetilde{R}^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda} \widetilde{R}^{\mu\lambda} = \widetilde{T}^{\alpha\mu\nu} R_{\mu\alpha}, \tag{6.11}$$

$$\partial_\mu R^{\mu\nu} + \Gamma^\mu{}_{\mu\lambda} R^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda} R^{\mu\lambda} = T^{\alpha\mu\nu} R_{\mu\alpha}. \tag{6.12}$$

These equations can be abbreviated as

$$\partial_\mu \widetilde{R}^{\mu\nu} = j^\nu, \tag{6.13}$$

$$\partial_\mu R^{\mu\nu} = J^\nu, \tag{6.14}$$

where:

$$j^\nu = \widetilde{T}^{\alpha\mu\nu} R_{\mu\alpha} - \Gamma^\mu{}_{\mu\lambda} \widetilde{R}^{\lambda\nu} - \Gamma^\nu{}_{\mu\lambda} \widetilde{R}^{\mu\lambda}, \tag{6.15}$$

$$J^\nu = T^{\alpha\mu\nu} R_{\mu\alpha} - \Gamma^\mu{}_{\mu\lambda} R^{\lambda\nu} - \Gamma^\nu{}_{\mu\lambda} R^{\mu\lambda}. \tag{6.16}$$

We now define new axioms for transforming the geometrical elements $R^{\mu\nu}$ and $\widetilde{R}^{\mu\nu}$ into the electromagnetic fields $F^{\mu\nu}$ and $\widetilde{F}^{\mu\nu}$:

$$F^{\mu\nu} := W^{(0)} R^{\mu\nu}, \tag{6.17}$$

$$\widetilde{F}^{\mu\nu} := W^{(0)} \widetilde{R}^{\mu\nu}, \tag{6.18}$$

where $W^{(0)}$ is a scalar with units of magnetic flux (Tesla·m^2 or V·s or Weber). Then equations (6.13, 6.14) take the same form as (4.47, 4.48), but without the index a of tangent space:

$$\partial_\mu \widetilde{F}^{\mu\nu} = W^{(0)} j^\nu, \tag{6.19}$$

$$\partial_\mu F^{\mu\nu} = W^{(0)} J^\nu. \tag{6.20}$$

The electromagnetic fields are defined as in standard theory (2.175, 2.188), and both are in Tesla units here:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{bmatrix}, \tag{6.21}$$

$$\widetilde{F}^{\mu\nu} = \begin{bmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & -E^3/c & E^2/c \\ -B^2 & E^3/c & 0 & -E^1/c \\ -B^3 & -E^2/c & E^1/c & 0 \end{bmatrix}. \tag{6.22}$$

We identify the contravariant tensor elements with the cartesian elements of the electric and magnetic fields, as usual. With the current vector definitions

$$(j^\mu) = \begin{bmatrix} j^0 \\ j_X \\ j_Y \\ j_Z \end{bmatrix} = \begin{bmatrix} j^0 \\ \mathbf{j} \end{bmatrix}, \tag{6.23}$$

$$(J^\mu) = \begin{bmatrix} J^0 \\ J_X \\ J_Y \\ J_Z \end{bmatrix} = \begin{bmatrix} J^0 \\ \mathbf{J} \end{bmatrix}, \tag{6.24}$$

we then obtain (as explained in Examples 2.11 and 2.12, in detail) the field equations in vector form:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = W^{(0)} j^0, \tag{6.25}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = c W^{(0)} \mathbf{j}, \tag{6.26}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = c W^{(0)} J^0, \tag{6.27}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = W^{(0)} \mathbf{J}. \tag{6.28}$$

Written in this form, the units of j^ν and J^ν are $1/\text{m}^3$, so that all right sides have the correct physical units. These field equations, which are valid in a spacetime with torsion and curvature, have thus taken the form that we developed in Chapter 5. The homogeneous current j^ν vanishes when

$$\widetilde{T}^{\alpha\mu\nu} R_{\mu\alpha} = \Gamma^\mu{}_{\mu\lambda} \widetilde{R}^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda} \widetilde{R}^{\mu\lambda}, \tag{6.29}$$

and the electric current J^ν is zero when

$$T^{\alpha\mu\nu}R_{\mu\alpha} = \Gamma^{\mu}{}_{\mu\lambda}R^{\lambda\nu} + \Gamma^{\nu}{}_{\mu\lambda}R^{\mu\lambda}. \tag{6.30}$$

In this representation from pure curvature, the field equations do not have more indices than Maxwell's equations, in particular, there is no index of tangent space. Because the axioms (6.17, 6.18) are based on curvature only, this is reminiscent of Einstein's theory. However, the geometrical current definitions (6.15, 6.16) contain a torsion term, whereby Cartan geometry enters.

We now show, through an example, what is required for a purely curvature-based theory to produce convincing numerical results, although not as effectively or comprehensively as ECE/ECE2 theory.

■ **Example 6.1** In this example, we will discuss an early attempt at unification between geometrical and electromagnetic equations. Such an approach was developed quite soon after Einstein's general relativity, by Rainich [50]. It is known under the name *Einstein-Maxwell Equations*. Its basis is the Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{E\mu\nu}, \tag{6.31}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, k the Einstein constant, and $T_{E\mu\nu}$ the energy-momentum tensor of the system under consideration. The *Ricci tensor* and *Ricci scalar* are defined by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \tag{6.32}$$

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{6.33}$$

These are contractions of the Riemann tensor $R^{\lambda}{}_{\mu\rho\nu}$, which was derived from a symmetric Christoffel connection, in Einstein's general relativity. It has been shown that the Einstein field equation, although not directly compatible with ECE theory, can be considered as an approximation, at least in cosmological problems [51].

The Einstein-Maxwell theory uses the electromagnetic energy-momentum tensor of the form [52]:

$$T_{E\mu\nu} = F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{6.34}$$

$F_{\mu\nu}$ is the covariant, antisymmetric electromagnetic field tensor as defined by Eq. (2.177), in electric units. Its covariant form is given by Eq. (6.21). Bruchholz [53] neglected the scalar curvature in Einstein's field equation (6.31), and consequently equated the Ricci tensor with the energy-momentum tensor:

$$R_{\mu\nu} = -kT_{E\mu\nu}. \tag{6.35}$$

Because of energy conservation, the covariant divergence of the energy-momentum tensor has to vanish:

$$D_\nu T_{E\mu}{}^{\nu} = 0, \tag{6.36}$$

which implies that

$$D_\nu R_{\mu}{}^{\nu} = 0. \tag{6.37}$$

Rainich assumed that masses and charges are concentrated in point masses, and exempted these points from the space to be considered. In the "vacuum" regions around the point masses, the covariant divergence of the field tensor $F^{\mu\nu}$ has to vanish also:

$$D_\nu F^{\mu\nu} = 0. \tag{6.38}$$

For the electromagnetic field in vacuo, the first two Maxwell equations can be written according to Example 2.11:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0, \tag{6.39}$$

or in form notation:

$$d \wedge F = 0. \tag{6.40}$$

The other two Maxwell equations have to be expressed by the Hodge dual according to Eq. (2.185) in Example 2.12:

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0, \tag{6.41}$$

and are not considered in Einstein-Maxwell theory. Eq. (6.39) is automatically satisfied, if the electromagnetic field tensor is expressed by potentials A_μ, as is done in Einstein-Maxwell theory:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.42}$$

We see that the field $F_{\mu\nu}$ is defined within special relativity, in a spacetime without curvature and torsion. In Einsteinian relativity, the energy-momentum tensor of the gravitational field vanishes in regions of the vacuum, but the corresponding electromagnetic tensor does not. This is a deficiency of Einstein-Maxwell theory, but it may be irrelevant here because electromagnetic forces are stronger than gravitational forces by at least 21 orders of magnitude.

In total, the equations to be solved are (6.34, 6.35) and (6.38):

$$R_{\mu\nu} = k\left(\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} F_\nu{}^\alpha\right), \tag{6.43}$$

$$\partial_\nu F^{\mu\nu} + \Gamma^\mu{}_{\nu\lambda} F^{\lambda\nu} + \Gamma^\nu{}_{\nu\lambda} F^{\mu\lambda} = 0, \tag{6.44}$$

where $F_{\mu\nu}$ is defined by the potentials in (6.42). The Γ's follow from the metric, and there are 10 independent components of the metric and 4 components of the electromagnetic potential to be determined, in total. There are only 10 equations available, so 4 of 14 components remain undetermined. The corresponding result also holds in ECE theory (see Example 4.1). The equations are nonlinear in their variables, which produces chaotic solutions.

Ulrich Bruchholz found an unrivaled way to determine properties of elementary particles from the Einstein-Maxwell equations without having to solve them directly [54–56]. This is described immediately below. Although elementary particles belong to the realm of quantum mechanics, Bruchholz succeeded in describing their properties with a classical method.

He developed a computation scheme for elementary physical qualities like particle masses, charges, spin and magnetic moments, using this classical method. Wave equations are used as first approximations of the field equations. Non-zero solutions stem from integration constants of these wave equations. These integration constants are defined by the particle qualities mentioned above.

When we try to solve these geometrical equations numerically, the Einstein-Maxwell equations can be fairly difficult to handle. Instead of this, Bruchholz used sampling methods, going from known geometrical regions through unknown regions up to a geometrical limit, by finite steps. For

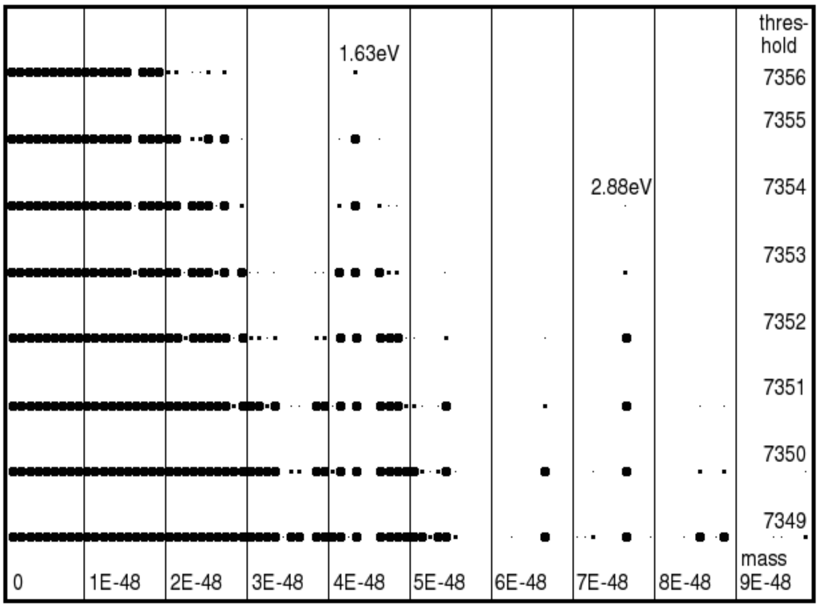


Figure 6.1: Bruchholz results for the electron neutrino, masses < 4eV [55].

central problems, he integrated the solution from an outer starting radius to a somewhat arbitrary inner radius. The end radius is reached when the solutions exceed a certain limit, i.e., when they begin to diverge.

The integration constants are parameters that are inserted into the wave solutions to serve as initial values for the numerical integration of the field equations. The initial metric elements for the coordinates (t, r, θ, ϕ) are defined by

$$g_{00} = 1 - \frac{c_1}{r} + \frac{1}{2}\left(\left(\frac{c_3}{r}\right)^2 + \left(\frac{c_4}{r^2}\right)^2 \cos^2\theta\right), \tag{6.45}$$

$$g_{11} = 1 + \frac{c_1}{r} - \frac{1}{2}\left(\frac{c_3}{r}\right)^2 + \frac{1}{10}\left(\frac{c_4}{r^2}\right)^2 \left(1+\sin^2\theta\right), \tag{6.46}$$

$$g_{22} = r^2\left(1 + \left(\frac{c_4}{r^2}\right)^2\left(\frac{1}{3}\sin^2\theta - \frac{3}{10},\right)\right), \tag{6.47}$$

$$g_{33} = r^2\sin^2\theta\left(1 + \left(\frac{c_4}{r^2}\right)^2\left(\frac{\sin^2\theta}{15} - \frac{3}{10},\right)\right), \tag{6.48}$$

$$g_{23} = r\sin^2\theta\left(\frac{c_2}{r^2} - \frac{1}{2}\frac{c_3c_4}{r^3}\right), \tag{6.49}$$

and the electromagnetic potentials are given by

$$A_0 = \frac{c_3}{r}, \tag{6.50}$$

$$A_3 = r\sin^2\theta\frac{c_4}{r^2}, \tag{6.51}$$

with constants

$$c_1 = \frac{k}{4\pi} m, \tag{6.52}$$

$$c_2 = \frac{ks}{4\pi c}, \tag{6.53}$$

$$c_3 = \frac{\sqrt{k\mu_0}}{4\pi} Q, \tag{6.54}$$

$$c_4 = \frac{\sqrt{k\varepsilon_0}}{4\pi} M, \tag{6.55}$$

for mass m, spin s, charge Q and magnetic moment M. k is the Einstein constant.

The integration constants have to be varied (modified in the sense of variational calculus), and the desired solutions are the discrete values of the integration constants that produce minima with respect to the geometrical limits (the inner end-radii mentioned above). One can numerically determine these discrete values through repeated calculations. These results differ from experimentally known values (e.g., of the electron, nuclei, etc.) by not more than 5%. While these differences may be (comparatively) significant, the large number of constants of Nature that have been determined in this way supports the validity of the method. Moreover, the masses of neutrinos have been predicted [55] (an example is shown in Fig. 6.1), and the predictions are, in the most significant cases, 1.63 eV and 2.88 eV. The abscissa describes the mass values, and the thickness of points describes convergence strength. The more iterations that can be made until divergence, the better the convergence, and the thicker the point. The initial radius was chosen to be $5 \cdot 10^{-15}$m. The horizontally stacked lines indicate different numbers of computational steps (see the right-most column). The results are in the range of assumed mass energies of neutrinos. Reliable values still remain to be found through experiments that are being carried out by several institutions. ■

6.1.2 Enhanced development from the Jacobi-Cartan-Evans identity

In the preceding section, we defined new ECE axioms based only on curvature. This leads to a formulation of ECE2 theory that is based on the tensors of the base manifold, but without reference to the tangent space. The benefit is that no interpretation of polarization indices of the tangent space is required, and the Maxwell-like equations are formally identical to those of Maxwell-Heaviside theory. However, there is no possibility of introducing potentials, which are essential for ECE theory because they have a physical meaning. Potentials were developed on the basis of the first Maurer-Cartan structure equation (2.268), which only involves torsion. In this section, we extend ECE2 theory using the second structure equation (2.282), which relates to curvature [57, 58], and allows us to define potentials. We will see later that the tangent space indices can be removed for simplicity.

As before, we start with the Jacobi-Cartan-Evans identity (6.1) and its Hodge dual, Eq. (6.2). These can be rewritten in covariant form (see Eqs. (6.3, 6.4)):

$$D_\mu \widetilde{R}^{a}{}_{\lambda}{}^{\mu\nu} = \widetilde{T}^{\alpha\mu\nu} R^{a}{}_{\lambda\mu\alpha}, \tag{6.56}$$

$$D_\mu R^{a}{}_{\lambda}{}^{\mu\nu} = T^{\alpha\mu\nu} R^{a}{}_{\lambda\mu\alpha}. \tag{6.57}$$

Then we define a new curvature 2-form $R^{a}{}_{b}{}^{\mu\nu}$ by

$$R^{a}{}_{b}{}^{\mu\nu} := q^{\lambda}{}_{b} R^{a}{}_{\lambda}{}^{\mu\nu}, \tag{6.58}$$

and its Hodge-dual by

$$\widetilde{R}^{a}{}_{b}{}^{\mu\nu} := q^{\lambda}{}_{b} \widetilde{R}^{a}{}_{\lambda}{}^{\mu\nu}. \tag{6.59}$$

The subsequent derivations are very similar to those in Section 6.1.1. By multiplying Eqs. (6.56, 6.57) by $q^{\lambda}{}_{b}$ and using the tetrad postulate, we can formally replace the λ index with the b index of tangent space, giving

$$D_\mu \widetilde{R}^{a\ \mu\nu}_{\ b} = \widetilde{T}^{\alpha\mu\nu} R^{a}{}_{b\mu\alpha}, \tag{6.60}$$
$$D_\mu R^{a\ \mu\nu}_{\ b} = T^{\alpha\mu\nu} R^{a}{}_{b\mu\alpha}. \tag{6.61}$$

From the definition of the covariant derivative of a 2-form, it follows for both equations that

$$\partial_\mu \widetilde{R}^{a\ \mu\nu}_{\ b} + \omega^{a}{}_{\mu c} \widetilde{R}^{c\ \mu\nu}_{\ b} - \omega^{c}{}_{\mu b} \widetilde{R}^{a\ \mu\nu}_{\ c} = \widetilde{T}^{\alpha\mu\nu} R^{a}{}_{b\mu\alpha}, \tag{6.62}$$
$$\partial_\mu R^{a\ \mu\nu}_{\ b} + \omega^{a}{}_{\mu c} R^{c\ \mu\nu}_{\ b} - \omega^{c}{}_{\mu b} R^{a\ \mu\nu}_{\ c} = T^{\alpha\mu\nu} R^{a}{}_{b\mu\alpha}. \tag{6.63}$$

We will now define a new axiom for transforming the geometric curvature 2-form $R^{a\ \mu\nu}_{\ b}$ into a 2-form of the electromagnetic field $F^{a\ \mu\nu}_{\ b}$, and an equivalent new axiom for the Hodge dual:

$$F^{a\ \mu\nu}_{\ b} := W^{(0)} R^{a\ \mu\nu}_{\ b}, \tag{6.64}$$
$$\widetilde{F}^{a\ \mu\nu}_{\ b} := W^{(0)} \widetilde{R}^{a\ \mu\nu}_{\ b}. \tag{6.65}$$

As before, $W^{(0)}$ is a scalar with units of magnetic flux (Tesla·m^2 or Weber), as in the definitions (6.17, 6.18). Note that the expression for the Hodge dual is not independent, because it follows from the definition of $F^{a\ \mu\nu}_{\ b}$.

The electromagnetic field equations then follow from (6.60, 6.61):

$$D_\mu \widetilde{F}^{a\ \mu\nu}_{\ b} = \widetilde{T}^{\alpha\mu\nu} F^{a}{}_{b\mu\alpha}\ , \tag{6.66}$$
$$D_\mu F^{a\ \mu\nu}_{\ b} = T^{\alpha\mu\nu} F^{a}{}_{b\mu\alpha}\ . \tag{6.67}$$

The vector forms of these equations can be developed, but both the electric and magnetic field vectors have two indices a and b, making physical interpretation quite difficult. However, there is a method to remove these indices, as we will see later.

Torsion appears as a tensor of the base manifold, so the situation is different, compared to the original derivation of the ECE field equations in earlier chapters, where torsion appeared as a 2-form in the definitions of the physical fields.

Using the axioms (6.64, 6.65), Eqs. (6.62, 6.63) can be expressed by fields:

$$\partial_\mu \widetilde{F}^{a\ \mu\nu}_{\ b} + \omega^{a}{}_{\mu c} \widetilde{F}^{c\ \mu\nu}_{\ b} - \omega^{c}{}_{\mu b} \widetilde{F}^{a\ \mu\nu}_{\ c} = \widetilde{T}^{\alpha\mu\nu} F^{a}{}_{b\mu\alpha}\ , \tag{6.68}$$
$$\partial_\mu F^{a\ \mu\nu}_{\ b} + \omega^{a}{}_{\mu c} F^{c\ \mu\nu}_{\ b} - \omega^{c}{}_{\mu b} F^{a\ \mu\nu}_{\ c} = T^{\alpha\mu\nu} F^{a}{}_{b\mu\alpha}\ . \tag{6.69}$$

Again, these equations can be abbreviated as

$$\partial_\mu \widetilde{F}^{a\ \mu\nu}_{\ b} = \mu_0 j^{a\ \nu}_{\ b}, \tag{6.70}$$
$$\partial_\mu F^{a\ \mu\nu}_{\ b} = \mu_0 J^{a\ \nu}_{\ b}, \tag{6.71}$$

with current definitions

$$j^{a\ \nu}_{\ b} = \frac{1}{\mu_0}\left(\widetilde{T}^{\alpha\mu\nu} F^{a}{}_{b\mu\alpha} - \omega^{a}{}_{\mu c} \widetilde{F}^{c\ \mu\nu}_{\ b} + \omega^{c}{}_{\mu b} \widetilde{F}^{a\ \mu\nu}_{\ c}\right), \tag{6.72}$$
$$J^{a\ \nu}_{\ b} = \frac{1}{\mu_0}\left(T^{\alpha\mu\nu} F^{a}{}_{b\mu\alpha} - \omega^{a}{}_{\mu c} F^{c\ \mu\nu}_{\ b} + \omega^{c}{}_{\mu b} F^{a\ \mu\nu}_{\ c}\right). \tag{6.73}$$

So far, these currents, the homogeneous and inhomogeneous currents, are 1-forms as before; however, they are augmented with two indices of tangent space.

The electromagnetic fields are also defined as before, but with two indices of tangent space:

$$F^{a\ \mu\nu}_{\ b} = \begin{bmatrix} 0 & -E^{a\ 1}_{\ b}/c & -E^{a\ 2}_{\ b}/c & -E^{a\ 3}_{\ b}/c \\ E^{a\ 1}_{\ b}/c & 0 & -B^{a\ 3}_{\ b} & B^{a\ 2}_{\ b} \\ E^{a\ 2}_{\ b}/c & B^{a\ 3}_{\ b} & 0 & -B^{a\ 1}_{\ b} \\ E^{a\ 3}_{\ b}/c & -B^{a\ 2}_{\ b} & B^{a\ 1}_{\ b} & 0 \end{bmatrix}, \tag{6.74}$$

$$\widetilde{F}^{a\ \mu\nu}_{\ b} = \begin{bmatrix} 0 & B^{a\ 1}_{\ b} & B^{a\ 2}_{\ b} & B^{a\ 3}_{\ b} \\ -B^{a\ 1}_{\ b} & 0 & -E^{a\ 3}_{\ b}/c & E^{a\ 2}_{\ b}/c \\ -B^{a\ 2}_{\ b} & E^{a\ 3}_{\ b}/c & 0 & -E^{a\ 1}_{\ b}/c \\ -B^{a\ 3}_{\ b} & -E^{a\ 2}_{\ b}/c & E^{a\ 1}_{\ b}/c & 0 \end{bmatrix}. \tag{6.75}$$

The new field equations (6.70, 6.71) can be expressed in vector form using a procedure that is completely analogous to the one described in Sections 4.2.2 and 4.3.3. What this gives us is the field equations in vector form (4.72 - 4.75), but with two indices of tangent space:

$$\boldsymbol{\nabla}\cdot\mathbf{B}^a_{\ b} = -\mu_0 j^{a\ 0}_{\ b}, \tag{6.76}$$

$$\frac{\partial \mathbf{B}^a_{\ b}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{E}^a_{\ b} = c\,\mu_0 \mathbf{j}^a_{\ b}, \tag{6.77}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E}^a_{\ b} = \frac{\rho^{a\ 0}_{\ b}}{\varepsilon_0}, \tag{6.78}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}^a_{\ b}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B}^a_{\ b}, = \mu_0 \mathbf{J}^a_{\ b}. \tag{6.79}$$

The current vectors are defined as in Eqs. (6.23, 6.24) by

$$(j^a_{\ b})^\mu = \begin{bmatrix} j^{a\ 0}_{\ b} \\ j^a_{\ bX} \\ j^a_{\ bY} \\ j^a_{\ bZ} \end{bmatrix} = \begin{bmatrix} j^{a\ 0}_{\ b} \\ \mathbf{j}^a_{\ b} \end{bmatrix}, \tag{6.80}$$

$$(J^a_{\ b})^\mu = \begin{bmatrix} J^{a\ 0}_{\ b} \\ J^a_{\ bX} \\ J^a_{\ bY} \\ J^a_{\ bZ} \end{bmatrix} = \begin{bmatrix} J^{a\ 0}_{\ b} \\ \mathbf{J}^a_{\ b} \end{bmatrix}. \tag{6.81}$$

Removing tangent space indices

The indices a, b of tangent space in the field tensor $F^{a\ \mu\nu}_{\ b}$ can be removed by multiplication with the basis vectors. The basis vectors of tangent space are the covariant 4-vectors e_a:

$$e_{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{6.82}$$

and the contravariant vectors e^a:

$$e^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e^{(1)} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad e^{(2)} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad e^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \tag{6.83}$$

where the lower indices have been raised by the Minkowski metric, as usual. Their scalar product is a constant:

$$e^a e_a = 1 - 1 - 1 - 1 = -2. \tag{6.84}$$

We can apply this to remove the Latin indices in the torsion and curvature forms. In the original ECE theory developed in Chapter 4, we can define a vector potential and field tensor by a contraction process with unit 4-vectors:

$$A_\mu = e_a A^a{}_\mu = A^{(0)} e_a q^a{}_\mu = A^{(0)} q_\mu, \tag{6.85}$$

where q_μ is a 4-vector in the space of the base manifold, derived from the tetrad matrix $q^a{}_\mu$. For the electromagnetic field we obtain

$$F_{\mu\nu} = e_a F^a{}_{\mu\nu} = A^{(0)} e_a T^a{}_{\mu\nu} = A^{(0)} T_{\mu\nu}\,, \tag{6.86}$$

which corresponds to a kind of reduced torsion form:

$$T_{\mu\nu} = T^{(0)}{}_{\mu\nu} + T^{(1)}{}_{\mu\nu} + T^{(2)}{}_{\mu\nu} + T^{(3)}{}_{\mu\nu}\,. \tag{6.87}$$

In ECE2 theory, we have a double-indexed field tensor $F^a_{b\mu\nu}$. Therefore, we have to apply linear algebra to reduce this tensor to a field tensor with conventional structure:

$$F_{\mu\nu} = e_a e^b F^a_{b\mu\nu}\,, \tag{6.88}$$

and we can do the same for the contravariant tensor $F^{\mu\nu}$. Using axiom (6.64) we can also write

$$F^{\mu\nu} = W^{(0)} e_a e^b R^a{}_b{}^{\mu\nu}, \tag{6.89}$$

which relates F to the curvature form R.

We will now take a more detailed look at the summations in Eq. (6.88). Using the law of associativity, we can write

$$F_{\mu\nu} = e_a{}^T \left(F^a_{b\mu\nu}\, e^b \right), \tag{6.90}$$

where $e_a{}^T$ is a transposed unit vector. This equation has the form (omitting indices μ,ν):

$$\left([1,0,0,0] + \cdots + [0,0,0,1] \right) \left(F^a{}_{(0)} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots + F^a{}_{(3)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right). \tag{6.91}$$

F^a_b is a matrix, and the right parenthesis contains a sum of matrix-vector multiplications. Since the vectors are unit vectors, they produce terms only from the line of the matrix where the unit vector component is different from zero. Therefore, the matrix can be factored out and the unit vectors can be summed up. The same holds true for the left side:

$$[1,1,1,1] \left(F^a_b \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \right). \tag{6.92}$$

Therefore, when denoting the contracted unit vectors in this equation by $e_{(\mathrm{ctr})a}$ and $e_{(\mathrm{ctr})}{}^b$, we can write for (6.90):

$$F_{\mu\nu} = e_{(\mathrm{ctr})a}{}^T\, F^a_{b\mu\nu}\; e_{(\mathrm{ctr})}{}^b. \tag{6.93}$$

■ **Example 6.2** We compute the values of $F_{\mu\nu}$ by summing over indices a and b. For the single-indexed torsion form, it is simply the sum of the elements over the a index, as was done in Eqs. (6.86, 6.87). In the double-indexed case (6.93), it is more complicated.

The values of $F_{\mu\nu}$ are computed using computer algebra code [138]. The evaluation of the matrix-vector operations (omitting the indices μ, ν on the right side, again) leads to the result:

$$\begin{aligned} F_{\mu\nu} =& F^{(0)}{}_{(0)} - \left(F^{(0)}{}_{(1)} + F^{(0)}{}_{(2)} + F^{(0)}{}_{(3)}\right) \\ &+ F^{(1)}{}_{(0)} - \left(F^{(1)}{}_{(1)} + F^{(1)}{}_{(2)} + F^{(1)}{}_{(3)}\right) \\ &+ F^{(2)}{}_{(0)} - \left(F^{(2)}{}_{(1)} + F^{(2)}{}_{(2)} + F^{(2)}{}_{(3)}\right) \\ &+ F^{(3)}{}_{(0)} - \left(F^{(3)}{}_{(1)} + F^{(3)}{}_{(2)} + F^{(3)}{}_{(3)}\right). \end{aligned} \tag{6.94}$$

This sum has been written in matrix form. It can be seen that the first column is taken as positive summands and the other columns as negative summands. This is a consequence of the signs in the contravariant unit vector e^b. ■

The above simplifications can be applied to both sides of the field equations (6.70, 6.71), leading to reduced 1-forms of current on the right side. The field equations then take the familiar form:

$$\partial_\mu \widetilde{F}^{\mu\nu} = \mu_0 j^\nu, \tag{6.95}$$

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu. \tag{6.96}$$

This set can be handled in the same way as in Section 6.1.1, leading to the familiar vector form of electromagnetic field equations (6.25 - 6.28). In the present case, we have defined the currents in units of A/m^2, as usual; therefore, the constants on the right side differ. The results are the field equations in vector form (6.25 - 6.28) without any indices of tangent space:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = -\mu_0 j^0, \tag{6.97}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = c\,\mu_0 \mathbf{j}, \tag{6.98}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{6.99}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{B}, = \mu_0 \mathbf{J}. \tag{6.100}$$

If no magnetic monopoles are present, the homogeneous currents vanish. In this case, these equations are formally identical to the Maxwell-Heaviside equations, but they are valid in a space with torsion and curvature, thus exceeding the range of validity of Maxwell-Heaviside theory, by far.

6.1.3 ECE2 field equations in terms of potentials

The information contained in electromagnetic fields can be expanded significantly by considering the potentials, which are physical in ECE theory. In Section 4.4, the relationship between force fields and potentials was derived from the first Maurer-Cartan structure equation, Eq. (4.169), which connects the torsion with the tetrad and the spin connection:

$$T^a{}_{\mu\nu} = \partial_\mu q^a{}_\nu - \partial_\nu q^a{}_\mu + \omega^a{}_{\mu b} q^b{}_\nu - \omega^a{}_{\nu b} q^b{}_\mu. \tag{6.101}$$

The force fields then follow from the first ECE axiom:

$$F^a{}_{\mu\nu} = A^{(0)} T^a{}_{\mu\nu}. \tag{6.102}$$

In ECE2 theory, the force fields are derived from the axiom

$$F^a{}_{b\mu\nu} = W^{(0)} R^a{}_{b\mu\nu}, \tag{6.103}$$

which is based on curvature. Therefore, we have to use the second Maurer-Cartan structure equation, which connects curvature with the spin connection (see Eq. (2.282)). In tensor notation this equation reads:

$$R^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{\nu b} - \partial_\nu \omega^a{}_{\mu b} + \omega^a{}_{\mu c}\,\omega^c{}_{\nu b} - \omega^a{}_{\nu c}\,\omega^c{}_{\mu b}. \tag{6.104}$$

The left side describes the field tensor according to ECE2 axiom (6.103) above. The right side has to be equated to potentials. To make a consistent definition of ECE2 potentials, we can compare Eq. (6.104) with Eq. (6.101). The left sides give the fields according to the ECE axiom

$$F^a{}_{\mu\nu} = A^{(0)} T^a{}_{\mu\nu}, \tag{6.105}$$

and the ECE2 axiom

$$F^a{}_{b\mu\nu} = W^{(0)} R^a{}_{b\mu\nu}. \tag{6.106}$$

Since the left sides are physically equivalent, the right sides must be as well. In ECE theory, the potential is defined by

$$A^a{}_\mu = A^{(0)} q^a{}_\mu. \tag{6.107}$$

In ECE2 theory, it must be defined in a compatible way. This can be accomplished through the following identifications:

index $a \rightarrow$ indices a, b,
$q^a{}_\nu \rightarrow \omega^a{}_{\nu b}$.

This allows the second Maurer-Cartan structure equation (6.104) to be transformed formally into the first Maurer-Cartan structure equation (6.101). For consistency, the ECE2 potential, $W^a{}_{b\mu}$, has to be defined by

$$W^a{}_{b\mu} = W^{(0)} \omega^a{}_{\mu b}, \tag{6.108}$$

i.e., the ECE2 potential is the spin connection, augmented with the constant $W^{(0)}$. For convenience, the order of the lower indices has been interchanged. The ECE2 axioms then read:

$$\boxed{\begin{aligned} W^a{}_{b\mu} &= W^{(0)} \omega^a{}_{\mu b}, \quad (6.109)\\ F^a{}_{b\mu\nu} &= W^{(0)} R^a{}_{b\mu\nu}, \quad (6.110)\end{aligned}}$$

and the ECE2 field-potential relationship is

$$F^a{}_{b\mu\nu} = \partial_\mu W^a{}_{b\nu} - \partial_\nu W^a{}_{b\mu} + \omega^a{}_{\mu c} W^c{}_{b\nu} - \omega^a{}_{\nu c} W^c{}_{b\mu}. \tag{6.111}$$

The remaining ω's are from the Maurer-Cartan structure equation. In order to give this equation a unified structure, we replace the remaining spin connections with the potentials as well:

$$\boxed{F^a{}_{b\mu\nu} = \partial_\mu W^a{}_{b\nu} - \partial_\nu W^a{}_{b\mu} + \frac{1}{W^{(0)}}\left(W^a{}_{c\mu} W^c{}_{b\nu} - W^a{}_{c\nu} W^c{}_{b\mu}\right).} \tag{6.112}$$

Remarkably, the fields depend only on the potentials, without the explicit appearance of a spin connection, but in a non-linear way.

Next, we will transform this equation into a vector representation. By using the ECE2 field-potential relationship in the form of (6.111), we can proceed analogously to Section 4.4, starting with Eq. (4.178). For $\mu = 0$, we obtain

$$F^a{}_{b01} = \partial_0 W^a{}_{b1} - \partial_1 W^a{}_{b0} + \omega^a{}_{0c} W^c{}_{b1} - \omega^a{}_{1c} W^c{}_{b0}\,, \tag{6.113}$$
$$F^a{}_{b02} = \partial_0 W^a{}_{b2} - \partial_2 W^a{}_{b0} + \omega^a{}_{0c} W^c{}_{b2} - \omega^a{}_{2c} W^c{}_{b0}\,, \tag{6.114}$$
$$F^a{}_{b03} = \partial_0 W^a{}_{b3} - \partial_3 W^a{}_{b0} + \omega^a{}_{0c} W^c{}_{b3} - \omega^a{}_{3c} W^c{}_{b0}\,. \tag{6.115}$$

We introduce a scalar potential $\phi_W{}^a{}_b$ analogously to ϕ^a:

$$W^a{}_{b0} = \frac{\phi_W{}^a{}_b}{c}. \tag{6.116}$$

It then follows that (in vector form):

$$\mathbf{E}^a{}_b = -\boldsymbol{\nabla}\phi_W{}^a{}_b - \frac{\partial \mathbf{W}^a{}_b}{\partial t} - c\omega^a{}_{0c}\mathbf{W}^c{}_b + \boldsymbol{\omega}^a{}_c \phi_W{}^c{}_b\,, \tag{6.117}$$

where $\boldsymbol{\omega}^a{}_b$ is a vector as defined by Eq. (4.189). The magnetic field is computed in vector form analogously to Eqs. (4.190 ff.), giving

$$\mathbf{B}^a{}_b = \boldsymbol{\nabla}\times\mathbf{W}^a{}_b - \boldsymbol{\omega}^a{}_c \times \mathbf{W}^c{}_b. \tag{6.118}$$

Replacing the ω's with W, we obtain the vector form of Eq. (6.112):

$$\mathbf{E}^a{}_b = -\boldsymbol{\nabla}\phi_W{}^a{}_b - \frac{\partial \mathbf{W}^a{}_b}{\partial t} + \frac{1}{W^{(0)}}\left(-\phi_W{}^a{}_c \mathbf{W}^c{}_b + \phi_W{}^c{}_b \mathbf{W}^a{}_c\right), \tag{6.119}$$
$$\mathbf{B}^a{}_b = \boldsymbol{\nabla}\times\mathbf{W}^a{}_b - \frac{1}{W^{(0)}}\mathbf{W}^a{}_c \times \mathbf{W}^c{}_b. \tag{6.120}$$

We can now remove the indices a, b of tangent space, as we did in Section 6.1.2. The index c is a dummy index, so the procedure of multiplying by unit vectors e_a and e^b can be applied directly, and then the nonlinear terms in W cancel out. The result is

$$\mathbf{E} = -\boldsymbol{\nabla}\phi_W - \frac{\partial \mathbf{W}}{\partial t}, \tag{6.121}$$
$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{W}. \tag{6.122}$$

This remarkable result means that in ECE2 theory the potentials ϕ_W and $\mathbf{W}$ play the same role as the scalar potential ϕ and vector potential $\mathbf{A}$ in Maxwellian theory. ECE2 can be applied formally in the same way as Maxwellian theory, but with a much greater scope in generally relativistic spacetime.

6.1.4 Combining ECE2 and ECE theory

So far, we have used the current 1-forms (6.72, 6.73), which have two tangent space indices in the same way as the ECE2 fields $\mathbf{E}^a{}_b$ and $\mathbf{B}^a{}_b$. In the case where we summed over these indices, we obtained the Maxwell-like equations (6.97 - 6.100) with current vectors $\mathbf{j}$ and $\mathbf{J}$, as well as charge densities j^0 and ρ. In the following discussion, we develop the charge and current expressions in vector form. We will see that this leads to a combination of double-indexed vectors, like $\mathbf{E}^a{}_b$, while those of the original ECE theory will continue to have only one index, like $\mathbf{E}^a$.

We start by revisiting the original ECE theory equations, which have one tangent space index. According to Eqs. (4.49, 4.50), the 1-forms for currents are

$$j^{a\nu} = \frac{1}{\mu_0}\left(A^{(0)}\widetilde{R}^{a}{}_{\mu}{}^{\mu\nu} - \omega_{(\Lambda)}{}^{a}{}_{\mu b}\widetilde{F}^{b\mu\nu}\right), \tag{6.123}$$

$$J^{a\nu} = \frac{1}{\mu_0}\left(A^{(0)}R^{a}{}_{\mu}{}^{\mu\nu} - \omega^{a}{}_{\mu b}F^{b\mu\nu}\right). \tag{6.124}$$

The currents contain the spin connections, which can be transformed into ECE2 potentials, as was done in the previous section. The field tensor has one Latin index and can be replaced with electric and magnetic field vector components according to Eqs. (4.56, 4.65). We will develop the vector forms of the currents as described in the Notes for [58]. Computer algebra code for evaluating the current components is available [139].

We start with the homogeneous current $j^{a\nu}$. First, the curvature tensor $\widetilde{R}^{a}{}_{\mu}{}^{\mu\nu}$ needs to be transformed by replacing the lower Greek index μ with an index of tangent space in the following way:

$$\widetilde{R}^{a}{}_{\mu}{}^{\mu\nu} = q^{b}{}_{\mu}\widetilde{R}^{a}{}_{b}{}^{\mu\nu}. \tag{6.125}$$

$\widetilde{R}^{a}{}_{b}{}^{\mu\nu}$ is a curvature element of ECE2 theory, corresponding to vector components of $\mathbf{E}^{a}{}_{b}$ and $\mathbf{B}^{a}{}_{b}$ as defined in Eq. (6.75). By applying the ECE and ECE2 axioms:

$$A^{a}{}_{\nu} := A^{(0)}q^{a}{}_{\nu}, \tag{6.126}$$

$$\widetilde{F}^{a}{}_{b}{}^{\mu\nu} := W^{(0)}\widetilde{R}^{a}{}_{b}{}^{\mu\nu}, \tag{6.127}$$

we obtain

$$\begin{aligned} j^{a\nu} &= \frac{1}{\mu_0}\left(A^{b}{}_{\mu}\widetilde{R}^{a}{}_{b}{}^{\mu\nu} - \omega_{(\Lambda)}{}^{a}{}_{\mu b}\widetilde{F}^{b\mu\nu}\right) \\ &= \frac{1}{\mu_0}\left(\frac{1}{W^{(0)}}A^{b}{}_{\mu}\widetilde{F}^{a}{}_{b}{}^{\mu\nu} - \omega_{(\Lambda)}{}^{a}{}_{\mu b}\widetilde{F}^{b\mu\nu}\right). \end{aligned} \tag{6.128}$$

To obtain vector representations, we proceed as in Section 4.2.3. For $\nu = 0$, $\mu = 1,2,3$, we obtain the current component

$$\begin{aligned} j^{a0} = \frac{1}{\mu_0}\Bigg(&\frac{1}{W^{(0)}}\left(A^{b}{}_{1}\widetilde{F}^{a}{}_{b}{}^{10} + A^{b}{}_{2}\widetilde{F}^{a}{}_{b}{}^{20} + A^{b}{}_{3}\widetilde{F}^{a}{}_{b}{}^{30}\right) \\ &-\omega_{(\Lambda)}{}^{a}{}_{1b}\widetilde{F}^{b10} - \omega_{(\Lambda)}{}^{a}{}_{2b}\widetilde{F}^{b20} - \omega_{(\Lambda)}{}^{a}{}_{3b}\widetilde{F}^{b30}\Bigg). \end{aligned} \tag{6.129}$$

From (4.56) and (6.75) it follows that $\widetilde{F}^{a10} = -B^{a1}$, ... and $\widetilde{F}^{a}{}_{b}{}^{10} = -B^{a}{}_{b}{}^{1}$, ... , and therefore:

$$\begin{aligned} j^{a0} = \frac{1}{\mu_0}\Bigg(&\frac{1}{W^{(0)}}\left(-A^{b}{}_{1}B^{a}{}_{b}{}^{1} - A^{b}{}_{2}B^{a}{}_{b}{}^{2} - A^{b}{}_{3}B^{a}{}_{b}{}^{3}\right) \\ &+\omega_{(\Lambda)}{}^{a}{}_{1b}B^{b1} + \omega_{(\Lambda)}{}^{a}{}_{2b}B^{b2} + \omega_{(\Lambda)}{}^{a}{}_{3b}B^{b3}\Bigg). \end{aligned} \tag{6.130}$$

Raising the indices of the vector potential and the spin connection gives a sign change:

$$\begin{aligned} j^{a0} = \frac{1}{\mu_0}\Bigg(&\frac{1}{W^{(0)}}\left(A^{b1}B^{a}{}_{b}{}^{1} + A^{b2}B^{a}{}_{b}{}^{2} + A^{b3}B^{a}{}_{b}{}^{3}\right) \\ &-\omega_{(\Lambda)}{}^{a1}{}_{b}B^{b1} - \omega_{(\Lambda)}{}^{a2}{}_{b}B^{b2} - \omega_{(\Lambda)}{}^{a3}{}_{b}B^{b3}\Bigg). \end{aligned} \tag{6.131}$$

In vector notation, this equation is

$$j^{a0} = \frac{1}{\mu_0}\left(\frac{1}{W^{(0)}}\mathbf{A}^b \cdot \mathbf{B}^a{}_b - \boldsymbol{\omega}_{(\Lambda)}{}^a{}_b \cdot \mathbf{B}^b\right). \tag{6.132}$$

All terms on the right side contain a double sum over b. This could be simplified as described in Example 6.2, where all components of $F^a_{\ b}$ were summed up. Alternatively, we can assume that there is a common value of $F^a_{\ b}$ for all a and b, say F. Either way, the double sum gives

$$F^a_{\ b} \rightarrow e^b F^a_{\ b} = -2F^a. \tag{6.133}$$

We can apply this to the scalar products in Eq. (6.132). Furthermore, the sum over a gives

$$F^a \rightarrow e_a F^a = 4F, \tag{6.134}$$

and is applied to both sides of the equation so that the factor of 4 cancels out, which gives us the result:

$$j^0 = -\frac{2}{\mu_0}\left(\frac{1}{W^{(0)}}\mathbf{A} \cdot \mathbf{B} - \boldsymbol{\omega}_{(\Lambda)} \cdot \mathbf{B}\right) \tag{6.135}$$

with indexless vectors. The Gauss Law (6.97) then takes the form

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 2\left(\frac{1}{W^{(0)}}\mathbf{A} - \boldsymbol{\omega}_{(\Lambda)}\right) \cdot \mathbf{B}. \tag{6.136}$$

The magnetic charge density vanishes if

$$\frac{1}{W^{(0)}}\mathbf{A} = \boldsymbol{\omega}_{(\Lambda)}, \tag{6.137}$$

which is usually the case, as is known from experiments. This means that the spin connection of the Λ connection (the Hodge dual of the Christoffel connection) is parallel to the vector potential. The Gauss law can then be rewritten as

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 2\left(\frac{A^{(0)}}{W^{(0)}}\mathbf{q} - \boldsymbol{\omega}_{(\Lambda)}\right) \cdot \mathbf{B} = 2\left(\frac{1}{r^{(0)}}\mathbf{q} - \boldsymbol{\omega}_{(\Lambda)}\right) \cdot \mathbf{B}, \tag{6.138}$$

where

$$r^0 = \frac{W^{(0)}}{A^{(0)}} \tag{6.139}$$

is a constant that has the dimension of length. In this notation, the magnetic charge density is described by geometrical quantities only, namely the tetrad and spin connection.

Next, we analyze the components of the homogeneous current (6.123) for $\nu = 1,2,3$. From Eq. (6.128), for $\nu = 1$, we obtain

$$\begin{aligned} j^{a1} = \frac{1}{\mu_0}\Bigg(&\frac{1}{W^{(0)}}\left(A^b{}_0 \widetilde{F}^a_{\ b}{}^{01} + A^b{}_2 \widetilde{F}^a_{\ b}{}^{21} + A^b{}_3 \widetilde{F}^a_{\ b}{}^{31}\right) \\ &-\omega_{(\Lambda)}{}^a{}_{0b}\widetilde{F}^{b01} - \omega_{(\Lambda)}{}^a{}_{2b}\widetilde{F}^{b21} - \omega_{(\Lambda)}{}^a{}_{3b}\widetilde{F}^{b31}\Bigg), \end{aligned} \tag{6.140}$$

and by inserting the field components:

$$\begin{aligned} j^{a1} = \frac{1}{\mu_0}\Bigg(&\frac{1}{W^{(0)}}\left(A^b{}_0 B^a{}_b{}^1 + A^b{}_2 E^a{}_b{}^3/c - A^b{}_3 E^a{}_b{}^2/c\right) \\ &-\omega_{(\Lambda)}{}^a{}_{0b}B^{b1} - \omega_{(\Lambda)}{}^a{}_{2b}E^{b3}/c + \omega_{(\Lambda)}{}^a{}_{3b}E^{b2}/c\Bigg), \end{aligned} \tag{6.141}$$

and by raising lower indices (please notice that raising the index 0 does not give a sign change) we get

$$j^{a1} = \frac{1}{\mu_0} \left(\frac{1}{W^{(0)}} \left(A^{b0} B^a{}_b{}^1 - A^{b2} E^a{}_b{}^3/c + A^{b3} E^a{}_b{}^2/c \right) - \omega_{(\Lambda)}{}^{a0}{}_b B^{b1} + \omega_{(\Lambda)}{}^{a2}{}_b E^{b3}/c - \omega_{(\Lambda)}{}^{a3}{}_b E^{b2}/c \right). \tag{6.142}$$

After proceeding in the same way for $\nu = 2, 3$, we obtain the cumulative result in vector form:

$$\mathbf{j}^a = \frac{1}{\mu_0} \left(\frac{1}{W^{(0)}} \left(A^{b0} \mathbf{B}^a{}_b - \mathbf{A}^b \times \mathbf{E}^a{}_b/c \right) - \omega_{(\Lambda)}{}^{a0}{}_b \mathbf{B}^b + \boldsymbol{\omega}_{(\Lambda)}{}^a{}_b \times \mathbf{E}^b/c \right). \tag{6.143}$$

Removing the indices a, b, as before (see Eq. (6.84)) gives a factor of -2 on the right side:

$$\mathbf{j} = -\frac{2}{\mu_0} \left(\frac{1}{W^{(0)}} \left(A^0 \mathbf{B} - \mathbf{A} \times \mathbf{E}/c \right) - \omega_{(\Lambda)}{}^0 \mathbf{B} + \boldsymbol{\omega}_{(\Lambda)} \times \mathbf{E}/c \right). \tag{6.144}$$

With

$$A^0 = \frac{\phi}{c}, \tag{6.145}$$

this becomes

$$\mathbf{j} = -\frac{2}{\mu_0 c} \left(\left(\frac{1}{W^{(0)}} \phi - c\omega_{(\Lambda)}{}^0 \right) \mathbf{B} - \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega}_{(\Lambda)} \right) \times \mathbf{E} \right). \tag{6.146}$$

The inhomogeneous current (6.124) can be evaluated in the same way. The Hodge-dual field tensors $\widetilde{F}$ have to be replaced with the original tensors F, and the Λ-based spin connection with the Γ-based spin connection. The replacement rules, according to Eqs. (6.74) and (6.75), are

$$\begin{aligned}
\widetilde{F}^a{}_b{}^{10} = -B^a{}_b{}^1 \quad &\rightarrow \quad E^a{}_b{}^1/c = F^a{}_b{}^{10} \\
\widetilde{F}^a{}_b{}^{20} = -B^a{}_b{}^2 \quad &\rightarrow \quad E^a{}_b{}^2/c = F^a{}_b{}^{20} \\
\widetilde{F}^a{}_b{}^{30} = -B^a{}_b{}^3 \quad &\rightarrow \quad E^a{}_b{}^3/c = F^a{}_b{}^{30} \\
\widetilde{F}^a{}_b{}^{01} = B^a{}_b{}^1 \quad &\rightarrow \quad -E^a{}_b{}^1/c = F^a{}_b{}^{01} \\
\widetilde{F}^a{}_b{}^{21} = E^a{}_b{}^3/c \quad &\rightarrow \quad B^a{}_b{}^3 = F^a{}_b{}^{21} \\
\widetilde{F}^a{}_b{}^{31} = -E^a{}_b{}^2/c \quad &\rightarrow \quad -B^a{}_b{}^2 = F^a{}_b{}^{31}
\end{aligned} \tag{6.147}$$

Therefore, we can transform Eq. (6.129) (with additional index raising) to

$$\begin{aligned}
J^{a0} &= \frac{1}{\mu_0 c} \left(\frac{1}{W^{(0)}} \left(-A^b{}_1 E^a{}_b{}^1 - A^b{}_2 E^a{}_b{}^2 - A^b{}_3 E^a{}_b{}^3 \right) + \omega^a{}_{1b} E^{b1} + \omega^a{}_{2b} E^{b2} + \omega^a{}_{3b} E^{b3} \right) \\
&= \frac{1}{\mu_0 c} \left(\frac{1}{W^{(0)}} \left(A^{b1} E^a{}_b{}^1 + A^{b2} E^a{}_b{}^2 + A^{b3} E^a{}_b{}^3 \right) - \omega^{a1}{}_b E^{b1} - \omega^{a2}{}_b E^{b2} - \omega^{a3}{}_b E^{b3} \right),
\end{aligned} \tag{6.148}$$

which leads to the vector form representation

$$J^0 = -\frac{2}{\mu_0 c} \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega} \right) \cdot \mathbf{E}. \tag{6.149}$$

With

$$J^0 = c\rho, \tag{6.150}$$

the electric charge density becomes

$$\rho = -2\varepsilon_0 \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega} \right) \cdot \mathbf{E} \tag{6.151}$$

or, in geometrical terms,

$$\rho = -2\varepsilon_0 \left(\frac{1}{r^{(0)}} \mathbf{q} - \boldsymbol{\omega} \right) \cdot \mathbf{E}. \tag{6.152}$$

Then, the Coulomb law becomes

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} = -2 \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega} \right) \cdot \mathbf{E}. \tag{6.153}$$

For the 1-component of the current vector $\mathbf{J}$, it follows from Eqs. (6.124) and (6.147) that

$$\begin{aligned} J^{a1} = \frac{1}{\mu_0} \Bigg(& \frac{1}{W^{(0)}} \left(-A^{b0} E^a{}_b{}^1 / c - A^{b2} B^a{}_b{}^3 + A^{b3} B^a{}_b{}^2 \right) \\ & + \omega^{a0}{}_b E^{b1} / c + \omega^{a2}{}_b B^{b3} - \omega^{a3}{}_b B^{b2} \Bigg). \end{aligned} \tag{6.154}$$

Again, please notice that raising the index 0 does not give a sign change. By proceeding in the same way for the two other components, we obtain the vector form of the electric current

$$\mathbf{J} = -\frac{2}{\mu_0} \left(\frac{1}{W^{(0)}} \left(-A^0 \mathbf{E}/c - \mathbf{A} \times \mathbf{B} \right) + \omega^0 \mathbf{E}/c + \boldsymbol{\omega} \times \mathbf{B} \right). \tag{6.155}$$

With

$$A^0 = \frac{\phi}{c}, \tag{6.156}$$

this becomes

$$\mathbf{J} = \frac{2}{\mu_0} \left(\left(\frac{1}{c^2 W^{(0)}} \phi - \frac{1}{c} \omega^0 \right) \mathbf{E} + \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega} \right) \times \mathbf{B} \right). \tag{6.157}$$

The factors in front of $\mathbf{E}$ can be interpreted as conductivity terms, as discussed in connection with Eq. (4.54).

The following is the full set of ECE2 field equations with expanded current terms:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 2 \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega}_{(\Lambda)} \right) \cdot \mathbf{B}, \tag{6.158}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = 2 \left(\left(\frac{1}{W^{(0)}} \phi - c\omega_{(\Lambda)}{}^0 \right) \mathbf{B} - \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega}_{(\Lambda)} \right) \times \mathbf{E} \right), \tag{6.159}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = -2 \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega} \right) \cdot \mathbf{E}, \tag{6.160}$$

$$-\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{B} = 2 \left(\left(\frac{1}{c^2 W^{(0)}} \phi - \frac{1}{c} \omega^0 \right) \mathbf{E} + \left(\frac{1}{W^{(0)}} \mathbf{A} - \boldsymbol{\omega} \right) \times \mathbf{B} \right). \tag{6.161}$$

Notice that the Λ-based spin connection is associated with the homogeneous current (this spin connection was introduced in Chapter 4 for the Hodge-dual field equation, see Eq. (4.89)). The formulas for the inhomogeneous current are very similar but contain the "usual" Γ-based spin connection.

A sufficient condition for the magnetic charge density to vanish is Eq. (6.137). The magnetic current density also becomes zero when the additional condition $\frac{1}{W^{(0)}}\phi = c\omega_{(\Lambda)}{}^0$ is true. In free space, the electric charge density vanishes as well, and then we have $\frac{1}{W^{(0)}}\phi = c\omega^0$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_{(\Lambda)}$.

The field equations can be simplified further by introducing wave numbers (in scalar and vector form) defined by

$$\kappa_{(\Lambda)0} = \frac{1}{cW^{(0)}}\phi - \omega_{(\Lambda)}{}^0, \tag{6.162}$$

$$\boldsymbol{\kappa}_{(\Lambda)} = \frac{1}{W^{(0)}}\mathbf{A} - \boldsymbol{\omega}_{(\Lambda)}, \tag{6.163}$$

$$\kappa_0 = \frac{1}{cW^{(0)}}\phi - \omega^0, \tag{6.164}$$

$$\boldsymbol{\kappa} = \frac{1}{W^{(0)}}\mathbf{A} - \boldsymbol{\omega}. \tag{6.165}$$

Then, Eqs. (6.158 - 6.161) can be written as

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 2\boldsymbol{\kappa}_{(\Lambda)}\cdot\mathbf{B}, \tag{6.166}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{E} = 2\left(c\kappa_{(\Lambda)0}\mathbf{B} - \boldsymbol{\kappa}_{(\Lambda)}\times\mathbf{E}\right), \tag{6.167}$$

$$\boldsymbol{\nabla}\cdot\mathbf{E} = -2\boldsymbol{\kappa}\cdot\mathbf{E}, \tag{6.168}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B} = 2\left(\frac{1}{c}\kappa_0\mathbf{E} + \boldsymbol{\kappa}\times\mathbf{B}\right). \tag{6.169}$$

The homogeneous currents vanish, for example, when both Λ-based wave numbers are zero. Another case where they vanish is when $\boldsymbol{\kappa}_{(\Lambda)}$ is parallel to **E**, and **B** is zero.

Instead of using the original spin connections of Cartan geometry in Eqs. (6.162 - 6.165), we can use the W potentials of ECE2 theory (see Eqs. (6.109) and (6.116)), with Latin indices removed:

$$\Phi_W = cW_0 = cW^{(0)}\omega^0, \tag{6.170}$$

$$\mathbf{W} = W^{(0)}\boldsymbol{\omega}. \tag{6.171}$$

Then, Eqs. (6.162 - 6.165) take the form

$$\kappa_{(\Lambda)0} = \frac{1}{cW^{(0)}}\left(\phi - \phi_{(\Lambda)W}\right), \tag{6.172}$$

$$\boldsymbol{\kappa}_{(\Lambda)} = \frac{1}{W^{(0)}}\left(\mathbf{A} - \mathbf{W}_{(\Lambda)}\right), \tag{6.173}$$

$$\kappa_0 = \frac{1}{cW^{(0)}}\left(\phi - \phi_W\right), \tag{6.174}$$

$$\boldsymbol{\kappa} = \frac{1}{W^{(0)}}\left(\mathbf{A} - \mathbf{W}\right). \tag{6.175}$$

Because we have two types of spin connections (with and without Λ), we have two types of W potentials. However, the Λ-based potentials play a role only for the homogeneous currents. They vanish if

$$\phi = \phi_{(\Lambda)W} \quad \text{and} \tag{6.176}$$

$$\mathbf{A} = \mathbf{W}_{(\Lambda)}. \tag{6.177}$$

In the same way, the inhomogeneous currents vanish if

$$\phi = \phi_W \quad \text{and} \tag{6.178}$$
$$\mathbf{A} = \mathbf{W}. \tag{6.179}$$

In this case, we have an electromagnetic field in free space without charges and currents.

■ **Example 6.3** The electromagnetic fields with one or two tangent space indices can be interpreted as geometrical quantities with spin and orbital character [59]. We define the vector parts of torsion and curvature by

$$\mathbf{B}^a = A^{(0)}\mathbf{T}^a(\text{spin}), \tag{6.180}$$
$$\mathbf{E}^a = cA^{(0)}\mathbf{T}^a(\text{orbital}), \tag{6.181}$$
$$\mathbf{B}^a{}_b = W^{(0)}\mathbf{R}^a{}_b(\text{spin}), \tag{6.182}$$
$$\mathbf{E}^a{}_b = cW^{(0)}\mathbf{R}^a{}_b(\text{orbital}). \tag{6.183}$$

The components of geometrical spin and orbital vectors can then be derived by comparing them with the field tensors (4.65) and (6.74):

$$F^{a\mu\nu} = \begin{bmatrix} 0 & -E^{a1}/c & -E^{a2}/c & -E^{a3}/c \\ E^{a1}/c & 0 & -B^{a3} & B^{a2} \\ E^{a2}/c & B^{a3} & 0 & -B^{a1} \\ E^{a3}/c & -B^{a2} & B^{a1} & 0 \end{bmatrix} \tag{6.184}$$
$$= A^{(0)} \begin{bmatrix} 0 & -T^{a1}(\text{spin}) & -T^{a2}(\text{spin}) & -T^{a3}(\text{spin}) \\ T^{a1}(\text{spin}) & 0 & -T^{a3}(\text{orbital}) & T^{a2}(\text{orbital}) \\ T^{a2}(\text{spin}) & T^{a3}(\text{orbital}) & 0 & -T^{a1}(\text{orbital}) \\ T^{a3}(\text{spin}) & -T^{a2}(\text{orbital}) & T^{a1}(\text{orbital}) & 0 \end{bmatrix},$$

$$F^a{}_b{}^{\mu\nu} = \begin{bmatrix} 0 & -E^a{}_b{}^1/c & -E^a{}_b{}^2/c & -E^a{}_b{}^3/c \\ E^a{}_b{}^1/c & 0 & -B^a{}_b{}^3 & B^a{}_b{}^2 \\ E^a{}_b{}^2/c & B^a{}_b{}^3 & 0 & -B^a{}_b{}^1 \\ E^a{}_b{}^3/c & -B^a{}_b{}^2 & B^a{}_b{}^1 & 0 \end{bmatrix} \tag{6.185}$$
$$= W^{(0)} \begin{bmatrix} 0 & -R^a{}_b{}^1(\text{spin}) & -R^a{}_b{}^2(\text{spin}) & -R^a{}_b{}^3(\text{spin}) \\ R^a{}_b{}^1(\text{spin}) & 0 & -R^a{}_b{}^3(\text{orbital}) & R^a{}_b{}^2(\text{orbital}) \\ R^a{}_b{}^2(\text{spin}) & R^a{}_b{}^3(\text{orbital}) & 0 & -R^a{}_b{}^1(\text{orbital}) \\ R^a{}_b{}^3(\text{spin}) & -R^a{}_b{}^2(\text{orbital}) & R^a{}_b{}^1(\text{orbital}) & 0 \end{bmatrix}.$$

The field equations can be formulated in terms of these geometrical representations of the field tensors. Details can be found in [59]. ■

Consequences for the ECE potentials

In Section 4.4, the ECE potentials were introduced by applying the first Maurer-Cartan structure equation. This led to Eq. (4.172):

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + \omega^a{}_{\mu b} A^b{}_\nu - \omega^a{}_{\nu b} A^b{}_\mu. \tag{6.186}$$

The last two terms are sums over the tangent space index b. This summation is retained in the vector representations of the electric and magnetic fields, Eqs. (4.197 - 4.198). When no polarizations are present, the Latin indices have been omitted, leading to the simplified field-potential relations (4.211 - 4.212):

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} - c\omega_0 \mathbf{A} + \boldsymbol{\omega}\phi, \tag{6.187}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} - \boldsymbol{\omega} \times \mathbf{A}. \tag{6.188}$$

Alternatively, we can apply a summation process over the b index, as was done earlier in this section. Then, a factor of -2 is obtained:

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + 2\left(c\omega_0 \mathbf{A} - \boldsymbol{\omega}\phi\right), \tag{6.189}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} + 2\boldsymbol{\omega} \times \mathbf{A}. \tag{6.190}$$

By equating this with the ECE2 potentials (6.121 - 6.122), we obtain

$$\mathbf{E} = -\boldsymbol{\nabla}\phi_W - \frac{\partial \mathbf{W}}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + 2\left(c\omega_0 \mathbf{A} - \boldsymbol{\omega}\phi\right), \tag{6.191}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{W} = \boldsymbol{\nabla} \times \mathbf{A} + 2\boldsymbol{\omega} \times \mathbf{A}, \tag{6.192}$$

which define the relationship between the ECE potentials $\phi, \mathbf{A}$ and the ECE2 potentials $\phi_W, \mathbf{W}$. For further consistency, the spin connection occurrences in the ECE part can be replaced with the ECE2 potentials. By using

$$\phi_W = W^{(0)} c\omega_0, \tag{6.193}$$

$$\mathbf{W} = W^{(0)} \boldsymbol{\omega}, \tag{6.194}$$

we obtain

$$\mathbf{E} = -\boldsymbol{\nabla}\phi_W - \frac{\partial \mathbf{W}}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + \frac{2}{W^{(0)}}\left(\phi_W \mathbf{A} - \mathbf{W}\phi\right), \tag{6.195}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{W} = \boldsymbol{\nabla} \times \mathbf{A} + \frac{2}{W^{(0)}} \mathbf{W} \times \mathbf{A}. \tag{6.196}$$

Under the free space conditions (6.178, 6.179), we see that the right sides of these equations reduce to the ECE2 equations, which are formally equal to the Maxwell-Heaviside case:

$$\mathbf{E} = -\boldsymbol{\nabla}\phi_W - \frac{\partial \mathbf{W}}{\partial t}, \tag{6.197}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{W}. \tag{6.198}$$

From Eq. (6.196), it follows for the Gauss law without magnetic monopoles that

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \mathbf{A} + \frac{2}{W^{(0)}} \mathbf{W} \times \mathbf{A}\right) = 0, \tag{6.199}$$

and from the above equation:

$$\boldsymbol{\nabla} \cdot (\mathbf{W} \times \mathbf{A}) = 0. \tag{6.200}$$

In geometrical quantities and with indices re-inserted, we ultimately get:

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\omega}^a{}_b \times \mathbf{q}^b\right) = 0. \tag{6.201}$$

This is an additional condition for the vanishing of magnetic monopoles.

Myron Evans writes in Note 9 of Paper 317 [58]:
So everything that is known about electrodynamics can be derived from the Cartan and Cartan-Evans identities with basic axioms... . The fundamental philosophical difference is that ECE2 is a generally covariant unified field theory, whereas Maxwell-Heaviside is special relativity. ECE2 has more information than Maxwell-Heaviside, given by the relation between field and potential. ... The spin connection is the key difference between ECE2 and Maxwell-Heaviside.

■ **Example 6.4** We will use the Coulomb potential as an example to illustrate how to determine the wave vector $\boldsymbol{\kappa}$ and selected spin connections. At radial distance r from a point charge q, the electric Coulomb field has only the radial component

$$E_r = \frac{q}{4\pi\varepsilon_0 r^2} \,. \tag{6.202}$$

The Coulomb law (6.168) then reads:

$$\frac{\partial E_r}{\partial r} = -2\kappa_r E_r, \tag{6.203}$$

where κ_r is the radial component of the wave vector $\boldsymbol{\kappa}$. Evaluating this equation gives

$$-2\frac{q}{4\pi\varepsilon_0 r^3} = -2\kappa_r \frac{q}{4\pi\varepsilon_0 r^2} \,, \tag{6.204}$$

from which we then get

$$\kappa_r = \frac{1}{r}. \tag{6.205}$$

The full Cartan geometry of the Coulomb potential was developed in Example 4.1, where the radial Λ-based spin connection (the 1-component) for Latin indices $a = b = 0$ was shown in Eq. (4.105):

$$\omega_{(\Lambda)}{}^{(0)}{}_{1(0)} = \frac{1}{r}, \tag{6.206}$$

and is identical to κ_r. By applying the condition of vanishing homogeneous currents, from (6.158) we obtain

$$\frac{1}{W^{(0)}}\mathbf{A} - \boldsymbol{\omega}_{(\Lambda)} = \mathbf{0}. \tag{6.207}$$

Then, with $A_r/W^{(0)} = q_r/r^{(0)}$, we get

$$\frac{q_r}{r^{(0)}} = \omega_{(\Lambda)r}\,, \tag{6.208}$$

which then gives us

$$q_r = \frac{r^{(0)}}{r} \tag{6.209}$$

for the radial tetrad element. For the Coulomb field, we have no magnetic field and no time dependence of the electric field; therefore, from the Ampère-Maxwell law (6.169) we see that

$$\kappa_0 = 0, \tag{6.210}$$

which implies that $\phi = \phi_W$, from (6.174). Consequently,

$$\phi = \frac{q}{4\pi\varepsilon_0 r} = \phi_W = cW^{(0)}\omega^0 \tag{6.211}$$

and

$$\omega^0 = \frac{q}{4\pi\varepsilon_0 c W^{(0)} r} \,, \tag{6.212}$$

which has no direct counterpart in Example 4.1. ■

6.2 Beltrami solutions in electrodynamics

Toward the end of the nineteenth century, the Italian mathematician Eugenio Beltrami developed a system of equations for the description of hydrodynamic flow, in which the curl of a vector is proportional to the vector itself [61]. An example is the use of the velocity vector. For a long time, this solution was not used outside of the field of hydrodynamics, but in the 1950s it started to be used by researchers such as Alfven and Chandrasekhar in the area of cosmology, notably whirlpool galaxies. The Beltrami field, as it came to be known, has been observed in plasma vortices and, as argued by Reed [62, 63], is indicative of the type of electrodynamics that is characterized by ECE theory. Both ECE theory and Beltrami theory are based on geometry and are ubiquitous throughout nature at all scales. This section will discuss the ways in which ECE electrodynamics reduces to Beltrami electrodynamics (which can itself be viewed as a sub-theory of ECE theory, see [60] and Chapter 3 of [4]).

In ordinary electrodynamics, it is assumed that electromagnetic waves are transverse, which means that the electric field vector is always perpendicular to the magnetic field vector, and the curl of the fields is perpendicular to the fields themselves:

$$\boldsymbol{\nabla}\times\mathbf{E}\perp\mathbf{E}, \tag{6.213}$$
$$\boldsymbol{\nabla}\times\mathbf{B}\perp\mathbf{B}. \tag{6.214}$$

Deviations from these properties are commonly accepted only for fields in materials, in particular where material properties are anisotropic, for example, where the permeability and permittivity are tensors with directional dependence. However, we will see that the properties of transverse waves do not hold in general, and that there are large classes of solutions of the field equations where the opposite is true. These are the Beltrami solutions.

6.2.1 Beltrami solutions of the field equations

It has been known for more than a hundred years that the curl of a vector field and the field itself need not be perpendicular to one another. In particular, they can be parallel. Such solutions of the Maxwell-Heaviside equations are called *Beltrami solutions* and have the properties

$$\boldsymbol{\nabla}\times\mathbf{E} = \kappa\,\mathbf{E}, \tag{6.215}$$
$$\boldsymbol{\nabla}\times\mathbf{B} = \kappa\,\mathbf{B}. \tag{6.216}$$

The curl of the electric field and of the magnetic field is proportional to the respective field itself. κ is a scalar factor that has the dimension of inverse meters and is a wave number in principle[1].

We will consider the Faraday and Ampère-Maxwell laws from the set of field equations (6.97-6.100), without polarization indices or homogeneous currents:

$$\frac{\partial\mathbf{B}}{\partial t}+\boldsymbol{\nabla}\times\mathbf{E}=0, \tag{6.217}$$
$$-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}+\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}. \tag{6.218}$$

For magnetostatics or a slowly varying electric field, the second equation becomes

$$\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}. \tag{6.219}$$

With the Beltrami condition for **B**, this reduces to

$$\mathbf{B}=\frac{\mu_0}{\kappa}\mathbf{J}. \tag{6.220}$$

[1] According to Beltrami theory, κ is allowed to be a non-constant function, but we restrict consideration to κ=const.

Applying the curl operator gives

$$\nabla \times \mathbf{B} = \frac{\mu_0}{\kappa} \nabla \times \mathbf{J}. \tag{6.221}$$

After again applying the Beltrami condition for **B**, this becomes

$$\mathbf{B} = \frac{\mu_0}{\kappa^2} \nabla \times \mathbf{J}. \tag{6.222}$$

Comparing Eqs. (6.220) and (6.222) shows that

$$\nabla \times \mathbf{J} = \kappa \mathbf{J}, \tag{6.223}$$

i.e., the current **J** also obeys a Beltrami condition.

Applying a reverse Beltrami condition to the Gauss law,

$$\nabla \cdot \mathbf{B} = 0, \tag{6.224}$$

gives

$$\frac{1}{\kappa} \nabla \cdot (\nabla \times \mathbf{B}) = 0, \tag{6.225}$$

which is always satisfied for a vanishing magnetic charge density, since the divergence of the curl of a vector field must always be zero:

$$\nabla \cdot (\nabla \times \mathbf{B}) = 0. \tag{6.226}$$

However, from the Coulomb law,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{6.227}$$

it follows that

$$\frac{1}{\kappa} \nabla \cdot (\nabla \times \mathbf{E}) = \frac{\rho}{\varepsilon_0}. \tag{6.228}$$

The above vector condition is satisfied only for $\rho = 0$. Therefore, the electric field can be a Beltrami field only in free space.

We will now show that the vector potential and vector spin connection in free space are also Beltrami fields. In this case, we have

$$\mathbf{A} = \mathbf{W} = W^{(0)} \boldsymbol{\omega}. \tag{6.229}$$

Therefore, $\boldsymbol{\omega}$ is parallel to **A**, and the cross product $\boldsymbol{\omega} \times \mathbf{A}$ vanishes. Then, the magnetic field (6.188) is simply

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{6.230}$$

Applying the Beltrami condition to **B** gives

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \kappa \mathbf{B} = \kappa \nabla \times \mathbf{A} = \nabla \times (\kappa \mathbf{A}). \tag{6.231}$$

Comparing the second and last terms of the chain shows that

$$\nabla \times \mathbf{A} = \kappa \mathbf{A}, \tag{6.232}$$

i.e., the vector potential is a Beltrami field. From (6.229) it follows that the vector spin connection is also a Beltrami field:

$$\boldsymbol{\nabla} \times \boldsymbol{\omega} = \kappa \boldsymbol{\omega}. \tag{6.233}$$

This means that spacetime itself has a Beltrami structure. This will be relevant to what is called "aether flow" or "fluid spacetime", as we will see in later chapters. Another consequence is that the magnetic field is parallel to the vector potential:

$$\mathbf{B} = \kappa \mathbf{A}. \tag{6.234}$$

From the definition of Beltrami fields, it follows directly that these fields are divergenceless in free space:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \boldsymbol{\nabla} \cdot \mathbf{B} = \boldsymbol{\nabla} \cdot \mathbf{A} = \boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0. \tag{6.235}$$

In addition, these fields (except for the electric field, but including the current in magnetostatics), are all parallel:

$$\mathbf{B} \parallel \mathbf{A} \parallel \boldsymbol{\omega} \parallel \mathbf{J}. \tag{6.236}$$

In the case of longitudinal fields (see Example 6.7, below), the electric field is also parallel to the fields mentioned above. This follows from the Faraday law:

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \kappa \mathbf{E} = \mathbf{0}. \tag{6.237}$$

If $\mathbf{B}$ is longitudinal, then $\partial \mathbf{B}/\partial t \parallel \mathbf{B}$, and it further follows that

$$\mathbf{E} \parallel \mathbf{B}. \tag{6.238}$$

Another consequence of Beltrami fields is the possibility of spin connection resonance. For a fixed current $\mathbf{J}$ and vanishing scalar potentials, i.e.:

$$\phi = 0, \quad \phi_W = 0, \tag{6.239}$$

Eq. (6.197) becomes

$$\mathbf{E} = -\frac{\partial}{\partial t} \mathbf{W}. \tag{6.240}$$

Assuming a Beltrami vector potential with $\boldsymbol{\nabla} \times \mathbf{W} = \kappa \mathbf{W}$, the Ampère-Maxwell law reads:

$$-\frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E} + \boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{W} + \kappa^2 \mathbf{W} = \mu_0 \mathbf{J}. \tag{6.241}$$

If the current is of the form $\mathbf{J} = \mathbf{J}_0 \cos(\omega t)$, with a time frequency ω, this becomes a three-component equation of Euler-Bernoulli resonances:

$$\frac{\partial^2}{\partial t^2} \mathbf{W} + \omega_0^2\, \mathbf{W} = \frac{1}{\varepsilon_0} \mathbf{J} \tag{6.242}$$

with resonance frequency

$$\omega_0 = c\kappa. \tag{6.243}$$

This shows that resonance solutions of Beltrami fields are possible in electrodynamics.

6.2.2 Continuity equation

The time derivative of the Coulomb law (6.227) is

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{E}=\frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t}. \tag{6.244}$$

Applying the divergence operator to the Ampère-Maxwell law (6.100) gives

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{E}+\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{B}=\mu_0\boldsymbol{\nabla}\cdot\mathbf{J}. \tag{6.245}$$

By rewriting the latter equation (while observing that the divergence of a curl vanishes), we get

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{E}=-c^2\mu_0\boldsymbol{\nabla}\cdot\mathbf{J}. \tag{6.246}$$

Equating (6.244) with (6.246) gives

$$\frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t}=-c^2\mu_0\boldsymbol{\nabla}\cdot\mathbf{J}, \tag{6.247}$$

and ultimately leads to

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{J}=0, \tag{6.248}$$

which is the continuity equation. This holds for ECE and ECE2 theory in general, as well as for special cases, such as the Beltrami fields.

6.2.3 Helmholtz equation

We will now show that the Helmholtz equation can be derived directly from a Beltrami condition. By applying the curl operator twice to **E** and **B**, we obtain

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E}=\kappa\boldsymbol{\nabla}\times\mathbf{E}=\kappa^2\mathbf{E}, \tag{6.249}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{B}=\kappa\boldsymbol{\nabla}\times\mathbf{B}=\kappa^2\mathbf{B}, \tag{6.250}$$

which are also called *Trkalian equations*. By applying a vector-analysis theorem, the left sides can be rewritten as

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E})-\nabla^2\mathbf{E}=\kappa^2\mathbf{E}, \tag{6.251}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{B}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{B})-\nabla^2\mathbf{B}=\kappa^2\mathbf{B}. \tag{6.252}$$

In free space, all fields are divergence-free, and the equations become

$$\nabla^2\mathbf{E}+\kappa^2\mathbf{E}=0, \tag{6.253}$$

$$\nabla^2\mathbf{B}+\kappa^2\mathbf{B}=0, \tag{6.254}$$

which are the *Helmholtz equations*. These are wave equations with oscillating solutions, whose time dependence is harmonic, i.e., it is a multiplicative factor. For example, plane waves

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0\,e^{i(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)}, \tag{6.255}$$

$$\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0\,e^{i(\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t)}, \tag{6.256}$$

with a wave vector

$$\kappa = \begin{bmatrix} \kappa_X \\ \kappa_Y \\ \kappa_Z \end{bmatrix}, \tag{6.257}$$

and constant amplitudes $\mathbf{E}_0, \mathbf{B}_0$, satisfy the Helmholtz equations. Differentiating twice gives

$$\nabla^2 \mathbf{E} = -\left(\kappa_X^2 + \kappa_Y^2 + \kappa_Z^2\right) \mathbf{E} \tag{6.258}$$

or

$$\nabla^2 \mathbf{E} = -\kappa^2 \mathbf{E}, \tag{6.259}$$

and an analogous procedure gives

$$\nabla^2 \mathbf{B} = -\kappa^2 \mathbf{B}. \tag{6.260}$$

Inserting these expressions into Eqs. (6.253, 6.254) shows that the Helmholtz equations are satisfied by these solutions. The same procedure can be applied to the vector potential, which gives us the Helmholtz equation:

$$\nabla^2 \mathbf{A} + \kappa^2 \mathbf{A} = \mathbf{0}. \tag{6.261}$$

6.2.4 Wave equation

The wave or d'Alembert equation for the magnetic field is obtained in the following way. In free space, the curl of the Ampère-Maxwell law is

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}. \tag{6.262}$$

Replacing the curl of $\mathbf{E}$ by means of the Faraday law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}, \tag{6.263}$$

gives

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{B} = \mathbf{0}. \tag{6.264}$$

Replacing the double curl by using a vector analysis theorem as before, leads to

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{B} - \nabla^2 \mathbf{B} = \mathbf{0} \tag{6.265}$$

or, when written with the d'Alembert operator,

$$\square\, \mathbf{B} = \mathbf{0}. \tag{6.266}$$

An analogous derivation can be performed to obtain the wave equation for the electric field:

$$\square\, \mathbf{E} = \mathbf{0}. \tag{6.267}$$

The wave equation for the vector potential has already been derived in Section 4.3, which for the standard electromagnetic theory reads:

$$\square\, \mathbf{A} = \mathbf{0}. \tag{6.268}$$

We can apply the Trkalian equation with the double curl (6.250) to Eq. (6.264), to obtain an alternative form of the wave equation:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}+\kappa^2\mathbf{B}=\mathbf{0}. \tag{6.269}$$

This form of the wave equation is specific to Beltrami fields. The Helmholtz equation is for the space part of the fields, while the wave equation includes the time development. In the same way, we can obtain the Beltrami wave equation of the electric field:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}+\kappa^2\mathbf{E}=\mathbf{0}. \tag{6.270}$$

An equivalent equation for the vector potential is obtained by replacing the magnetic field in Eq. (6.269) with

$$\mathbf{B}=\kappa\mathbf{A}, \tag{6.271}$$

which gives

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}+\kappa^2\mathbf{A}=\mathbf{0}. \tag{6.272}$$

Gauge invariance and additional properties

Gauge invariance is invalidated by Beltrami fields. The vector potential of Maxwell-Heaviside theory has the property that the field equations are unaltered when the vector potential is augmented with the gradient of a scalar function $\psi(\mathbf{r})$:

$$\mathbf{A}\rightarrow\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\psi, \tag{6.273}$$

for example,

$$\boldsymbol{\nabla}\times\mathbf{A}\rightarrow\boldsymbol{\nabla}\times(\mathbf{A}+\boldsymbol{\nabla}\psi)=\boldsymbol{\nabla}\times\mathbf{A}, \tag{6.274}$$

because the curl of a gradient field vanishes. However, for Beltrami fields, according to Eq. (6.271):

$$\mathbf{B}=\kappa\mathbf{A}\rightarrow\kappa(\mathbf{A}+\boldsymbol{\nabla}\psi). \tag{6.275}$$

This means that the magnetic field depends on changes in the vector potential. This destroys the gauge invariance, which is also called U(1) symmetry. Gauge invariance is also broken by the Proca equation (4.167):

$$\left(\square+\left(\frac{m_0c}{\hbar}\right)^2\right)\mathbf{A}=\mathbf{0}. \tag{6.276}$$

Gauge invariance is broken because a gauge operation on the vector potential gives a different solution due to the presence of the constant term in parentheses.

By adding the Proca equation (6.276) and the Helmholtz equation (6.261) we get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\kappa_0^2+\kappa^2\right)\mathbf{A}=\mathbf{0} \tag{6.277}$$

with

$$\kappa_0=\frac{m_0c}{\hbar}, \tag{6.278}$$

where m_0 is a particle mass. In particular, this can be identified with the photon rest mass. A solution of Eq. (6.277) is

$$\mathbf{A} = \mathbf{A_0} e^{i(\omega t - \boldsymbol{\kappa}\cdot\mathbf{x})} \tag{6.279}$$

with time frequency ω, wave vector $\boldsymbol{\kappa}$, and space coordinate vector $\mathbf{x}$. By inserting this solution into Eq. (6.277) we get

$$\omega^2 = c^2(\kappa_0^2 + \kappa^2). \tag{6.280}$$

The energy of a particle with rest mass m_0 and momentum $\hbar\kappa$ is

$$E = \hbar\omega = \sqrt{m_0^2 c^4 + c^2\hbar^2\kappa^2}. \tag{6.281}$$

ω is the de Broglie frequency or, in the case of a photon, the frequency of its electromagnetic oscillation. In this way, the ECE wave equation for Beltrami fields is connected with the quantum-mechanical realm. In particular, the photon has a rest mass and the U(1) symmetry of electrodynamics does not exist.

The inhomogeneous d'Alembert equation of classical physics can be derived in the following way. Inserting the potentials (6.197, 6.198) into Eq. (6.241) gives

$$\kappa^2\mathbf{W} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi_W - \frac{\partial}{\partial t}\mathbf{W}\right), \tag{6.282}$$

which can be rewritten as

$$\kappa^2\mathbf{W} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{W} + \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial\phi_W}{\partial t} = \mu_0\mathbf{J}. \tag{6.283}$$

In space regions outside of the current distribution, we can apply $\phi_W = \phi$ and $\mathbf{W} = \mathbf{A}$, and use the Helmholtz equation (6.261) to get

$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} + \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial\phi}{\partial t} = \mu_0\mathbf{J}. \tag{6.284}$$

The Lorenz condition of ECE theory is

$$\partial_\mu A^{a\mu} = 0, \tag{6.285}$$

and its space part (without polarization index a) reads:

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{6.286}$$

Since $\mathbf{A}$ is a Beltrami field, $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, i.e., the scalar potential is static. Inserting this condition into (6.284) gives

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} = \mu_0\mathbf{J}, \tag{6.287}$$

which is the d'Alembert or classical wave equation with a source term:

$$\Box\,\mathbf{A} = \mu_0\mathbf{J}. \tag{6.288}$$

This equation is connected with the Proca equation [64] in the following way. If we identify the inhomogeneous term in the Proca equation (6.276) with the current density:

$$\mathbf{J} = -\frac{1}{\mu_0}\left(\frac{m_0 c}{\hbar}\right)^2 \mathbf{A}, \tag{6.289}$$

we immediately obtain the classical wave equation (6.288) with a source term on the right side. This source term is proportional to the vector potential itself.

We can proceed in a similar way with the 0-component of the Proca equation (4.167):

$$\left(\Box + \left(\frac{m_0 c}{\hbar}\right)^2\right) A^{a0} = 0. \tag{6.290}$$

Analogously to (6.289), we define

$$J^{a0} = -\frac{1}{\mu_0}\left(\frac{m_0 c}{\hbar}\right)^2 A^{a0}. \tag{6.291}$$

Without the a index, this can be written as

$$J^0 = -\frac{1}{\mu_0}\left(\frac{m_0 c}{\hbar}\right)^2 A^0 \tag{6.292}$$

or, with

$$A^0 = \frac{\rho}{c}, \quad J^0 = c\rho, \tag{6.293}$$

it becomes

$$\rho = -\frac{1}{\mu_0 c^2}\left(\frac{m_0 c}{\hbar}\right)^2 \phi = -\varepsilon_0\left(\frac{m_0 c}{\hbar}\right)^2 \phi. \tag{6.294}$$

The scalar part (6.290) of the Proca equation then takes the form:

$$\Box\, \phi = \frac{1}{\varepsilon_0}\rho. \tag{6.295}$$

This is an inhomogeneous wave equation for the electric potential.

In the discussion above, charge densities and currents were defined via potentials, and not via conventional charged masses. Therefore, they can be considered to be vacuum structures, conglomerating into matter with a rest mass m_0. The vacuum potentials themselves are sources of charges and currents. In a philosophical sense, matter may be considered to consist of vacuum or spacetime structures. A photon with mass fits well into this approach, which is fundamentally different from that of quantum electrodynamics (special relativity only).

6.2.5 Interpretation of Beltrami fields, with examples

Now that we have described the theoretical aspects of Beltrami fields in the previous sections, we will clarify their essential properties using practical examples.

First, we consider Beltrami solutions in vacuo with harmonic time dependence. The Faraday and Ampère-Maxwell laws in vacuo are

$$\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}, \tag{6.296}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{0}. \tag{6.297}$$

We use the following two approaches for electric and magnetic fields:

1) Time dependence:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{i\omega t}, \tag{6.298}$$
$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r})e^{i\omega t}, \tag{6.299}$$

with a time frequency ω.

2) Beltrami field assumptions for the space parts:

$$\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}) = \kappa\, \mathbf{E}(\mathbf{r}), \tag{6.300}$$
$$\boldsymbol{\nabla} \times \mathbf{B}(\mathbf{r}) = \kappa\, \mathbf{B}(\mathbf{r}), \tag{6.301}$$

with κ being a constant wave number.

Applying both approaches to (6.296 - 6.297) gives

$$\kappa\mathbf{E} + i\omega\mathbf{B} = \mathbf{0}, \tag{6.302}$$
$$\kappa\mathbf{B} - i\frac{\omega}{c^2}\mathbf{E} = \mathbf{0}. \tag{6.303}$$

From Eq. (6.302) it follows that

$$\mathbf{B} = i\frac{\kappa}{\omega}\mathbf{E}, \tag{6.304}$$

and inserting this into (6.303) gives

$$i\frac{\kappa^2}{\omega}\mathbf{E} - i\frac{\omega}{c^2}\mathbf{E} = \mathbf{0} \tag{6.305}$$

or

$$i\left(\frac{\kappa^2}{\omega} - \frac{\omega}{c^2}\right)\mathbf{E} = \mathbf{0}. \tag{6.306}$$

Since $\mathbf{E}$ is not zero in general, the factor in parentheses must vanish:

$$\frac{\kappa^2}{\omega} - \frac{\omega}{c^2} = 0. \tag{6.307}$$

This gives

$$\kappa^2 = \frac{\omega^2}{c^2} \tag{6.308}$$

or

$$\kappa = \frac{\omega}{c}, \tag{6.309}$$

which is the usual definition of the wave number belonging to the time frequency ω.

The time-dependence approach (6.298, 6.299) means that $\mathbf{E}$ and $\mathbf{B}$ describe standing waves. For example, a wave of type

$$E_0 \cos(\kappa X)\cos(\omega t) \tag{6.310}$$

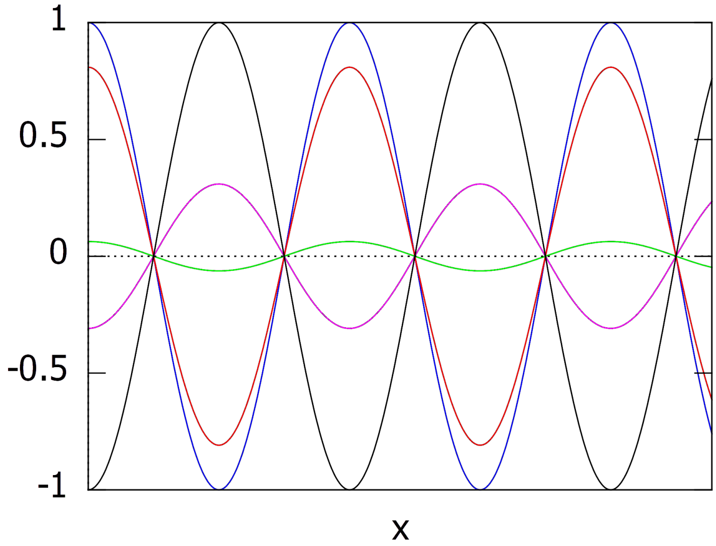


Figure 6.2: Example of a standing wave: $A = \cos(X)\sin(t)$ for different t values.

is a standing wave in the X direction, modulated by time. Thus, the space part of Beltrami fields represents standing waves, if the time dependence can be separated, as in the above equation. An example is graphed in Fig. 6.2.

It has even been shown, by the potential-based approaches, (6.289) and (6.294), that the ECE potential is a Beltrami field in general [60].

Next, we will consider some concrete examples.

■ **Example 6.5** In Example 4.2, circularly polarized plane waves, which are fundamental to the B(3) field and O(3) electrodynamics, were discussed. The three polarization vectors of the vector potential, $\mathbf{A}^{(i)}$, $i = 1, 2, 3$, represent a rotating circular basis. It was shown that these vectors obey Beltrami conditions (Eqs. (4.156 - 4.158)):

$$\boldsymbol{\nabla} \times \mathbf{A}^{(1)} = \kappa\, \mathbf{A}^{(1)}, \tag{6.311}$$

$$\boldsymbol{\nabla} \times \mathbf{A}^{(2)} = \kappa\, \mathbf{A}^{(2)}, \tag{6.312}$$

$$\boldsymbol{\nabla} \times \mathbf{A}^{(3)} = 0 \cdot \mathbf{A}^{(3)}. \tag{6.313}$$

The third equation has the wave number $\kappa = 0$, and represents a special case, because the magnetic field in the Z direction, derived from these polarization vectors, is constant. The non-vanishing κ values describe the space oscillation of the waves (see Eqs. (4.150 - 4.151) for definitions). This shows that O(3) electrodynamics is essentially a Beltrami theory. ■

■ **Example 6.6** Reed [62, 63] has shown that the most general Beltrami field $\mathbf{v}$ can be described by

$$\mathbf{v} = \kappa\, \boldsymbol{\nabla} \times (\psi \mathbf{a}) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\psi \mathbf{a}), \tag{6.314}$$

where ψ is an arbitrary function, κ is a constant and $\mathbf{a}$ is a constant vector. We will illustrate this by presenting two cases. First, we set

$$\psi = \frac{1}{L^3} XYZ \tag{6.315}$$

with

$$\mathbf{a} = [0, 0, 1]. \tag{6.316}$$

The field resulting from Eq. (6.314) is graphed in Fig. 6.3. The field has only XY components and describes a hyperbolic vortex. Nevertheless, the divergence is zero. For details, see computer algebra code [140].

For the second case, we set

$$\psi = \sin(\kappa X)\,\sin(\kappa Y)\,\cos(\kappa Z). \tag{6.317}$$

This field has a more complicated structure, which can be seen in Fig. 6.4. The projection of several Z levels on the XY plane is shown in Fig. 6.5. We see that the vectors rotate around the Z axis. Such rotational structures in all three dimensions are typical for Beltrami fields. ■

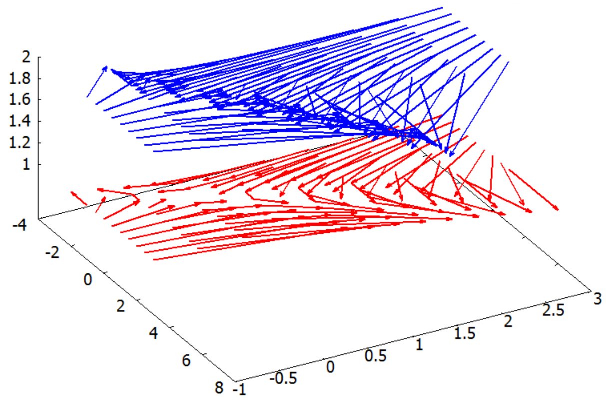


Figure 6.3: General Beltrami field of Eqs. (6.314 - 6.316).

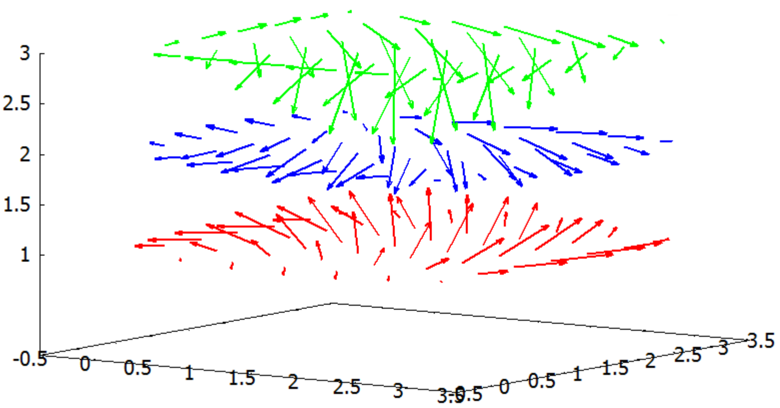


Figure 6.4: General Beltrami field of Eqs. (6.314, 6.316 - 6.317).

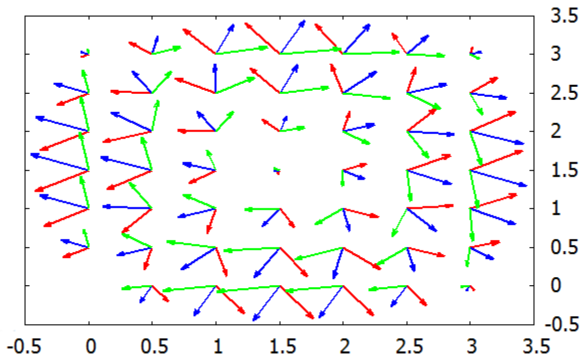


Figure 6.5: General Beltrami field of Eqs. (6.314, 6.316 - 6.317) – projected on the XY plane.

■ **Example 6.7** Marsh [65] defines a general Beltrami field with cylindrical geometry by

$$\mathbf{B} = \begin{bmatrix} 0 \\ B_\theta(r) \\ B_Z(r) \end{bmatrix} \tag{6.318}$$

with cylindrical coordinates r, θ, Z. There is only an r dependence of the field components. For this to be a Beltrami field, the Beltrami condition in cylindrical coordinates

$$\nabla\times\mathbf{B} = \begin{bmatrix} \frac{1}{r}\frac{\partial B_Z}{\partial\theta} - \frac{\partial B_\theta}{\partial Z} \\ \frac{\partial B_r}{\partial Z} - \frac{\partial B_Z}{\partial r} \\ \frac{1}{r}\left(\frac{\partial(rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial\theta}\right) \end{bmatrix} = \kappa\,\mathbf{B} \tag{6.319}$$

must hold. The divergence in cylindrical coordinates is

$$\nabla\cdot\mathbf{B} = \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{\partial B_Z}{\partial Z}. \tag{6.320}$$

When we insert (6.318) into (6.320), we see that this field is divergence-free, which is a prerequisite to being a Beltrami field. Eq. (6.319) simplifies to

$$\nabla\times\mathbf{B} = \begin{bmatrix} 0 \\ -\frac{\partial B_Z}{\partial r} \\ \frac{\partial B_\theta}{\partial r} + \frac{1}{r}B_\theta. \end{bmatrix} = \kappa\begin{bmatrix} 0 \\ B_\theta \\ B_Z \end{bmatrix}. \tag{6.321}$$

We now consider the case of a constant κ. From the second component of Eq. (6.321) we get

$$-\frac{\partial}{\partial r}B_Z = \kappa B_\theta, \tag{6.322}$$

and from the third component we get

$$r\frac{\partial}{\partial r}B_\theta + B_\theta = \kappa r B_Z. \tag{6.323}$$

Integrating Eq. (6.322) and inserting the result for B_Z into (6.323) gives

$$\frac{\partial}{\partial r}B_\theta + \frac{B_\theta}{r} = -\kappa^2\int B_\theta\,dr, \tag{6.324}$$

and differentiating this equation leads to the second order differential equation

$$r^2 \frac{\partial^2}{\partial r^2} B_\theta + r \frac{\partial}{\partial r} B_\theta + \kappa^2 r^2 B_\theta - B_\theta = 0. \tag{6.325}$$

Finally, we change the variable r to κr, which leads to Bessel's differential equation

$$r^2 \frac{d^2}{d r^2} B_\theta (\kappa r) + r \frac{d}{d r} B_\theta (\kappa r) + \left(\kappa^2 r^2 - 1\right) B_\theta (\kappa r) = 0. \tag{6.326}$$

The solution is the Bessel function

$$B_\theta(r) = B_0 J_1(\kappa r) \tag{6.327}$$

with a constant B_0, and from (6.322) it follows that

$$B_Z(r) = B_0 J_0(\kappa r) \tag{6.328}$$

(see computer algebra code [141]). This is the known solution of Reed/Marsh, scaled by the wave number κ, and with longitudinal components. It is graphed in Fig. 6.6. It can be seen that the vector field changes from transverse to longitudinal when approaching the Z axis. However, there are always longitudinal components at certain distances, as well. This is evident in Fig. 6.7, where the Beltrami field of Bessel functions has been decomposed into transverse and longitudinal vectors and plotted separately. According to the oscillating nature of the Bessel functions (with zero crossings), the field changes periodically from transverse to longitudinal, with increasing radius.

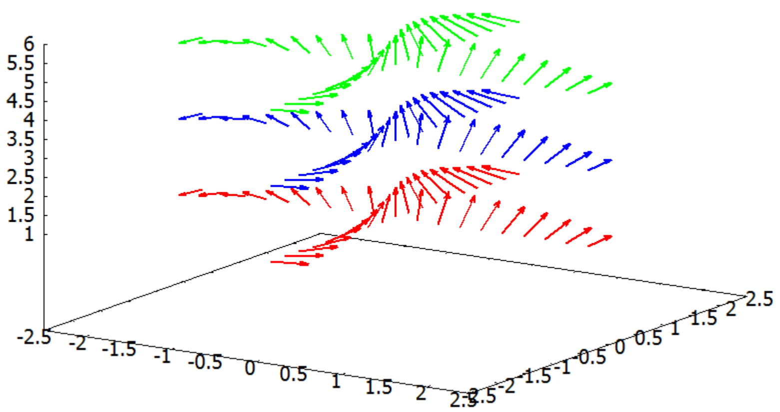


Figure 6.6: Beltrami field of Bessel functions.

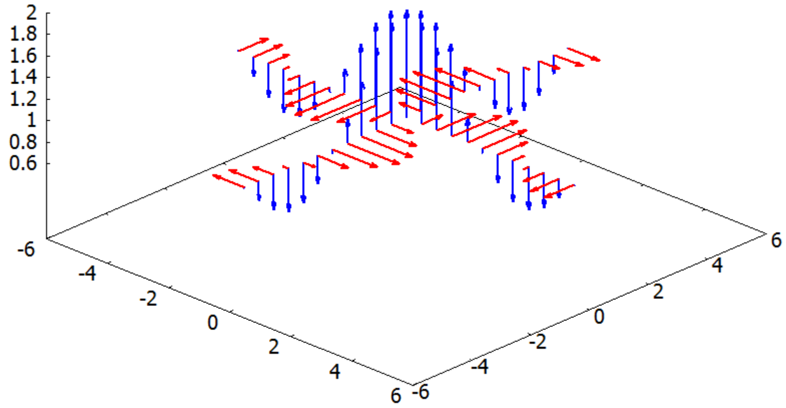


Figure 6.7: Beltrami field of Bessel functions, decomposed into transverse (red) and longitudinal (blue) vectors.

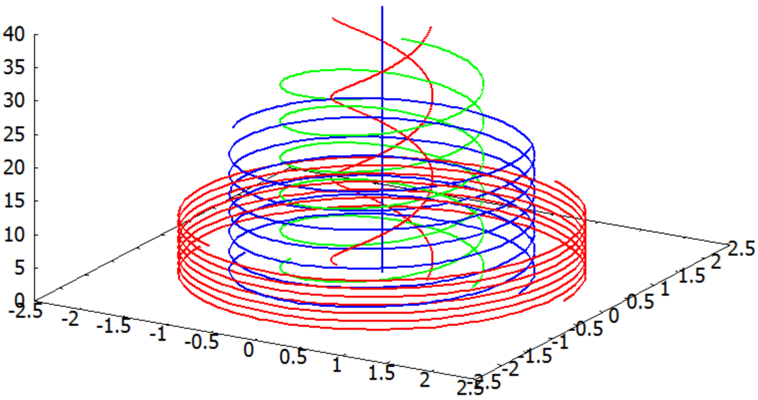


Figure 6.8: Beltrami field of Bessel functions, streamlines.

The flow of test particles in a field (if it is assumed to be hydrodynamic) can be best seen in a picture with streamlines (Fig. 6.8). Streamlines show how a test particle moves in the vector field, which is considered to be a velocity field. The particle is transported according to

$$\mathbf{x} + \Delta\mathbf{x} = \mathbf{x} + \mathbf{v}(\mathbf{x})\ \Delta t \tag{6.329}$$

with a velocity $\mathbf{v}$. In Fig. 6.8, all streamlines start with nine points in parallel on the X axis. At the center, the flow is fast and longitudinal, while at the periphery (near the first zero crossing of the Bessel function J_0) the flow is circling and has only a small Z component.

There is a remarkable similarity to the technical innovations of Nicola Tesla. In Fig. 6.8, the transverse parts resemble the current distribution in a Tesla flat coil. The longitudinal part in the middle corresponds to the current going to the sphere in a Tesla transmitter (for example, see Tesla patent [66]). According to Eq. (6.223), the current density for producing a Beltrami field is a Beltrami field also. Therefore, it should be possible to transmit free Beltrami fields by constructing a transmitter that has a current distribution as shown, for example, in Fig. 6.8.

A flat coil is only a rough approximation, since the current density within it is constant and does not depend on the radius. An improvement could be to use concentric conducting rings with differing currents, and with a dipole-like structure perpendicular to the rings at the center.

The spatial dimensions are determined by the wave number, which enters Eq. (6.326) through $\kappa = 2\pi/\lambda$. The wavelength λ is defined by the frequency, which should not be too high, otherwise the Maxwell displacement current of Eq. (6.218) has to be taken into account. ■

■ **Example 6.8** The last example shows a longitudinal vortex flow of Victor Schauberger (Fig. 6.9, as cited by Reed [62]). Schauberger described natural phenomena that often consisted of vortex phenomena. Similar vortex structures are mentioned by Reed in connection with interstellar magnetic fields without forces. Fig. 6.9 is highly similar to the Beltrami flow of the Bessel function example, which is shown in Fig. 6.8. In addition, there are small toroidal counter-vortices near the surrounding pipe. This is a transport phenomenon in fluids. ■

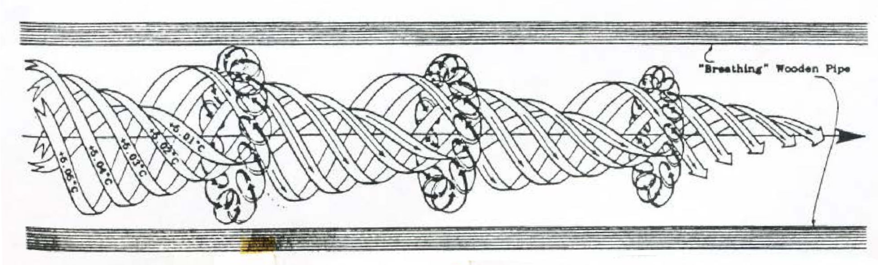


Figure 6.9: Helical flow according to Schauberger [62].

Some additional examples of Beltrami fields are described in [60] and [4]. These are examples of longitudinal solutions of the field equations, even in the case of Maxwell-Heaviside equations of special relativity. In particular, it is remarkable that standing waves can be generated from only one side of a transmission. Normally, two sides with "fixed ends" are required for this. It is not clear how communication mechanisms could be established by standing waves. It has been argued by mathematicians that changes in standing waves propagate instantaneously, and this could even enable superluminal communication.

Part Three: Dynamics and Gravitation

7. ECE dynamics

7.1 ECE dynamics and mechanics

In the preceding part of this textbook, we discussed the details of ECE electrodynamics. We now turn to the subject of mechanics, which includes several areas. Dynamics, whose physical laws were first formulated by Isaac Newton, is the area where time-dependent processes are handled, and includes Lagrange theory. Another area of mechanics is statics, for example, structural mechanics. One of the most important mechanical forces is gravitation, which defines a realm of its own, with relativistic effects playing a significant role. All of these areas are based on the mechanical equations of motion, and therefore they will be derived first.

7.1.1 Field equations of dynamics

Since the same approach will be used, we will first review how the field equations of electrodynamics were derived. These equations are based on the alternative form of the Cartan-Bianchi and Cartan-Evans identities, Eqs. (4.40, 4.41):

$$D_\mu \widetilde{T}^{a\mu\nu} = \widetilde{R}^{a}{}_{\mu}{}^{\mu\nu}, \tag{7.1}$$
$$D_\mu T^{a\mu\nu} = R^{a}{}_{\mu}{}^{\mu\nu}, \tag{7.2}$$

which are equations of Cartan geometry, with torsion form $T^{a\mu\nu}$ and curvature form $R^{a}{}_{\mu}{}^{\mu\nu}$, in contravariant representation. Electromagnetism is introduced by the ECE axioms, which connect the tetrad $q^a{}_\mu$ with the potential $A^a{}_\mu$, and the torsion $T^a{}_{\mu\nu}$ with the electromagnetic field $F^a{}_{\mu\nu}$:

$$A^a{}_\mu := A^{(0)} q^a{}_\mu, \tag{7.3}$$
$$F^a{}_{\mu\nu} := A^{(0)} T^a{}_{\mu\nu}, \tag{7.4}$$

where $A^{(0)}$ is a constant with physical units. Then, after inserting the axioms into Eqs. (7.1, 7.2), the field equations follow directly:

$$D_\mu \widetilde{F}^{a\mu\nu} = A^{(0)} \widetilde{R}^{a}{}_{\mu}{}^{\mu\nu}, \tag{7.5}$$
$$D_\mu F^{a\mu\nu} = A^{(0)} R^{a}{}_{\mu}{}^{\mu\nu}. \tag{7.6}$$

These are the electromagnetic field equations in contravariant form (see Eqs. (4.42, 4.43)). From these equations, we can derive the vector equations in Maxwell-Heaviside notation, as we have seen from the detailed explanation in Section 4.2.3.

Since ECE is a unified field theory, we will proceed in the same way for the mechanical sector. After defining the appropriate axioms, we will eventually arrive at field equations for dynamics that are equivalent to (7.5, 7.6). We start by comparing Newton's law of gravitation with the Coulomb law of electrostatics. Both have the same radial dependence and are formally identical. We will see that Newton's law is a part of the more general field equations of ECE dynamics. The Coulomb law is

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, \tag{7.7}$$

while Newton's gravitational law reads:

$$\mathbf{g} = -\frac{MG}{r^2}\hat{\mathbf{r}}. \tag{7.8}$$

Here G the gravitational constant, q is the electric charge and M is the gravitational mass. $\hat{\mathbf{r}}$ is the unit vector in the direction of a probe charge or mass, respectively, where r is its modulus. It can be seen that M takes the role of a "gravitational charge" in this law. Both fields decrease with $1/r^2$ over distance. The force on a probe charge e or mass m is, correspondingly:

$$\mathbf{F}_{\text{el}} = e\mathbf{E}, \tag{7.9}$$

$$\mathbf{F}_{\text{grav}} = m\mathbf{g}. \tag{7.10}$$

Newton's equivalence principle (see Section 7.2.4), which says that the dynamical force of acceleration on a mass is identical to the force of the gravitational field, follows from the second law. **E** and **g** correspond to one another and represent a unification of fields. The related field equation in the electromagnetic case is the Coulomb law for distributed charges ρ, written in divergence form:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}. \tag{7.11}$$

Therefore, an equivalent equation should exist in the case of dynamics. Rewriting the Coulomb law by using the scalar potential ϕ,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi, \tag{7.12}$$

gives us

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}, \tag{7.13}$$

which is the Poisson equation. This equation is also present in classical mechanics, where the acceleration field is derived, in an analogous way, from the mechanical potential Φ by

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi. \tag{7.14}$$

Therefore, the Poisson equation of dynamics is

$$\nabla^2\Phi = \alpha\rho_m, \tag{7.15}$$

where ρ_m is the mass density and α is a constant. This constant is found by comparison with electrodynamics, in the following way. The electrostatic field can be computed from the charge density ρ by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r' \tag{7.16}$$

(see [67]). This equation is derived from inserting what is called the Coulomb integral,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{7.17}$$

into Eq. (7.12). In the case of mechanics, analogously to (7.16), we can write

$$\mathbf{g}(\mathbf{r}) = -G\int \rho_m(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r' \tag{7.18}$$

with $-G$ being a constant. This constant provides the right physical units for $\mathbf{g}$, as the factor $1/(4\pi\varepsilon_0)$ does in the electric case. The minus sign stems from the fact that masses are always positive, but the force between them is attractive, not repulsive. This is different from electrodynamics, and appropriate adjustments must be made. For a point mass, ρ_m takes a Dirac delta function, which produces a factor of 4π so that this factor cancels out in the electrical Poisson equation [67]. In the case of mechanics, this factor is preserved, leading to $\alpha = 4\pi G$ in (7.15):

$$\nabla^2\Phi = 4\pi G\rho_m \tag{7.19}$$

or

$$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\rho_m. \tag{7.20}$$

This is one of the ECE field equations of dynamics.

The measurement of G, the gravitational constant, started with Bouguer (1738) and later Cavendish (1798), with Heyl (1930, 1942) publishing the first modern values, which have remained compatible with the many subsequent, ongoing higher-precision measurements.

For consistency with electrodynamics, there must be three additional field equations, so that a set of Maxwell-like equations is obtained for dynamics. To derive these equations, we proceed in the same way as we did for the field equations of electrodynamics. First, we define the ECE axioms analogously to Eqs. (7.3, 7.4). Instead of $A^a{}_\mu$, we use the 4-potential $Q^a{}_\mu$, and instead of $F^a{}_{\mu\nu}$, we use the field tensor $G^a{}_{\mu\nu}$. Both are defined to be identical to the tetrad and curvature, via a factor $Q^{(0)}$:

$$Q^a{}_\mu := Q^{(0)}q^a{}_\mu, \tag{7.21}$$
$$G^a{}_{\mu\nu} := Q^{(0)}T^a{}_{\mu\nu}\,. \tag{7.22}$$

Then, analogously to (7.5, 7.6), the field equations of dynamics in contravariant form read:

$$D_\mu\widetilde{G}^{a\mu\nu} = Q^{(0)}\widetilde{R}^a{}_\mu{}^{\mu\nu}, \tag{7.23}$$
$$D_\mu G^{a\mu\nu} = Q^{(0)}R^a{}_\mu{}^{\mu\nu}. \tag{7.24}$$

We have already seen from Eq. (7.20) that the gravitational acceleration field $\mathbf{g}$ corresponds to $\mathbf{E}$. The latter is of translational character, and therefore so is $\mathbf{g}$. In addition, there must be a mechanical field of rotational character in the field tensor, corresponding to $\mathbf{B}$. We call this field the *gravitomagnetic field* $\boldsymbol{\Omega}$. Both of these mechanical fields must have a polarization index a, as is the case in ECE electrodynamics. They are designated $\mathbf{g}^a$ and $\boldsymbol{\Omega}^a$, in ECE dynamics. Then, the gravitational field tensor, analogously to Eq. (4.65), has the form:

$$G^{a\mu\nu} = \begin{bmatrix} G^{a00} & G^{a01} & G^{a02} & G^{a03} \\ G^{a10} & G^{a11} & G^{a12} & G^{a13} \\ G^{a20} & G^{a21} & G^{a22} & G^{a23} \\ G^{a30} & G^{a31} & G^{a32} & G^{a33} \end{bmatrix} = \begin{bmatrix} 0 & -g^{a1}/c & -g^{a2}/c & -g^{a3}/c \\ g^{a1}/c & 0 & -\Omega^{a3} & \Omega^{a2} \\ g^{a2}/c & \Omega^{a3} & 0 & -\Omega^{a1} \\ g^{a3}/c & -\Omega^{a2} & \Omega^{a1} & 0 \end{bmatrix}, \tag{7.25}$$

and its Hodge dual, corresponding to (4.56), is

$$\widetilde{G}^{a\mu\nu} = \begin{bmatrix} \widetilde{G}^{a00} & \widetilde{G}^{a01} & \widetilde{G}^{a02} & \widetilde{G}^{a03} \\ \widetilde{G}^{a10} & \widetilde{G}^{a11} & \widetilde{G}^{a12} & \widetilde{G}^{a13} \\ \widetilde{G}^{a20} & \widetilde{G}^{a21} & \widetilde{G}^{a22} & \widetilde{G}^{a23} \\ \widetilde{G}^{a30} & \widetilde{G}^{a31} & \widetilde{G}^{a32} & \widetilde{G}^{a33} \end{bmatrix} = \begin{bmatrix} 0 & \Omega^{a1} & \Omega^{a2} & \Omega^{a3} \\ -\Omega^{a1} & 0 & -g^{a3}/c & g^{a2}/c \\ -\Omega^{a2} & g^{a3}/c & 0 & -g^{a1}/c \\ -\Omega^{a3} & -g^{a2}/c & g^{a1}/c & 0 \end{bmatrix}. \tag{7.26}$$

Analogously to the electromagnetic case, the $\mathbf{g}^a$ field is divided by c, the velocity of light in vacuo, so that all tensor elements have the same units, namely inverse seconds. Thus, it can be seen directly that $\mathbf{\Omega}^a$ is a rotational field. Since torsion has units of inverse meters, it follows from Eq. (7.22) that the constant $Q^{(0)}$ has units of m/s, i.e., it is a velocity. From Eq. (7.21) we see that the vector potential of dynamics, $Q^a{}_\mu$, also has the units of a velocity. It can be interpreted as a velocity of vacuum flux in classical mechanics. This property is completely unknown in standard theory.

The charge and current densities are defined analogously to Eqs. (4.45 - 4.50) by

$$\partial_\mu \widetilde{G}^{a\mu\nu} = Q^{(0)} \widetilde{R}^a{}_\mu{}^{\mu\nu} - \omega_{(\Lambda)}{}^a{}_{\mu b} \widetilde{G}^{b\mu\nu} =: j^{a\nu}, \tag{7.27}$$

$$\partial_\mu G^{a\mu\nu} = Q^{(0)} R^a{}_\mu{}^{\mu\nu} - \omega^a{}_{\mu b} G^{b\mu\nu} =: J^{a\nu}. \tag{7.28}$$

The field equations of dynamics can be transformed into vector form by observing all of the details in the comprehensive procedures presented in Section 4.2.3. We can use Eqs. (4.72 - 4.75) directly, by making the following replacements:

$$\begin{aligned} \mathbf{E}^a &\to \mathbf{g}^a, \\ \mathbf{B}^a &\to \mathbf{\Omega}^a, \\ \rho_h^a &\to \rho_{mh}^a, \\ \rho^a &\to \rho_m^a, \\ \mathbf{j}^a &\to \mathbf{j}_m^a, \\ \mathbf{J}^a &\to \mathbf{J}_m^a. \end{aligned} \tag{7.29}$$

In addition, we make the following replacements for the potentials:

$$\begin{aligned} \phi^a &\to \Phi^a, \\ \mathbf{A}^a &\to \mathbf{Q}^a. \end{aligned} \tag{7.30}$$

The constants on the right sides are adapted to Newton's law, as derived for Eq. (7.20). As described above, we have a sign change compared to the constants of electrodynamics. Finally, we arrive at the vector equations:

$$\nabla \cdot \mathbf{\Omega}^a = 4\pi G\, \rho_{mh}^a, \quad \text{Gauss law of dynamics} \tag{7.31}$$

$$\frac{\partial \mathbf{\Omega}^a}{\partial t} + \nabla \times \mathbf{g}^a = -\frac{4\pi G}{c} \mathbf{j}_m^a, \quad \text{Gravitomagnetic law} \tag{7.32}$$

$$\nabla \cdot \mathbf{g}^a = -4\pi G\, \rho_m^a, \quad \text{Newton's law (Poisson equation)} \tag{7.33}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{g}^a}{\partial t} + \nabla \times \mathbf{\Omega}^a = -\frac{4\pi G}{c^2} \mathbf{J}_m^a. \quad \text{Ampère-Maxwell law of dynamics} \tag{7.34}$$

The units of all components are listed in Table 7.1, together with the corresponding electromagnetic components, for comparison. In the dynamics case, the homogeneous and inhomogeneous currents have the same units. In the electromagnetic case, they differ by how they were defined in their respective unit system.

Electromagnetic			Mechanics/Dynamics		
Symbol	Field/Constant	Units	Symbol	Field/Constant	Units
$\mathbf{E}$	electric field	V/m	$\mathbf{g}$	gravitational field	m/s^2
$\mathbf{B}$	magnetic field	T=Vs/m^2	$\mathbf{\Omega}$	gravitomagnetic field	1/s
ϕ	scalar potential	V	Φ	gravitational potential	m^2/s^2
$\mathbf{A}$	vector potential	Vs/m	$\mathbf{Q}$	grav. vector potential	m/s
$A^{(0)}$	ECE constant	Vs/m	$Q^{(0)}$	grav. ECE constant	m/s
ρ	charge density	C/m^3	ρ_m	mass density	kg/m^3
$\mathbf{J}$	current density	C/(m^{2}s)	$\mathbf{J}_m$	mass current density	kg/(m^{2}s)
ρ_h	hom. charge density	A/m^2	ρ_{mh}	hom. mass density	kg/m^3
$\mathbf{J}_h = \mathbf{j}$	hom. current density	A/(ms)	$\mathbf{J}_{mh} = \mathbf{j}_m$	hom. mass current density	kg/(m^{2}s)
ε_0	vacuum permittivity	As/(Vm)	G	gravitational constant	m^3/(kg s^2)
μ_0	vacuum permeability	Vs/(Am)	k	Einstein constant	1/N=s^2/(kg m)

Table 7.1: Comparison between components of electromagnetism and gravitation.

The mechanical 4-currents are defined in detail by

$$(j^a)^\nu = \begin{bmatrix} c\rho^a_{mh} \\ j^{a1} \\ j^{a2} \\ j^{a3} \end{bmatrix} = \begin{bmatrix} c\rho^a_{mh} \\ \mathbf{j}^a \end{bmatrix}, \tag{7.35}$$

$$(J^a)^\nu = \begin{bmatrix} c\rho^a_{m} \\ J^{a1} \\ J^{a2} \\ J^{a3} \end{bmatrix} = \begin{bmatrix} c\rho^a_{m} \\ \mathbf{J}^a \end{bmatrix}. \tag{7.36}$$

Usually, the homogeneous currents vanish, as is known by experiment for the electromagnetic case.

We define the 4-potential analogously to Eq. (4.33):

$$(Q^a)^\nu = \begin{bmatrix} \Phi^a/c \\ Q^{a1} \\ Q^{a2} \\ Q^{a3} \end{bmatrix} = \begin{bmatrix} \Phi^a/c \\ \mathbf{Q}^a \end{bmatrix}. \tag{7.37}$$

With this definition, we can write the field-potential relations of dynamics in the same form as (4.197, 4.198):

$$\mathbf{g}^a = -\nabla\Phi^a - \frac{\partial \mathbf{Q}^a}{\partial t} - c\omega^a{}_{0b}\mathbf{Q}^b + \omega^a{}_b\Phi^b, \tag{7.38}$$

$$\mathbf{\Omega}^a = \nabla \times \mathbf{Q}^a - \omega^a{}_b \times \mathbf{Q}^b. \tag{7.39}$$

There is a significant difference when compared to standard theory, where only the gravitational scalar potential Φ and the gravitational field $\mathbf{g}$ are known, and are connected by:

$$\mathbf{g} = -\nabla\Phi. \tag{7.40}$$

ECE dynamics has a much richer structure. Specifically, it is a theory of general relativity with spacetime torsion and curvature. Therefore, the spin connections appear in these equations as they do in electromagnetism. The gravitomagnetic field $\mathbf{\Omega}$ has been observed experimentally. This will be described in a few examples, later. Even in Einsteinian general relativity, some authors speak of this field in terms of linear approximation to the Einstein field equations. In ECE theory, this is not an approximation, but rather a genuine field of nature.

7.1.2 Additional equations of the mechanical sector

As we have seen, there is a complete correspondence between mechanics and electromagnetism in ECE theory. Therefore, the mechanical sector can be developed further by adapting important laws of electrodynamics to dynamics. In this way, new laws of nature can be found that were not known hitherto.

An important equation, which follows from the field equations, is the generally covariant *continuity equation*. When its electromagnetic counterpart, Eq. (5.100), is adapted to dynamics, it reads:

$$\boxed{\frac{\partial \rho_m^a}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}_m^a = 0.} \tag{7.41}$$

It means that variation of a mass density in time is connected with a divergence of the mass current density. For example, removing mass from a volume results in a time-dependent density and, consequently, an equivalent mass current density.

We continue by deriving the Lorentz force equations from the Lorentz transform of special relativity. If an electromagnetic system ($\mathbf{E}$, $\mathbf{B}$) is observed in another frame of reference, moving with respect to the original Frame 1 with a velocity vector $\mathbf{v}$, then the electric and magnetic fields in Frame 2, denoted by $\mathbf{E}'$, $\mathbf{B}'$, are obtained according to the transformation equations [68]:

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{\gamma^2}{1+\gamma} \frac{\mathbf{v}}{c} \left(\frac{\mathbf{v}}{c} \cdot \mathbf{E} \right), \tag{7.42}$$

$$\mathbf{B}' = \gamma \left(\mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \right) - \frac{\gamma^2}{1+\gamma} \frac{\mathbf{v}}{c} \left(\frac{\mathbf{v}}{c} \cdot \mathbf{B} \right), \tag{7.43}$$

with the relativistic "gamma factor"

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{7.44}$$

In the nonrelativistic approximation,

$$v \ll c, \quad \gamma \to 1, \tag{7.45}$$

these equations reduce to

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{7.46}$$

$$\mathbf{B}' = \mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E}. \tag{7.47}$$

The first of these equations, when multiplied by a charge, is the well-known Lorentz force. The second equation is the less used magnetic counterpart to it. If there is no original electric field in the frame at rest, Eq. (7.46) can be written as

$$\mathbf{E}' = \mathbf{v} \times \mathbf{B}, \tag{7.48}$$

which is the form used in most applications.

The ECE dynamics counterparts are the gravitational *Lorentz force*

$$\boxed{\mathbf{g}' = \mathbf{g} + \mathbf{v} \times \boldsymbol{\Omega},} \tag{7.49}$$

and its gravitomagnetic equivalent

$$\boxed{\boldsymbol{\Omega}' = \boldsymbol{\Omega} - \frac{1}{c^2}\mathbf{v} \times \mathbf{g},} \tag{7.50}$$

also called the *gravitomagnetic equation*. $\boldsymbol{\Omega}$ is quite small in general, and the effects of these forces cannot be experienced in daily life, in contrast to the electromagnetic Lorentz force, which is a basis for many types of technical applications. The term $\mathbf{v} \times \mathbf{g}$ in Eq. (7.50) is not as small, but it is weighted with $1/c^2$; therefore, it acts as a relativistic correction, and its scale also makes it undetectable by standard measurement methods.

From the geometrical Evans lemma (4.159), via postulate (7.21), we get the *ECE wave equation* of dynamics:

$$\boxed{\square\, Q^a{}_\nu + R\, Q^a{}_\nu = 0} \tag{7.51}$$

with a scalar curvature R as described in Section 4.3. It is also possible to write the *Proca equation* with the mechanical potential $Q^a{}_\nu$:

$$\square\, Q^a{}_\nu + \left(\frac{m_0 c}{\hbar}\right)^2 Q^a{}_\nu = 0. \tag{7.52}$$

Finally, we will formulate the equations of ECE2 theory for mechanical fields and potentials. Denoting the scalar and vector ECE2 potentials of Section 6.1.3 by Φ_W and $\mathbf{Q}_W$, we obtain for the *ECE2 field-potential relations of dynamics*:

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi_W - \frac{\partial \mathbf{Q}_W}{\partial t}, \tag{7.53}$$

$$\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \mathbf{Q}_W. \tag{7.54}$$

The ECE2 field equations of electromagnetism, Eqs. (6.97 - 6.100), can be transformed directly into the *ECE2 field equations of dynamics*. For vanishing homogeneous currents, they are

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Omega} = 0, \qquad \text{Gauss law of dynamics} \tag{7.55}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{g} = \mathbf{0}, \qquad \text{Gravitomagnetic law} \tag{7.56}$$

$$\boldsymbol{\nabla} \cdot \mathbf{g} = -4\pi G\, \rho_m, \qquad \text{Newton's law (Poisson equation)} \tag{7.57}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{g}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{\Omega} = -\frac{4\pi G}{c^2}\mathbf{J}_m. \qquad \text{Ampère-Maxwell law of dynamics} \tag{7.58}$$

Compared to the field equations of dynamics, Eqs. (7.31 - 7.34), the only differences are that the tangent space index a has disappeared and no gravitomagnetic charges have been assumed.

Since the Cartan geometry is the same for electromagnetism and dynamics, the same antisymmetry laws exist for both field types. Therefore, we can directly translate the antisymmetry laws (5.14) and (5.18 - 5.20) of the electric and magnetic cases into the respective laws of dynamics. After the replacement rules (7.30) have been applied, the electric antisymmetry condition becomes the gravitational antisymmetry condition and reads:

$$\boxed{-\frac{\partial \mathbf{Q}^a}{\partial t} + \boldsymbol{\nabla}\Phi^a - c\,\omega^a{}_{0b}\mathbf{Q}^b - \omega^a{}_b\Phi^b = \mathbf{0},} \tag{7.59}$$

and the magnetic antisymmetry condition becomes the gravitomagnetic antisymmetry condition and consists of the following three equations with permutational structure:

$$-\partial_3 Q^{a2} - \partial_2 Q^{a3} + \omega^{a3}{}_b Q^{b2} + \omega^{a2}{}_b Q^{b3} = 0, \tag{7.60}$$

$$-\partial_1 Q^{a3} - \partial_3 Q^{a1} + \omega^{a1}{}_b Q^{b3} + \omega^{a3}{}_b Q^{b1} = 0, \tag{7.61}$$

$$-\partial_2 Q^{a1} - \partial_1 Q^{a2} + \omega^{a2}{}_b Q^{b1} + \omega^{a1}{}_b Q^{b2} = 0. \tag{7.62}$$

■ **Example 7.1** Gravity Probe B [69, 70] was a satellite-based experiment designed to verify two effects of general relativity. The satellite contained four gyroscopes, and the angular precession of the gyroscopic axes was measured by high-precision instruments. The larger effect is the geodetic precession, which stems from the fact that the Earth creates a spherical, non-Newtonian gravitational field. The smaller effect is the Lense-Thirring effect, which describes the frame dragging of spacetime, and should also be observable.

Figure 7.1: Polar satellite orbit around the Earth.

The satellite for Gravity Probe B was launched in 2004 and was operational until 2005. It orbited over the poles (see Fig. 7.1) at a height of 650 km above the Earth's surface, i.e., it was a low-orbit satellite, whose mean distance above the Earth's surface was small compared to the Earth's radius.

However, big problems arose during evaluation of the recorded data. The electromagnetic interaction of the gyroscopic spheres with the walls of the satellite had not been taken into account properly. In the first evaluation, the results were much less precise than expected, with a precession angle variance (error bar) of 0.1 rad/year. This unexpected variance also contained the Lense-Thirring effect, which is not observable separately.

ECE theory explains the frame dragging directly, through the gravitomagnetic field. When UFT Paper 117 [69] was being written around 2008, it seemed reasonable to expect a variance of 0.1 rad/year from the corresponding ECE calculation. The ECE result turned out to be 0.099 rad/year, which provided a consistent explanation of the experimental variance. Later, in 2011, a final report was published about Gravity Probe B, which stated that the precession result of the Lense-Thirring effect was 0.0372 ± 0.0072 rad/year [70]. To obtain this result, a lot of data corrections had to be developed (and without grant support), which is why it took six years for the final results to be made available. There was a concomitant theoretical explanation, in which a dipole approximation was made for the Earth's mass. However, a sphere of uniform mass has no multipoles except for the familiar monopole, the Newtonian potential. This makes the calculation of the Lense-Thirring effect at least questionable.

In ECE theory, the part of the gyroscopic precession that we are focusing on (the Lense-Thirring effect of Einsteinian theory) is explained by the gravitomagnetic field. This field, according to Eq. (7.50), is

$$\mathbf{\Omega} = -\frac{1}{c^2}\mathbf{v}\times\mathbf{g}, \tag{7.63}$$

where $\mathbf{v}$ is the velocity of the "observer", and $\mathbf{g}$ is the gravitational field of the Earth with mass M:

$$\mathbf{g} = -\frac{MG}{r^3}\mathbf{r}. \tag{7.64}$$

Inserting this into $\mathbf{\Omega}$ gives

$$\mathbf{\Omega} = -\frac{1}{c^2}\mathbf{v}\times\left(-\frac{MG}{r^3}\mathbf{r}\right) = -\frac{MG}{c^2r^3}\mathbf{r}\times\mathbf{v}. \tag{7.65}$$

This looks similar to an angular momentum, which is defined for a position $\mathbf{r}$ and velocity $\mathbf{v}$ of a mass m by

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\,\mathbf{r}\times\mathbf{v}, \tag{7.66}$$

where $\mathbf{p}$ is the linear momentum. Since the Earth is an extended body, the total angular momentum is the sum of its single mass elements m_i with their corresponding positions and velocities:

$$\mathbf{L} = \sum_i m_i\mathbf{r}_i\times\mathbf{v}_i. \tag{7.67}$$

This describes the angular momentum of the Earth, which is due to its rotating mass, and this is a spin momentum. Integration over all mass elements gives the well-known result

$$\mathbf{L}_{\text{sph}} = \frac{2}{5}MR^2\boldsymbol{\omega} \tag{7.68}$$

for a homogeneous sphere with radius R. The angular frequency $\omega = v/r$ is the rotational speed, 360 degrees in one day. The masses m_i sum to M, and therefore from Eq. (7.65) we get

$$\mathbf{\Omega} = -\frac{G}{c^2r^3}\mathbf{L}_{\text{sph}} \tag{7.69}$$

or

$$\Omega = \frac{2}{5}\frac{MGR^2}{c^2r^3}\omega. \tag{7.70}$$

This is the gravitomagnetic field of the Earth's spin. When we use the parameters listed under Eq. (24.21) in [69], the result becomes

$$\Omega = 1.5878\cdot10^{-14}\ \text{rad/s}. \tag{7.71}$$

With $t = 1$ year, the angular precession of the gyroscopic axes in the satellite then becomes

$$\theta = \Omega\cdot t = 0.0987\ \text{rad}. \tag{7.72}$$

All calculations can be found in the computer algebra code [143].

In the previously mentioned UFT Paper 117 (2008), this was interpreted as agreement with experiment, which had given a variance in measurement of 0.1 rad per year (see the discussion above). When we consider the newer result for the Lense-Thirring effect, which is 0.0372 ± 0.0072 rad/year, we see that the ECE result is larger, but of the same order of magnitude. For a precise comparison, we have to take into account the approximations made in the calculation, for example, the assumption that the Earth is a homogeneous sphere with homogeneous density. Another example is where the angular momentum vector is defined in space. Normally, it is assumed that this vector is valid at the position of the rotation axis (origin of the coordinate system). This would mean that it has the assumed value only when the satellite is over the poles. When it has moved to an angle of $\theta = 90°$ in the spherical coordinate system, the local vector of the angular momentum should be minimal. If we assume that it varies with the cosine function, the mean value of the gravitomagnetic field then becomes

$$\Omega_{\text{av}} = \Omega \cdot \frac{1}{\pi/2} \int_0^{\pi/2} \cos\theta d\theta = \Omega \cdot \frac{2}{\pi} \approx 0.637\ \Omega. \tag{7.73}$$

The angular precession of Eq. (7.72) then reduces to

$$\theta = 0.063 \text{ rad}, \tag{7.74}$$

which comes closer to the experimental result. Given the lengthy calculations and assumptions that led to the experimental value, this is a reasonable agreement. ■

■ **Example 7.2** Equinoctial precession [71] is a case in which Eq. (7.50) plays a role. The equinoxes are the two days during the year, one in spring and one in autumn, when day and night are of equal length. (The other planets in our solar system also have their own equinoxes.) As can be seen from Fig. 7.2, on these days the rotational axis of the Earth is perpendicular to the plane of motion of the Earth around the Sun, which is called the *ecliptic*. The appearance of seasons is a consequence of the fact that the rotational axis of the Earth is not perpendicular to the ecliptic, but is tilted by an angle of 23.5 degrees. The *celestial equator* is a projection of the Earth's equator (see Fig. 7.3), and its plane is inclined to the ecliptic by this angle.

The time between two vernal or autumnal equinoxes defines a year with respect to the ecliptic orbit of the Earth, the *tropical year*. Another definition is the *sideral year*, which is the time that elapses until the fixed stars are seen under the same angle again. If the axis of the Earth were fixed, both years would be identical, i.e., one would see the same background of the fixed stars during equinoxes. There is, however, a difference between them because of a precession of the Earth's axis. The background of fixed stars changes by 50.25 arcsec per year. In the past, attempts were made to explain this as influences by other planets in the Solar System; however, this was contradictory and led only to some empirical formulas to predict precession angles.

In [70] it is explained in detail why the true reason for equinoctial precession is the motion of the Sun in our galaxy. The Sun takes 230 million years to orbit the galactic center. If the ecliptic is dragged along the path of this motion, we arrive at the picture shown in Fig. 7.4. After one Earth year, the same position relative to the Sun leads to a change in the background of fixed stars due to this orbital motion of the Sun. When we compute this "precession" for one year, we get

$$\varepsilon = 2\pi \cdot \frac{1}{230 \cdot 10^6} = 0.005 \text{ arcsec}, \tag{7.75}$$

which is smaller than the observed 50.25 arcsec by several orders of magnitude. The period of the observed precession is only about 25,800 years. Therefore, another mechanism must be at work, and this mechanism is the gravitomagnetic field described by Eq. (7.50).

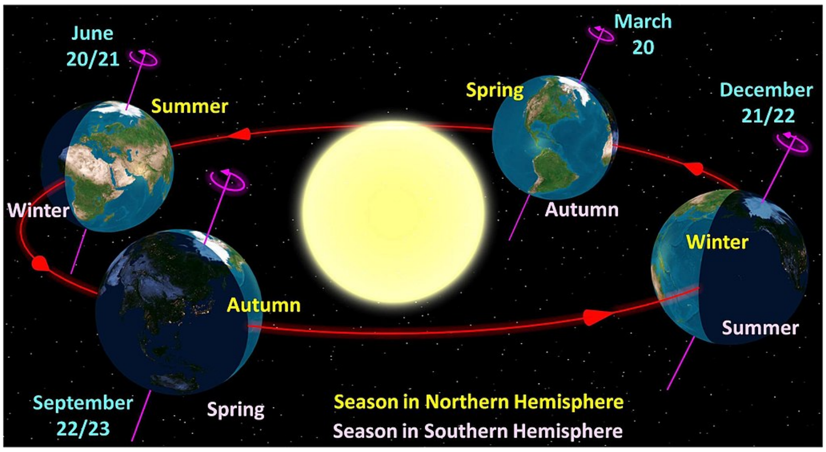


Figure 7.2: Terrestrial equinoxes in spring and autumn [185].

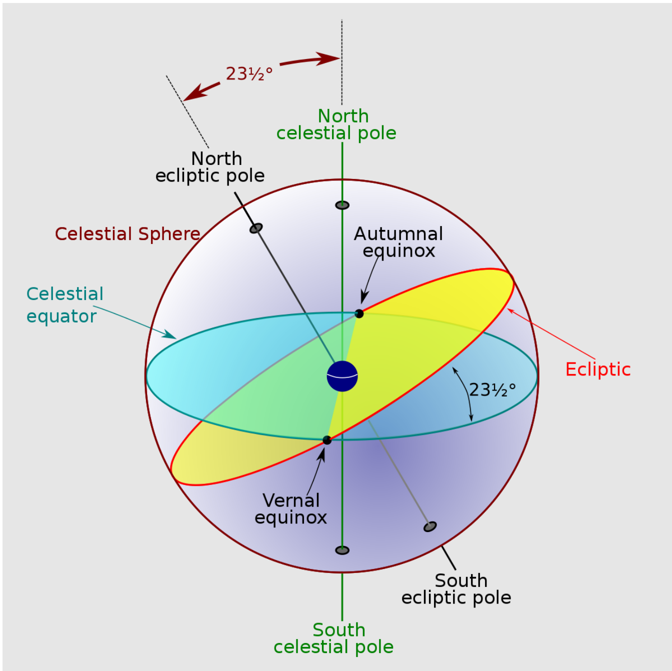


Figure 7.3: Celestial equator and ecliptic in the Earth-centered view [186].

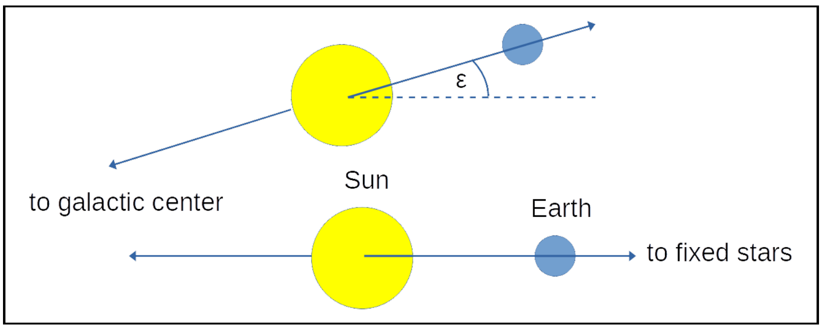


Figure 7.4: Directional effect due to the motion of the Sun around the galactic center.

The velocity in Eq. (7.50) is the orbital velocity of the Sun in the galactic orbit. The distance from the galactic center is 8,178 parsec, or $2.523 \cdot 10^{20}$ m. The orbital velocity of the Sun is then

$$v = \frac{2\pi}{230 \cdot 10^6 \text{ y}} \cdot 2.523 \cdot 10^{20} \text{ m} = 218\,448 \text{ m/s}, \tag{7.76}$$

which is very close to the value of 220 km/s obtained from other observations. All calculations are available in computer algebra code [144].

If there is no gravitomagnetic field in the rest system of the Sun, the gravitomagnetic equation (7.50) is simply

$$\boldsymbol{\Omega} = -\frac{1}{c^2}\mathbf{v} \times \mathbf{g}. \tag{7.77}$$

Its modulus is

$$|\boldsymbol{\Omega}| = \frac{1}{c^2} v g \sin(\theta), \tag{7.78}$$

where θ is the angle between the orbital velocity and the gravitational field in which the Earth rotates, which is mainly the acceleration field of the Earth. In [70] it is proposed that this is the gravitational field at the position of the observer, but the precession of the Earth's axis cannot depend on this position. Instead, it needs to be an average value over all directions of **g**, and even over the radius of the Earth, because each mass element will experience a different acceleration at its radius. With

$$v = 218\,448 \text{ m/s}, \quad g = 9.81 \text{ m/s}^2, \quad c = 2.9979 \cdot 10^8 \text{ m/s}, \tag{7.79}$$

when **v** and **g** are perpendicular to each other, we obtain as a maximum value:

$$\Omega = 2.38439 \cdot 10^{-11}/\text{s}. \tag{7.80}$$

By using this value, we would obtain a precession angle of

$$\Omega \cdot 1\,\text{year} = 155.2 \text{ arcsec/year}, \tag{7.81}$$

which is much more than the observed 55.25 arcsec/year. The average angle between **v** and **g** would then be

$$\theta = \arcsin \frac{55.25}{155.2} = 18.9^\circ, \tag{7.82}$$

if we used the value of $g = 9.81 \text{ m/s}^2$ at the Earth's surface. Alternatively, we can compute an effective gravitational field:

$$g_{\text{eff}} = g \sin\theta = 3.179 \text{ m/s}^2, \tag{7.83}$$

or an effective orbital velocity:

$$v_{\text{eff}} = v \sin\theta = 70\,782 \text{ m/s}. \tag{7.84}$$

The result $g_{\text{eff}} < g$ is reasonable. The gravitomagnetic field is a plausible, non-Newtonian explanation for the equinoctial precession that is based on a generally covariant theory of dynamics and not on the general relativity of Einstein. ∎

7.1.3 Mechanical polarization and magnetization

In Section 5.2, we introduced electrodynamic polarization and magnetization, which are well known in standard physics. When we transfer these concepts to dynamics, polarization is inferred from the gravitational field, and magnetization of matter is inferred from the gravitomagnetic field. In dynamics, we have the special situation that no "charges" of different sign are present, so polarization may work differently, if it even exists in the case of gravitation.

We define the analogs of electromagnetic displacement $\mathbf{D}$ and magnetic field $\mathbf{H}$ (not the induction) in the same way as in Section 5.2. For this, we map the electromagnetic fields to the fields of dynamics in the following way:

$$\begin{aligned} &\mathbf{E}^a \to \mathbf{g}_0^a, \\ &\mathbf{D}^a \to \mathbf{g}^a, \\ &\mathbf{B}^a \to \mathbf{\Omega}^a, \\ &\mathbf{H}^a \to \mathbf{\Omega}_0^a. \end{aligned} \tag{7.85}$$

For convenience, $\mathbf{g}$ is associated with the displacement field $\mathbf{D}$, and not with $\mathbf{E}$. The counterpart of the electric field $\mathbf{E}$, in the presence of polarization, is denoted by $\mathbf{g}_0$. The magnetic field $\mathbf{H}$ is associated with $\mathbf{\Omega}_0$, while the magnetic induction $\mathbf{B}$ is associated with $\mathbf{\Omega}$, as before.

The system of units in dynamics is simpler than in electrodynamics, because there are no existing conventions that must be accommodated. Therefore, we simply define a mechanical polarization $\mathbf{P}_m$ and magnetization $\mathbf{M}_m$ so that (omitting the polarization index a)

$$\mathbf{g} = \mathbf{g}_0 + \mathbf{P}_m \tag{7.86}$$

and

$$\mathbf{\Omega} = \mathbf{\Omega}_0 + \mathbf{M}_m. \tag{7.87}$$

The resulting equations of motion are presented in Table 7.2, together with their electromagnetic equivalents. In contrast to electrodynamics, the pairs of fields ($\mathbf{g}_0$, $\mathbf{g}$) and ($\mathbf{\Omega}_0$, $\mathbf{\Omega}$) have the same units.

When polarization and magnetization depend linearly on $\mathbf{g}_0$ and $\mathbf{\Omega}_0$, we can write (as in electrodynamics):

$$\mathbf{g} = \varepsilon_m \mathbf{g}_0, \tag{7.88}$$

$$\mathbf{\Omega} = \mu_m \mathbf{\Omega}_0. \tag{7.89}$$

Electromagnetic	Mechanical
$\nabla \cdot \mathbf{B}^a = 0$	$\nabla \cdot \mathbf{\Omega}^a = 0$
$\frac{1}{c^2}\frac{\partial \mathbf{H}^a}{\partial t} + \nabla \times \mathbf{D}^a = \mathbf{0}$	$\frac{\partial \mathbf{\Omega}_0^a}{\partial t} + \nabla \times \mathbf{g}^a = \mathbf{0}$
$\nabla \cdot \mathbf{D}^a = \frac{\rho^a}{\varepsilon_0}$	$\nabla \cdot \mathbf{g}^a = -4\pi G \rho_m^a$
$-c^2 \frac{\partial \mathbf{D}^a}{\partial t} + \nabla \times \mathbf{H}^a = \mathbf{J}^a$	$-\frac{1}{c^2}\frac{\partial \mathbf{g}^a}{\partial t} + \nabla \times \mathbf{\Omega}_0^a = -\frac{4\pi G}{c^2}\mathbf{J}_m^a$

Table 7.2: Equivalent electromagnetic and gravitational equations of polarization and magnetization.

Electromagnetic	Mechanical
$\boldsymbol{\nabla}\cdot\mathbf{B}^a=0$	$\boldsymbol{\nabla}\cdot\boldsymbol{\Omega}^a=0$
$\frac{1}{\mu_r}\frac{\partial\mathbf{B}^a}{\partial t}+\varepsilon_r\boldsymbol{\nabla}\times\mathbf{E}^a=\mathbf{0}$	$\frac{1}{\mu_m}\frac{\partial\boldsymbol{\Omega}^a}{\partial t}+\varepsilon_m\boldsymbol{\nabla}\times\mathbf{g}_0^a=\mathbf{0}$
$\boldsymbol{\nabla}\cdot\mathbf{E}^a=\frac{\rho^a}{\varepsilon_0\varepsilon_r}$	$\boldsymbol{\nabla}\cdot\mathbf{g}_0^a=-\frac{4\pi G}{\varepsilon_m}\rho_m^a$
$-\varepsilon_r\frac{\partial\mathbf{E}^a}{\partial t}+\frac{1}{\mu_r}\boldsymbol{\nabla}\times\mathbf{B}^a=\mu_0\mathbf{J}^a$	$-\frac{\varepsilon_m}{c^2}\frac{\partial\mathbf{g}_0^a}{\partial t}+\frac{1}{\mu_m}\boldsymbol{\nabla}\times\boldsymbol{\Omega}^a=-\frac{4\pi G}{c^2}\mathbf{J}_m^a$

Table 7.3: Equivalent electromagnetic and gravitational equations of polarization and magnetization for linear materials.

We do not need constants like ε_0 and μ_0, as we did in the electromagnetic case. Instead, we only need a relative mechanical permittivity ε_m and permeability μ_m, which we introduced in the equations directly above. Then, with these new constants, the equations of motion can be formulated analogously to the electromagnetic case, as listed in Table 7.3.

The question of when, or even if, these effects become evident is still open. It is not known which types of matter could produce such effects, because the gravitational field depends only on mass. However, the gravitational constant is known only to a few digits. This could be a hint that certain local deviations may depend on gravitational polarization. These effects could also exist for gravitational waves (see next section). Myron Evans has proposed that the spin connection term $\omega^a{}_b\times\mathbf{Q}^b$ in Eq. (7.39) is a magnetization effect of spacetime itself [72]. This effect could also appear in mechanics directly, as a gravitomagnetic moment (the counterpart to the magnetic moment) that could be defined as in the following example.

■ **Example 7.3** We compute the gravitomagnetic field that the Earth generates on its path around the Sun. A general method would be to solve the static Ampère-Maxwell law,

$$\boldsymbol{\nabla}\times\boldsymbol{\Omega}=-\frac{4\pi G}{c^2}\mathbf{J}_m, \tag{7.90}$$

for the current density $\mathbf{J}_m$ evoked by the Earth. However, the current consists of the motion of a single mass and there is no continuous current. Therefore, we choose a different approach. The angular momentum of the Earth's motion is an orbital angular momentum, in contrast to the spin momentum that we considered in Example 7.1. The angular momentum of the Earth's orbit is

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\mathbf{v}, \tag{7.91}$$

where m is the mass and $\mathbf{v}$ is the orbital velocity of the Earth. r is the distance from the Sun. Assuming a circular orbit, we have

$$\begin{aligned} m&=5.972\cdot10^{24}\text{ kg},\\ v&=2.978\cdot10^{4}\text{ m/s},\\ r&=6.371\cdot10^{6}\text{ m}, \end{aligned} \tag{7.92}$$

which leads to an orbital angular momentum of

$$L=1.133\cdot10^{36}\text{ kg m}^2/\text{s}. \tag{7.93}$$

From electrodynamics we know [73] that the angular momentum of an orbiting charge q, with mass M, is connected with a magnetic moment $\mathbf{m}$ by

$$\mathbf{m} = \frac{q}{2M}\mathbf{L}. \tag{7.94}$$

This result can be transferred directly to dynamics by replacing the charge q with the "charge of dynamics" M, so that the ratio of charge per mass is eliminated:

$$\mathbf{m}_m = \frac{1}{2}\mathbf{L}. \tag{7.95}$$

This *gravitomagnetic moment* is directly proportional to the angular momentum.

The gravitomagnetic field of the Earth's orbit can be derived in a dipole approximation. In the case of a magnetic dipole moment, the induction field is [73]:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{n}\,(\mathbf{n}\cdot\mathbf{m}) - \mathbf{m}}{|\mathbf{r}|^3} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{r})\right), \tag{7.96}$$

where $\mathbf{n} = \hat{\mathbf{r}}$ is the unit vector in direction of $\mathbf{r}$. The coordinate origin is assumed to be in the center of the dipole, the location of the Sun in our example. Dirac's δ function contributes to the divergence at $\mathbf{r} = \mathbf{0}$, and is required only for the correct volume integral over $\mathbf{B}$ [73].

This expression for $\mathbf{B}$ can be used directly for a gravitomagnetic dipole field. The factor μ_0 comes from the current density used in the derivation of the dipole field. In our case, it has to be replaced with

$$\mu_0 \rightarrow -\frac{4\pi G}{c^2}, \tag{7.97}$$

so that the gravitomagnetic field in dipole approximation is

$$\mathbf{\Omega} = -\frac{G}{c^2}\left(\frac{3\mathbf{n}\,(\mathbf{n}\cdot\mathbf{m}_m) - \mathbf{m}_m}{|\mathbf{r}|^3} + \frac{8\pi}{3}\mathbf{m}_m\delta(\mathbf{r})\right). \tag{7.98}$$

The term with the δ function can be omitted if only the field outside of the origin is considered. At the Earth's position, its order of magnitude is $1.3\cdot 10^{-25}$/s. Graphical examples of dipole fields can be found in [81]. ■

7.1.4 Gravitational waves

Physical waves of any kind can be described as solutions of their respective wave equations. The wave equations of dynamics are derived analogously to the wave equations of electrodynamics. We have already used this method of analogy several times. We start with the gravitomagnetic and Ampère-Maxwell laws of dynamics, Eqs. (7.56, 7.58):

$$\frac{\partial\mathbf{\Omega}}{\partial t} + \nabla\times\mathbf{g} = \mathbf{0}, \tag{7.99}$$

$$-\frac{1}{c^2}\frac{\partial\mathbf{g}}{\partial t} + \nabla\times\mathbf{\Omega} = -\frac{4\pi G}{c^2}\mathbf{J}_m, \tag{7.100}$$

and apply the same procedure as in Section 6.2.4. We take the curl of the gravitomagnetic law:

$$\nabla\times\frac{\partial\mathbf{\Omega}}{\partial t} = -\nabla\times\nabla\times\mathbf{g}, \tag{7.101}$$

and the time derivative of the Ampère-Maxwell law:

$$-\frac{1}{c^2}\frac{\partial^2 \mathbf{g}}{\partial t^2} + \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{\Omega}}{\partial t} = -\frac{4\pi G}{c^2}\frac{\partial \mathbf{J}_m}{\partial t}. \tag{7.102}$$

Inserting (7.101) into (7.102) gives

$$-\frac{1}{c^2}\frac{\partial^2 \mathbf{g}}{\partial t^2} - \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{g} = -\frac{4\pi G}{c^2}\frac{\partial \mathbf{J}_m}{\partial t}. \tag{7.103}$$

Replacing the double curl and reversing the sign of the equation leads to

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{g}}{\partial t^2} - \nabla^2 \mathbf{g} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{g}) = \frac{4\pi G}{c^2}\frac{\partial \mathbf{J}_m}{\partial t}. \tag{7.104}$$

Assuming that no independent mass density is present, the divergence of **g** vanishes, and we obtain

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{g}}{\partial t^2} - \nabla^2 \mathbf{g} = \frac{4\pi G}{c^2}\frac{\partial \mathbf{J}_m}{\partial t}, \tag{7.105}$$

which is the inhomogeneous wave equation. With the d'Alembert operator (2.253), it can be written in compact form:

$$\Box\, \mathbf{g} = \frac{4\pi G}{c^2}\frac{\partial \mathbf{J}_m}{\partial t}. \tag{7.106}$$

Applying the analogous procedure for the gravitomagnetic field gives the wave equation

$$\Box\, \boldsymbol{\Omega} = -\frac{4\pi G}{c^2}\boldsymbol{\nabla} \times \mathbf{J}_m. \tag{7.107}$$

The electric counterpart of (7.106), which is derivable in the same way, is

$$\Box\, \mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}}{\partial t}. \tag{7.108}$$

Solutions of the wave equations are, for example, plane waves. By comparing Eqs. (7.106) and (7.108), we see that gravitational waves obey the same laws and should show equivalent behavior.

These results allow us to answer the question of why it is so difficult to find gravitational waves experimentally. The answer can be found by comparing the numerical values of the factors for the currents in Eqs. (7.106) and (7.108). For gravitational waves, we have

$$\frac{4\pi G}{c^2} = 9.332 \cdot 10^{-27}\,\frac{\mathrm{m}}{\mathrm{kg}} \tag{7.109}$$

while, for the electromagnetic case, we have

$$\mu_0 = 4\pi \cdot 10^{-7}\,\frac{\mathrm{Vs}}{\mathrm{Am}} = 1.257 \cdot 10^{-6}\,\frac{\mathrm{Vs}}{\mathrm{Am}}. \tag{7.110}$$

Comparing the numerical values shows that there is a difference of 21 orders of magnitude!

The gravitational red shift, derived for electromagnetic plane waves in Example 5.2, can also be considered to be an effect of gravitational waves. The optical refraction index, according to Eq. (5.56), is

$$n^2 = \varepsilon_r \mu_r. \tag{7.111}$$

We can define a gravitational refraction index analogously:

$$n_m^2 = \varepsilon_m \mu_m. \tag{7.112}$$

Then, for example, the gravitomagnetic equation becomes

$$\frac{\partial \mathbf{\Omega}^a}{\partial t} + n_m^2 \, \boldsymbol{\nabla} \times \mathbf{g}_0^a = \mathbf{0}. \tag{7.113}$$

In Einsteinian general relativity, gravitational waves are difficult to describe. They follow from a linear approximation of the field equations and require – at least – a quadrupole moment as a source, because there are no waves emanating from a dipole. This is completely different from the results of ECE theory, wherein gravitational waves can arise from varying currents of any kind. Considering the formal equivalence between electromagnetic and gravitational waves, we can say that each electromagnetic wave is connected with a gravitational wave of the same form. Due to the large difference in factors, however, the gravitational counterpart is normally not detectable. This unified behavior should be observable in the universe when huge events, like a collision of heavy astronomical objects, lead to detectable gravitational waves directly. An electromagnetic pulse should then be detectable as a simultaneous event.

Gravitational waves are oscillations of spacetime itself, an effect of the curving and spinning of spacetime, as described by general relativity. They cannot be described by classical theory, because it is limited to Newton's gravitational law.

7.2 Generally covariant dynamics

Newton's laws, which introduced dynamics to physics, hold for linear motion. If rotational motion is considered, additional forces like the centrifugal force and the Coriolis force occur. In this section, we show that these are examples of spin connections, in a framework of generally covariant dynamics [75, 76].

7.2.1 Velocity

In classical dynamics, the velocity vector $\mathbf{v}$ is the time derivative of the position vector $\mathbf{r}$:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{7.114}$$

The natural way to extend this law to make it generally covariant is by using the covariant derivative as it appears, for example, in the first Maurer-Cartan structure equation. The torsion 2-form is defined as the derivative of the tetrad (see Eq. (2.283)):

$$T^a = D \wedge q^a. \tag{7.115}$$

In an analogous way, we define the covariant velocity to be

$$v^a = c(D \wedge r^a). \tag{7.116}$$

The factor c (velocity of light in vacuo) is included to get the right units, namely m/s. Eq. (7.116) is an equation of 2-forms, which in detail is

$$(v^a)_{\mu\nu} = c(D \wedge r^a)_{\mu\nu}, \tag{7.117}$$

so the covariant velocity is a two-index tensor of the base manifold. The position vector can be defined to be a tetrad of position, with the constant $r^{(0)}$ being a scaling factor with units of length:

$$r^a{}_\mu = r^{(0)} q^a{}_\mu. \tag{7.118}$$

Eq. (7.117), when written out in tensor form, analogously to Eq. (4.169), reads:

$$v^a{}_{\mu\nu} = c\left(\partial_\mu r^a{}_\nu - \partial_\nu r^a{}_\mu + \omega^a{}_{\mu b} r^b{}_\nu - \omega^a{}_{\nu b} r^b{}_\mu\right). \tag{7.119}$$

This is a valid equation of Cartan geometry. However, this tetrad is not the spacetime tetrad; therefore, the spin connection is not that of spacetime, but is specific to the velocity tensor.

As a result, we have the same situation as in electrodynamics or general dynamics. The field is an antisymmetric two-dimensional tensor, but we are used to dealing with field vectors. Therefore, we defined two types of field vectors as in Eq. (7.25), for example. According to this equation, the contravariant form of the velocity tensor can be written as

$$v^{a\mu\nu} = \begin{bmatrix} 0 & -v^{a1} & -v^{a2} & -v^{a3} \\ v^{a1} & 0 & -w^{a3} & w^{a2} \\ v^{a2} & w^{a3} & 0 & -w^{a1} \\ v^{a3} & -w^{a2} & w^{a1} & 0 \end{bmatrix} \tag{7.120}$$

with 4-velocities $v^{a\mu}$ and $w^{a\mu}$. The tensor, and thus the tensor elements, have a polarization index a of Cartan geometry, running from 0 to 3. As discussed earlier, the v elements are of translational character, while the w elements have a rotational character. They define velocity vectors $\mathbf{v}^a$ and $\mathbf{w}^a$ in the usual way. The upper indices 1, 2, 3 denote the space components. Another notation is

$$v^a{}_X = v^{a1}, \quad v^a{}_Y = v^{a2}, \quad \text{etc.} \tag{7.121}$$

Note that there is a sign change with lower indices:

$$v^a{}_X = -v^a{}_1, \quad v^a{}_Y = -v^a{}_2, \quad \text{etc.} \tag{7.122}$$

All of this has been discussed in detail in Chapter 4. To summarize, we have

$$\mathbf{v}^a = \begin{bmatrix} v^a{}_X \\ v^a{}_Y \\ v^a{}_Z \end{bmatrix} = \begin{bmatrix} v^{a1} \\ v^{a2} \\ v^{a3} \end{bmatrix} = \begin{bmatrix} -v^{a01} \\ -v^{a02} \\ -v^{a03} \end{bmatrix}, \tag{7.123}$$

$$\mathbf{w}^a = \begin{bmatrix} w^a{}_X \\ w^a{}_Y \\ w^a{}_Z \end{bmatrix} = \begin{bmatrix} w^{a1} \\ w^{a2} \\ w^{a3} \end{bmatrix} = \begin{bmatrix} -v^{a23} \\ v^{a13} \\ -v^{a12} \end{bmatrix}. \tag{7.124}$$

The spin connection has the vectorial form

$$\boldsymbol{\omega}^a{}_b = \begin{bmatrix} -\omega^a{}_{1b} \\ -\omega^a{}_{2b} \\ -\omega^a{}_{3b} \end{bmatrix} = \begin{bmatrix} \omega^{a1}{}_b \\ \omega^{a2}{}_b \\ \omega^{a3}{}_b \end{bmatrix}, \tag{7.125}$$

and the position 4-vector is, similarly,

$$(r^a)^\nu = \begin{bmatrix} r^{a0} \\ r^{a1} \\ r^{a2} \\ r^{a3} \end{bmatrix} = \begin{bmatrix} r^{a0} \\ \mathbf{r}^a \end{bmatrix}. \tag{7.126}$$

Eq. (7.119) can be written as two vector equations in the same way as Eqs. (7.38, 7.39):

$$\mathbf{v}^a = c\left(-\boldsymbol{\nabla} r^a{}_0 - \frac{1}{c}\frac{\partial \mathbf{r}^a}{\partial t} - \omega^a{}_{0b}\mathbf{r}^b + \boldsymbol{\omega}^a{}_b r^a{}_0\right), \tag{7.127}$$

$$\mathbf{w}^a = c\left(\boldsymbol{\nabla} \times \mathbf{r}^a - \boldsymbol{\omega}^a{}_b \times \mathbf{r}^b\right). \tag{7.128}$$

The total velocity vector is

$$\mathbf{v}^a_{\text{tot}} = \mathbf{v}^a + \mathbf{w}^a. \tag{7.129}$$

At the present time, it is not clear how to interpret the potential-like time component $r^a{}_0$. In the case of a flat space, all spin connections vanish. Furthermore, there is no polarization, and if $\mathbf{r}$ is a trajectory of a mass point as in classical mechanics, then there is no "scalar" potential $r^a{}_0$ and no curl of $\mathbf{r}$. The translational velocity then becomes

$$\mathbf{v} = -\frac{\partial \mathbf{r}}{\partial t}. \tag{7.130}$$

The minus sign is a convention and can be avoided by defining the position tetrad (7.118) with a negative sign:

$$r^a{}_\mu = -r^{(0)} q^a{}_\mu. \tag{7.131}$$

For trajectories of mass points, there is no spatial field dependence, and the partial time derivative is equal to the total time derivative, so that we have in total:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \tag{7.132}$$

which is in agreement with (7.114).

Rotation of the coordinate system corresponds to a spin connection vector $\boldsymbol{\omega}^a{}_b$. This vector can be transformed into an angular velocity vector of classical mechanics by setting

$$c\boldsymbol{\omega}^a{}_b \rightarrow \boldsymbol{\omega}, \tag{7.133}$$

where $\boldsymbol{\omega}$ has the units of 1/s, as usual. Under the approach described above, we now obtain a rotational velocity vector

$$\mathbf{w} = \boldsymbol{\omega} \times \mathbf{r}, \tag{7.134}$$

and the velocity, observed in the rest frame, is

$$\mathbf{v}_{\text{tot}} = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}. \tag{7.135}$$

This equation is identical to the result of rotational motion in classical mechanics, as we will see later.

Both vectors $\mathbf{v}^a$ and $\mathbf{w}^a$ are space-like components of 4-vectors (in covariant representation):

$$v^a{}_\mu = (v^a{}_0, -\mathbf{v}^a), \tag{7.136}$$

$$w^a{}_\mu = (w^a{}_0, -\mathbf{w}^a). \tag{7.137}$$

The components $v^a{}_0$, $w^a{}_0$, are time-like and have a special meaning in dynamics, which will be discussed later. According to the relativistic energy-momentum relation without rest mass,

$$E_r = cp \tag{7.138}$$

(where E_r is the relativistic energy), the 4-momentum can be written in the following form:

$$p^a{}_\mu = (\frac{E_r}{c}, -\mathbf{p}^a). \tag{7.139}$$

The 4-velocity is connected with the 4-momentum by $p^a{}_\mu = mv^a{}_\mu$. In particular, it is $p^a{}_0 = mv^a{}_0 = E_r/c$.

7.2.2 Acceleration

We define acceleration as the covariant external derivative of the velocity, analogously to Eq. (7.116), for both velocities:

$$\mathbf{a}^a = c(D \wedge v^a), \tag{7.140}$$
$$\alpha^a = c(D \wedge w^a). \tag{7.141}$$

$\mathbf{a}^a$ is the translational acceleration, and α^a is the rotational acceleration. Both have an orbital part and a spin part, according to Eqs. (7.127, 7.128). This gives four vector equations, which are derived analogously to the velocities (7.127), (7.128):

$$\mathbf{a}^a_{\text{orbital}} = c\boldsymbol{\nabla} v^a{}_0 + \frac{\partial \mathbf{v}^a}{\partial t} + c\omega^a{}_{0b}\,\mathbf{v}^b - c\boldsymbol{\omega}^a{}_b v^a{}_0, \tag{7.142}$$
$$\mathbf{a}^a_{\text{spin}} = c\left(-\boldsymbol{\nabla} \times \mathbf{v}^a + \boldsymbol{\omega}^a{}_b \times \mathbf{v}^b\right), \tag{7.143}$$
$$\alpha^a_{\text{orbital}} = c\boldsymbol{\nabla} w^a{}_0 + \frac{\partial \mathbf{w}^a}{\partial t} + c\omega^a{}_{0b}\,\mathbf{w}^b - c\boldsymbol{\omega}^a{}_b w^a{}_0, \tag{7.144}$$
$$\alpha^a_{\text{spin}} = c\left(-\boldsymbol{\nabla} \times \mathbf{w}^a + \boldsymbol{\omega}^a{}_b \times \mathbf{w}^b\right). \tag{7.145}$$

The zero components of the velocities $v^a{}_0$ and $w^a{}_0$ are potentials, whose gradient gives mechanical forces; in particular, we can identify $v^a{}_0$ with the gravitational potential Φ^a:

$$\Phi^a = cv^a{}_0. \tag{7.146}$$

If the factor c is omitted from the spin accelerations, they have the units of angular velocities and correspond to the gravitomagnetic field. In ECE theory, velocity is a tensor, consisting of vector fields. For example, the equations of fluid dynamics are based on a velocity field. This will be explained later in this book.

Comparison with classical mechanics

The equations for velocity and acceleration that we have just discussed can be compared to the classical motion of mass points in a rotating coordinate system, which is a subject of classical (or Newtonian) mechanics. The basic explanations can be found, for example, in [74], and they are summarized and discussed briefly in [75]. Here we will review only the results.

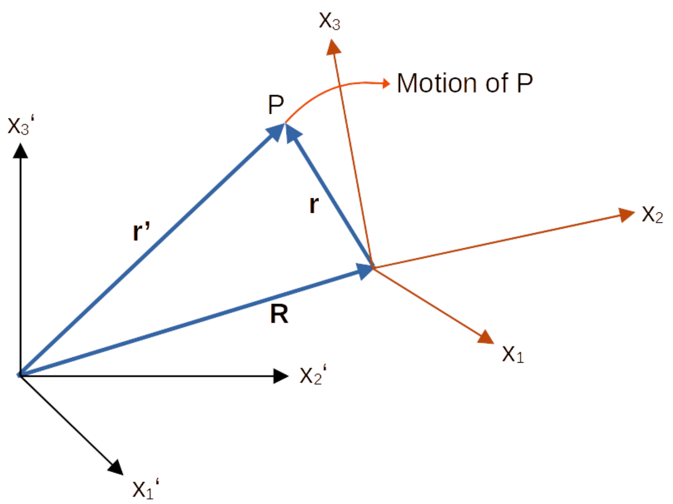


Figure 7.5: Rotating coordinate system.

In Fig. 7.5, a mass point P is drawn with respect to two coordinate systems, where the primed coordinates refer to the rest frame and the unprimed coordinates refer to the rotating frame. The origin of the rotating frame is displaced from the origin of the fixed frame by $\mathbf{R}$ and may also move in time with a velocity $\mathbf{V}$. The point P has the coordinate $\mathbf{r}'$ in the fixed frame and $\mathbf{r}$ in the rotating frame. Velocities are assigned to these coordinates using the following naming conventions:

$\mathbf{v}_f$ = Velocity relative to the fixed axes,
$\mathbf{v}_r$ = Velocity relative to the rotating axes,
$\boldsymbol{\omega}$ = Angular velocity relative to the rotating axes, and
$\mathbf{V}$ = Linear velocity of the moving origin.

The velocity in the fixed frame is

$$\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} \tag{7.147}$$

and is identical to Eq. (7.135), which was derived from ECE theory for a Minkowski space without curvature and torsion. The term $\boldsymbol{\omega} \times \mathbf{r}$ describes the rotation of P.

For acceleration, classical dynamics gives a more complicated result. With the total derivative d/dt being denoted by a dot, as usual, the relative acceleration between the frames alone is

$$\ddot{\mathbf{R}} = \frac{d\mathbf{V}}{dt} = \dot{\mathbf{V}}. \tag{7.148}$$

The total force acting on the mass point with mass m in the fixed frame is

$$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}} + m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \mathbf{v}_r. \tag{7.149}$$

The effective force acting on m is, therefore,

$$\mathbf{F}_{\text{eff}} = m\mathbf{a}_r = \mathbf{F} - m\ddot{\mathbf{R}} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r. \tag{7.150}$$

In this equation, three force terms appear that are a consequence of the rotation of the local frame of m. They are the force due to angular acceleration, the centrifugal force, and the Coriolis force (the latter appears only if the mass point moves in the rotating frame with a velocity $\mathbf{v}_r$):

$$\mathbf{F}_{\text{rot.frame}} = -m\dot{\boldsymbol{\omega}} \times \mathbf{r}, \tag{7.151}$$
$$\mathbf{F}_{\text{centrifugal}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \tag{7.152}$$
$$\mathbf{F}_{\text{Coriolis}} = -2m\boldsymbol{\omega} \times \mathbf{v}_r. \tag{7.153}$$

These forces are sometimes called "virtual forces", because they do not arise from a physical potential; but they are real, as we know from daily life. This representation arises solely from the attempt to extend the form of Newton's force law to a non-inertial system. In ECE theory, accelerations appear in the following way. The Coriolis force is part of Eq. (7.142) for $\mathbf{v}^b = \mathbf{v}_r$. When we insert Eq. (7.134) into Eq. (7.145), we obtain a term identical to the centrifugal force. The force of the rotating frame (7.151) appears as a part of Eq. (7.142), when we evaluate the time derivative

$$\frac{d\mathbf{v}_r}{dt} = \frac{d(\boldsymbol{\omega} \times \mathbf{r})}{dt} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_r. \tag{7.154}$$

This gives a Coriolis term, in addition to a term of the rotating frame.

We see that the ECE equations for acceleration (7.142 - 7.145) contain the terms of classical dynamics, plus many additional terms stemming from Cartan geometry of curving and twisting spacetime. It should be noticed that these equations are generally covariant, that they hold in any coordinate system. In contrast, this is not the case in Newtonian dynamics, in which one fixed frame and one moving/rotating frame are assumed. ECE theory does not distinguish between rest and moving frames, in accordance with the principle of relativity.

■ **Example 7.4** We compute the velocity and acceleration vectors in plane and spherical polar coordinates, and compare the result in spherical polar coordinates to that in the form of general covariance (i.e., a covariant derivative).

In two dimensions, the unit vectors of plane polar coordinates $(r,\ \phi)$ are

$$\mathbf{e}_r = \begin{bmatrix}\cos\phi \\ \sin\phi\end{bmatrix}, \quad \mathbf{e}_\phi = \begin{bmatrix}-\sin\phi \\ \cos\phi\end{bmatrix} \tag{7.155}$$

(also see Fig. 2.1). The position vector of any point in space is determined by the direction of its unit vector $\mathbf{e}_r$ and its length r:

$$\mathbf{r} = r\mathbf{e}_r. \tag{7.156}$$

The velocity vector is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{e}_r) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r. \tag{7.157}$$

Since the direction of $\mathbf{e}_r$ changes over time, this change has to be taken into account, in contrast to a cartesian coordinate system, where all unit vectors remain fixed. From Eqs. (7.155) it follows that

$$\dot{\mathbf{e}}_r = \dot{\phi}\mathbf{e}_\phi, \tag{7.158}$$

$$\dot{\mathbf{e}}_\phi = -\dot{\phi}\mathbf{e}_r, \tag{7.159}$$

and therefore:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi. \tag{7.160}$$

Similarly, the acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = \left(\ddot{r} - r\dot{\phi}^2\right)\mathbf{e}_r + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\mathbf{e}_\phi. \tag{7.161}$$

In three dimensions, we use a spherical coordinate system $(r,\ \theta,\ \phi)$, as illustrated in Fig. 2.3. The transformation of basis vectors has already been given in Eq. (2.32). Starting from unit vectors

$$\mathbf{e}_r = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{e}_\theta = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathbf{e}_\phi = \begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{7.162}$$

the cartesian unit vectors are the column vectors of the transformation matrix

$$(\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z) = \begin{pmatrix}\cos\phi\ \sin\theta & \sin\phi\ \sin\theta & \cos\theta \\ \cos\phi\ \cos\theta & \sin\phi\ \cos\theta & -\sin\theta \\ -\sin\phi & \cos\phi & 0\end{pmatrix} =: S. \tag{7.163}$$

In this example, we need the unit vectors of the spherical coordinate system, which are the column vectors of the inverse transformation:

$$(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi) = S^{-1} = S^T = \begin{pmatrix}\cos\phi\sin\theta & \cos\phi\cos\theta & -\sin\phi \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\phi \\ \cos\theta & -\sin\theta & 0\end{pmatrix}. \tag{7.164}$$

The position vector in a cartesian frame is

$$\mathbf{r} = X\mathbf{e}_X + Y\mathbf{e}_Y + Z\mathbf{e}_Z \tag{7.165}$$

with coordinate values, transformed to the cartesian frame, of

$$\begin{aligned} X &= r\sin\theta\cos\phi, \\ Y &= r\sin\theta\sin\phi, \\ Z &= r\cos\theta \end{aligned} \tag{7.166}$$

(see Eq. (2.29)). Then, the velocity in this frame is

$$\mathbf{v} = \dot{X}\mathbf{e}_X + \dot{Y}\mathbf{e}_Y + \dot{Z}\mathbf{e}_Z. \tag{7.167}$$

From (7.166), we have

$$\begin{aligned} \dot{X} &= \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi, \\ \dot{Y} &= \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi, \\ \dot{Z} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta. \end{aligned} \tag{7.168}$$

We insert these expressions into Eq. (7.167) and substitute the cartesian unit vectors from Eq. (7.163). The resulting lengthy calculation, which has been carried out by computer algebra (see computer algebra code [145]), arrives at the following result:

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_\phi\mathbf{e}_\phi \tag{7.169}$$

with

$$\begin{aligned} v_r &= \dot{r}, \\ v_\theta &= r\dot{\theta}, \\ v_\phi &= r\dot{\phi}\sin\theta. \end{aligned} \tag{7.170}$$

The acceleration vector in the cartesian frame is

$$\mathbf{a} = \ddot{X}\mathbf{e}_X + \ddot{Y}\mathbf{e}_Y + \ddot{Z}\mathbf{e}_Z. \tag{7.171}$$

In the same way, with the aid of computer algebra, an even more complicated calculation gives

$$\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi \tag{7.172}$$

with

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta, \\ a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta, \\ a_\phi &= (r\ddot{\phi} + 2\dot{r}\dot{\phi})\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta. \end{aligned} \tag{7.173}$$

Next, we try to write the acceleration in the form of a covariant derivative. As can be seen from Eqs. (7.172, 7.173), the acceleration is the partial derivative of the velocity plus additional terms, which can be written as a matrix-vector product with a matrix $\mathbf{\Omega}$:

$$\mathbf{a} = \frac{D}{Dt}\mathbf{v} = \frac{\partial}{\partial t}\mathbf{v} + \mathbf{\Omega}\mathbf{v}. \tag{7.174}$$

By insertion, it can be verified that this equation reads

$$\frac{D}{Dt}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\dot{\phi}\sin\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\dot{\phi}\sin\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} & -\dot{\phi}\sin\theta \\ \dot{\theta} & 0 & -\dot{\phi}\cos\theta \\ \dot{\phi}\sin\theta & \dot{\phi}\cos\theta & 0 \end{bmatrix}\begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ r\dot{\phi}\sin\theta \end{bmatrix}, \tag{7.175}$$

from which the form of $\boldsymbol{\Omega}$ can be derived. It is an antisymmetric matrix of the form

$$\boldsymbol{\Omega}=\begin{bmatrix}0 & \Omega_{12} & \Omega_{13}\\ -\Omega_{12} & 0 & \Omega_{23}\\ -\Omega_{13} & -\Omega_{23} & 0\end{bmatrix}=\begin{bmatrix}0 & -\dot{\theta} & -\dot{\phi}\sin\theta\\ \dot{\theta} & 0 & -\dot{\phi}\cos\theta\\ \dot{\phi}\sin\theta & \dot{\phi}\cos\theta & 0\end{bmatrix}. \tag{7.176}$$

So far, we have shown that the acceleration in the spherical polar coordinate system has the form of a covariant derivative (see Eq. (7.174)). However, instead of the matrix-vector operation $\boldsymbol{\Omega}\mathbf{v}$, comparison with the form of spin accelerations (7.143, 7.145) leads us to expect a vector product of the form $\boldsymbol{\omega}\times\mathbf{v}$:

$$\frac{D}{Dt}\mathbf{v}=\frac{\partial}{\partial t}\mathbf{v}+\boldsymbol{\omega}\times\mathbf{v}. \tag{7.177}$$

This implies that

$$\boldsymbol{\omega}\times\mathbf{v}=\boldsymbol{\Omega}\mathbf{v}, \tag{7.178}$$

which gives a linear equation system for determining the components of $\boldsymbol{\omega}$, when written out:

$$\begin{bmatrix}\omega_2 v_3-\omega_3 v_2\\ \omega_3 v_1-\omega_1 v_3\\ \omega_1 v_2-\omega_2 v_1\end{bmatrix}=\begin{bmatrix}\Omega_{13} v_3+\Omega_{12} v_2\\ \Omega_{23} v_3-\Omega_{12} v_1\\ -\Omega_{23} v_2-\Omega_{13} v_1\end{bmatrix}. \tag{7.179}$$

The computer algebra code shows that this equation system is of rank 2, i.e., it is underdetermined. The general solution is

$$\begin{aligned}\omega_1 &= -\frac{\Omega_{23}v_3-(\Omega_{12}+C_1)\,v_1}{v_3},\\ \omega_2 &= \frac{\Omega_{13}v_3+(\Omega_{12}+C_1)\,v_2}{v_3},\\ \omega_3 &= C_1,\end{aligned} \tag{7.180}$$

with an arbitrary constant C_1. We can set its value to $C_1=-\Omega_{12}$ so that the result becomes as simple as possible:

$$\begin{aligned}\omega_1 &= -\Omega_{23},\\ \omega_2 &= \Omega_{13},\\ \omega_3 &= -\Omega_{12}.\end{aligned} \tag{7.181}$$

Inserting this solution into Eq. (7.177) then gives the desired form of the covariant equation with the cross product:

$$\frac{D}{Dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\dot{\phi}\sin\theta\end{bmatrix}=\frac{\partial}{\partial t}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\dot{\phi}\sin\theta\end{bmatrix}+\begin{bmatrix}\dot{\phi}\cos\theta\\ -\dot{\phi}\sin\theta\\ \dot{\theta}\end{bmatrix}\times\begin{bmatrix}\dot{r}\\ r\dot{\theta}\\ r\dot{\phi}\sin\theta\end{bmatrix}. \tag{7.182}$$

$\boldsymbol{\omega}$ is the spin connection vector for acceleration. ■

7.2.3 Angular momentum and torque

In a fixed frame, the classical torque $\mathbf{N}$ is the time derivative of the angular momentum $\mathbf{L}$:

$$\mathbf{N}=\frac{d\mathbf{L}}{dt}=\dot{\mathbf{L}}. \tag{7.183}$$

In presence of a rotating frame, a term with the angular velocity vector is to be added:

$$\mathbf{N}_{\text{fixed}} = \dot{\mathbf{L}}_{\text{fixed}} + \boldsymbol{\omega} \times \mathbf{L}. \tag{7.184}$$

This equation holds for any vector considered in both frames, and not just for angular momentum [74]. It is close to the definition of the first Maurer-Cartan structure equation (2.283),

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b, \tag{7.185}$$

where the term $\omega^a{}_b \wedge q^b$ originates from the spinning of spacetime. We can define a generally covariant angular momentum 1-form J^a and torque 2-form N^a by

$$J^a{}_\mu := -J^{(0)} q^a{}_\mu, \tag{7.186}$$

$$N^a{}_{\mu\nu} := c J^{(0)} T^a{}_{\mu\nu}\,, \tag{7.187}$$

where T^a is the torsion form. A sign change was introduced so that it would conform with the classical result. This is analogous to Eqs. (7.118) and (7.116). As in Eq. (7.117), it then follows that

$$(N^a)_{\mu\nu} = c(D \wedge J^a)_{\mu\nu}, \tag{7.188}$$

which can be developed in two vectors of torque as before, a translational torque $\mathbf{N}^a_{\text{orbital}}$ and an intrinsic or spin torque $\mathbf{N}^a_{\text{spin}}$:

$$\mathbf{N}^a_{\text{orbital}} = c\boldsymbol{\nabla} J^a{}_0 + \frac{\partial \mathbf{J}^a}{\partial t} + c\omega^a{}_{0b}\mathbf{J}^b - c\boldsymbol{\omega}^a{}_b J^a{}_0, \tag{7.189}$$

$$\mathbf{N}^a_{\text{spin}} = c\left(-\boldsymbol{\nabla} \times \mathbf{J}^a + \boldsymbol{\omega}^a{}_b \times \mathbf{J}^b\right). \tag{7.190}$$

In a purely translational spacetime, the intrinsic torque vanishes. The classical limit of these equations is

$$\mathbf{N}^a = \mathbf{N}^a_{\text{orbital}} + \mathbf{N}^a_{\text{spin}} \quad \rightarrow \quad \frac{\partial \mathbf{J}^a}{\partial t} + c\boldsymbol{\omega}^a{}_b \times \mathbf{J}^b, \tag{7.191}$$

where the polarization indices have to be reduced to $a = b = 1$, as before. Myron Evans writes [75]:

ECE theory has a great deal more inherent information than the classical and non-relativistic Euler theory. The task is to reveal such information experimentally, using high accuracy experiments in the laboratory or in astronomy. These would amount to rigorous experimental tests of Einsteinian philosophy itself, because ECE theory completes the Einstein-Hilbert theory of 1916. They would therefore be important experiments.

7.2.4 Equivalence principle

Newton introduced the law that gravitational mass and inertial mass are identical. This is called the *equivalence principle* and has been proven experimentally to high precision. Here we derive this principle from ECE theory. In Newtonian theory, the equivalence principle says that the dynamical force of acceleration on a mass is equal to the force of the gravitational field:

$$m\mathbf{a} = m\mathbf{g} = -m\boldsymbol{\nabla}\Phi, \tag{7.192}$$

where Φ is the gravitational potential. In ECE theory, we start with the orbital acceleration (7.142):

$$\mathbf{a}^a_{\text{orbital}} = c\boldsymbol{\nabla} v^a{}_0 + \frac{\partial \mathbf{v}^a}{\partial t} + c\omega^a{}_{0b}\mathbf{v}^b - c\boldsymbol{\omega}^a{}_b v^a{}_0. \tag{7.193}$$

This equation is derived from the first Maurer-Cartan structure equation, and therefore the acceleration form can be written as a multiple of a torsion form [77]:

$$a^a{}_{\mu\nu} = a^{(0)}T^a{}_{\mu\nu} = cv^{(0)}T^a{}_{\mu\nu} \,. \tag{7.194}$$

For torsion, the antisymmetry laws were derived in Section 5.1, and we can transform Eq. (5.13),

$$-\frac{1}{c}\frac{\partial \mathbf{A}^a}{\partial t} + \boldsymbol{\nabla} A^a{}_0 - \omega^a{}_{0b}\mathbf{A}^b - \boldsymbol{\omega}^a{}_b A^b{}_0 = \mathbf{0}, \tag{7.195}$$

into the equivalent equation of dynamics, directly. By substituting $\mathbf{A}^a \to \mathbf{v}^a$, $cv^a{}_0 \to \Phi$, we obtain

$$-\frac{\partial \mathbf{v}^a}{\partial t} + \boldsymbol{\nabla}\Phi^a - c\omega^a{}_{0b}\mathbf{v}^b - \boldsymbol{\omega}^a{}_b\Phi^b = \mathbf{0}. \tag{7.196}$$

For the flat space of classical mechanics, the spin connections vanish, and this equation reads:

$$m\frac{\partial \mathbf{v}^a}{\partial t} = m\boldsymbol{\nabla}\Phi^a. \tag{7.197}$$

This is the equivalence principle of classical mechanics (with a sign convention change). Alternatively, when it is assumed that the orbital acceleration (i.e., the sum of all forces) in (7.193) vanishes, we get the equation:

$$\mathbf{a}^a_{\text{orbital}} = \mathbf{0}. \tag{7.198}$$

In this case, even the sign of the potential comes out in the usual form:

$$m\frac{\partial \mathbf{v}^a}{\partial t} = -m\boldsymbol{\nabla}\Phi^a. \tag{7.199}$$

If no additional forces are present (no rotational forces, for example), a mass moves in a force-free equilibrium. Examples are free fall or satellites, which are in “free fall around the Earth”.

In ECE theory, the equivalence principle is a consequence of the theory, while Newton had to introduce it as an axiom. In Einsteinian general relativity, the equivalence principle exists in two forms, weak and strong. According to the *weak equivalence principle*, the mass of a body alone (i.e., the measure of its inertia) determines which force of gravity acts on it in a given homogeneous gravitational field. Its other properties, such as chemical composition, size, shape, etc., have no influence.

According to the *strong equivalence principle*, gravitational and inertial forces are equivalent on small distance and time scales, in the sense that their effects cannot be distinguished from each other by mechanical means or any other form of observation. The weak equivalence principle follows from the strong equivalence principle, which is founded on the fact that the Einsteinian Lagrange density is independent of the choice of coordinate system.

7.3 Lagrange theory

In classical mechanics, Lagrange theory plays a very important role in the computation of the dynamics of mass points. It is based on Newtonian mechanics with the extensions of Euler for rotational motion, so it is applicable to any “machine” that can be reduced to the motion of mass points with constraints. We only give a short overview here. Relativistic extensions of this theory will be developed later in this book. The basics of Lagrange theory can be found in textbooks on mechanics, for example [78].

Lagrange theory uses generalized coordinates, denoted by q_i. These can be length and angular coordinates, connected with linear and angular momenta. Constraints of motion are used to reduce the number of coordinates so that only independent coordinates and their momenta remain, whose number is identical to the degree of freedom of a mechanical system. There is no general method for finding these coordinates, so this becomes a task for the modeler. The equations of motion are obtained from the *Lagrange function* or *Lagrangian*, which is the difference between kinetic energy T and potential energy U:

$$\mathscr{L} = T - U. \tag{7.200}$$

Both energies have to be expressed by generalized coordinates q_i. The kinetic energy will become a complicated expression, in most cases. Therefore, it is advisable to write the kinetic energy in cartesian coordinates,

$$T = \frac{1}{2}\sum_i m_i \dot{x}_i^2, \tag{7.201}$$

and insert the coordinate transformations $x_i(q_j),\ i,j = 1,\ \ldots,\ N$ so that an expression for $T(q_i, \dot{q}_i)$ is produced. This expression will be suitable for use in the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \frac{\partial \mathscr{L}}{\partial q_i} = 0. \tag{7.202}$$

These are N equations for a system with degree of freedom N. They lead to differential equations of motion with time derivatives of second order and can be transformed into $2N$ equations of first order by replacing each time derivative of q_i with a new variable v_i:

$$v_i = \dot{q}_i. \tag{7.203}$$

The v_i are handled as independent variables, as is done in the Hamilton equations, for example. This is beneficial for numerical solution of the equations of motion, because numerical solvers like Runge-Kutta are designed for solving ordinary differential equations of first order only.

The Euler-Lagrange theory was developed from the principle of energy conservation. Therefore, energy is conserved for all solutions. It is possible, however, to introduce external forces. Then energy is not conserved but added or extracted by these forces. They are called *generalized forces* because they can be ordinary forces or torques, depending on the type of coordinates with which they are connected. When denoted by Q_i, the Lagrange equations take the generalized form

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \frac{\partial \mathscr{L}}{\partial q_i} = Q_i. \tag{7.204}$$

Terms that remain conserved during motion can be derived from the Lagrange equations. If the time derivative of the Lagrangian with respect to a variable $\dot{q}_i$ vanishes, it follows that

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = 0 \quad \rightarrow \quad \frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \text{const.} \tag{7.205}$$

This describes *constants of motion*. Normally, these are linear momenta and angular momenta. Although Newtonian theory says that linear momenta in free space are always conserved, this is not the case for machines when, for example, only angular coordinates are present. There is, however, no guarantee that all constants of motion of a system will be found, because there is a freedom of choice for coordinates, and any choice can inadvertently veil some constants of motion.

■ **Example 7.5** As an example of Lagrange theory in classical mechanics, we consider the dynamics of a gyroscope. Although this needs to be described as a rigid body and not by mass points alone, it still belongs to the class of Lagrangian mechanics. However, there are two problems. First, the equations of motion are so complicated that it is cumbersome to derive them by hand. Therefore, the full equation set (for all coordinates), for a complete motion in three dimensions, is nowhere to be found in textbooks. Second, analytical solutions are not known even for subsets of the equations. Therefore, no analytical solutions exist for the general equation set, which is described in this example. The equations have to be solved numerically on a computer. With the aid of computer algebra and numerical solution methods, we were able to determine the full equation set, and present solutions without approximations (see computer algebra code [146]).

We compute the motion of a symmetric top with one point fixed (for example, a spinning top on a table), by using a Lagrangian formulation based on ECE2 theory [79]. The basics, which are very extensive, are described in detail in [80]. In particular, there is a detailed description of how rigid bodies can be modeled by point mechanics. Rigid bodies are characterized by their moments of inertia for rotations about their major body axes. A symmetric spinning top has two such moments of inertia, one for rotation around the vertical axis and two identical moments for rotations about the axes perpendicular to the vertical axis. They are designated I_3 and $I_1 = I_2$. As explained in Section 7.2, the motion in the coordinate system centered on the body has to be transformed to the observer's coordinate system. For the gyroscope, this is accomplished by introducing Eulerian angles (see Fig. 7.6).

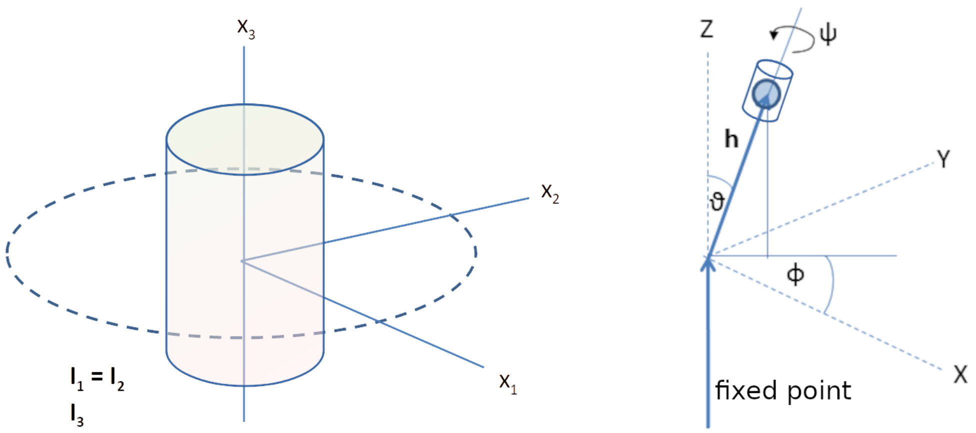


Figure 7.6: Rotational axes and Eulerian angles of a gyroscope with one point fixed.

The fixed point is assumed to be the center of a coordinate system consisting of three Eulerian angles θ, ϕ, ψ. The latter describes the rotation around the Z axis x_3 of the spinning top. θ and ϕ are identical to angles of a spherical coordinate system (polar and azimuthal angles, see Fig. 2.3). ψ is the rotation angle around the x_3 body axis. The spinning top exhibits a rotation around the Z axis by the angle ϕ, which is called the *precession angle*. In addition, there is a "nodding" described by θ, the *nutation angle*.

According to the Lagrange calculus, the body coordinates are to be transformed to the (θ, ϕ, ψ) coordinate system. The kinetic energy is then purely rotational:

$$T_{rot} = \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin(\theta)^2 + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\phi} \cos(\theta) + \dot{\psi} \right)^2, \tag{7.206}$$

where $I_{12} = I_1 = I_2$ and I_3 are the moments of inertia around the three principle axes (for details see [80]). The potential energy is defined by the gravitational field at the Earth's surface:

$$U = m g Z = m g h \cos(\theta) \tag{7.207}$$

with constant gravitational acceleration g, and mass m of the gyroscope. The Lagrangian is

$$\begin{aligned}\mathscr{L} &= T_{rot} - U \qquad (7.208)\\ &= \frac{1}{2} I_{12} \left(\dot{\phi}^2 \sin(\theta)^2 + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\dot{\phi}\cos(\theta) + \dot{\psi}\right)^2 - m g h \cos(\theta).\end{aligned}$$

The three Euler-Lagrange equations for the angular coordinates q_j,

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathscr{L}}{\partial q_j} = 0, \qquad (7.209)$$

lead to three equations containing first and second time derivatives of the angular coordinates, which can be rearranged to give the ordinary differential equation system:

$$\ddot{\theta} = \frac{\left((I_{12} - I_3)\,\dot{\phi}^2 \cos(\theta) - I_3\,\dot{\phi}\,\dot{\psi} + mgh\right)\sin(\theta)}{I_{12}}, \qquad (7.210)$$

$$\ddot{\phi} = -\frac{\left((2I_{12} - I_3)\,\dot{\phi}\cos(\theta) - I_3\,\dot{\psi}\right)\dot{\theta}}{I_{12}\sin(\theta)}, \qquad (7.211)$$

$$\ddot{\psi} = \frac{\left((I_{12} - I_3)\,\dot{\phi}\cos(\theta)^2 + I_{12}\dot{\phi} - I_3\,\dot{\psi}\cos(\theta)\right)\dot{\theta}}{I_{12}\sin(\theta)}. \qquad (7.212)$$

These equations can be solved numerically, in principle. Fortunately, there is additional information present in the Lagrange equations (7.209). There are two constants of motion, representing the angular momenta around the Z axis and the body axis:

$$L_\phi = I_{12}\,\dot{\phi}\sin(\theta)^2 + I_3\cos(\theta)\left(\dot{\phi}\cos(\theta) + \dot{\psi}\right), \qquad (7.213)$$

$$L_\psi = I_3\left(\dot{\phi}\cos(\theta) + \dot{\psi}\right). \qquad (7.214)$$

These equations contain only the first time derivatives of ϕ and ψ. Using these equations instead of Eqs. (7.211, 7.212) leads to the simpler differential equation system:

$$\ddot{\theta} = \frac{\left((I_{12} - I_3)\,\dot{\phi}^2 \cos(\theta) - I_3\,\dot{\phi}\,\dot{\psi} + mgh\right)\sin(\theta)}{I_{12}}, \qquad (7.215)$$

$$\dot{\phi} = \frac{L_\phi - L_\psi\cos(\theta)}{I_{12}\sin(\theta)^2}, \qquad (7.216)$$

$$\dot{\psi} = \frac{L_\psi - I_3\,\dot{\phi}\cos(\theta)}{I_3}. \qquad (7.217)$$

The constants L_ϕ and L_ψ have to be chosen appropriately for a solution. Even for this simpler equation system, a numerical solution procedure is required. The difference from Eqs. (7.211, 7.212) is that no initial conditions for $\dot{\phi}$ and $\dot{\psi}$ are required.

Equations (7.215 - 7.217) have been solved numerically in [146]. In Figs. 7.7 and 7.8, the 3D motion of the center of mass is plotted for different parameter settings. Precession is always present, and nutation may be either periodic (Fig. 7.7), or superimposed with a back-and-forth precession that leads to a spiraling motion (Fig. 7.8).

External forces can be applied to a gyroscope in the form of additional torques upon the rotation axes. Then, the constants of motion are no longer valid, and one has to apply the original equation set (7.210 - 7.212). With external torques Q_θ, Q_ϕ, Q_ψ, the equations read:

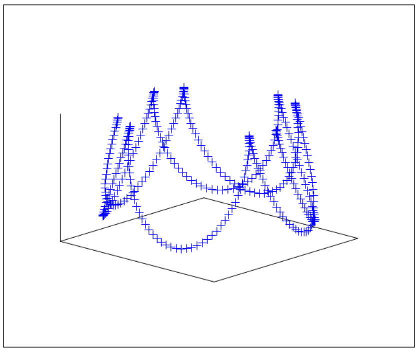


Figure 7.7: Precession of the center of mass for a gyroscope (spikes).

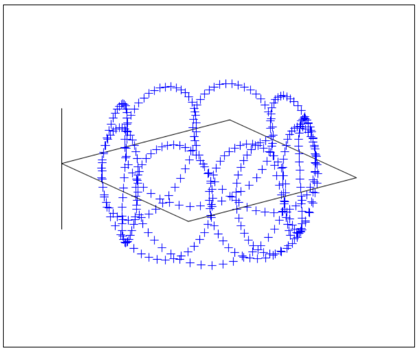


Figure 7.8: Precession of the center of mass for a gyroscope (spiraling).

$$\ddot{\theta} = \frac{\left((I_{12}-I_3)\,\dot{\phi}^2\cos(\theta) - I_3\,\dot{\phi}\,\dot{\psi} + mgh\right)\sin(\theta)}{I_{12}} + \frac{Q_\theta}{I_{12}}, \tag{7.218}$$

$$\ddot{\phi} = -\frac{\left((2I_{12}-I_3)\,\dot{\phi}\cos(\theta) - I_3\,\dot{\psi}\right)\dot{\theta}}{I_{12}\sin(\theta)} + \frac{Q_\phi - Q_\psi\cos(\theta)}{I_{12}\sin(\theta)^2}, \tag{7.219}$$

$$\ddot{\psi} = \frac{\left((I_{12}-I_3)\,\dot{\phi}\cos(\theta)^2 + I_{12}\dot{\phi} - I_3\,\dot{\psi}\cos(\theta)\right)\dot{\theta}}{I_{12}\sin(\theta)} + \frac{\left(I_{12}\sin(\theta)^2 + I_3\cos(\theta)^2\right)Q_\psi - I_3\,Q_\phi\cos(\theta)}{I_{12}I_3\sin(\theta)^2}, \tag{7.220}$$

as derived using computer algebra code [147]. It can be seen that the ϕ and ψ coordinates are coupled with both Q_ϕ and Q_ψ. For example, when a gyroscope is driven on its body axis, this will have an effect on its precession and vice versa. The results for an external torque Q_ϕ around the Z axis are graphed in Figs. 7.9 and 7.10. The torque is negative, i.e., contrary to the "natural" direction of motion; therefore, the initial angular velocity $\dot{\phi}$ goes to zero in an oscillation (Fig. 7.9), and continues in a strong negative spike before recovering, but then continues to increase in negative angular velocity with oscillations. Due to the coupling, the spike is also present for $\dot{\psi}$. The effects are not so dramatic in the angular trajectories themselves (Fig. 7.10). The change in the direction of ϕ rotation is due mainly to the external torque. The oscillations in all three angles (and angular velocities) increase in frequency with increasing precessional speed.

The theory of gyroscopes can be extended further for one that is free-falling. For it, the typical precession and nutation effects vanish. It can even be proven that a falling gyroscope can move upward for a short period of time, if suitable initial conditions are given. There are points in time when the vertical acceleration is zero. At these points, the gyroscope can be handled like a weightless mass, as was shown in experiments by Laithwaite (see UFT Paper 369 [79] for details and equations of motion). An example of the vertical motion is graphed in Fig. 7.11. It can be seen that the gyroscope rises briefly at the beginning of motion. ■

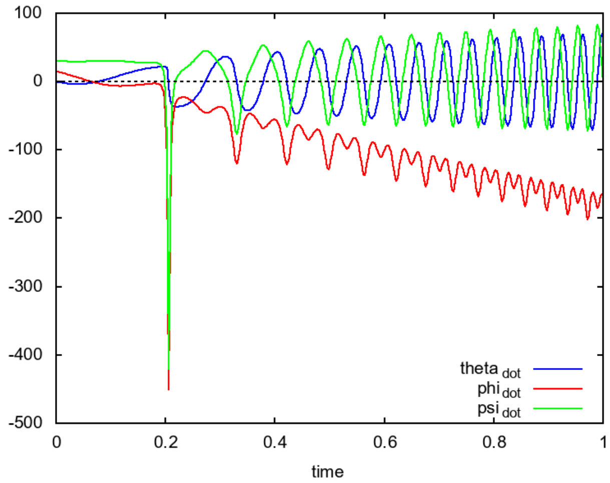


Figure 7.9: Time development of angular velocities for a gyroscope driven by Q_ϕ.

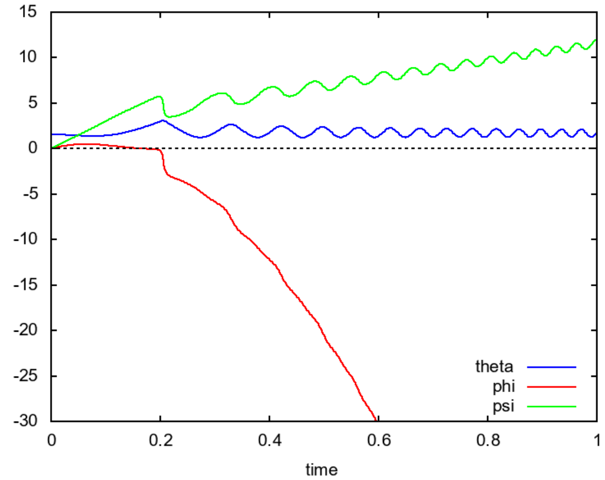


Figure 7.10: Time development of angles for a gyroscope driven by Q_ϕ.

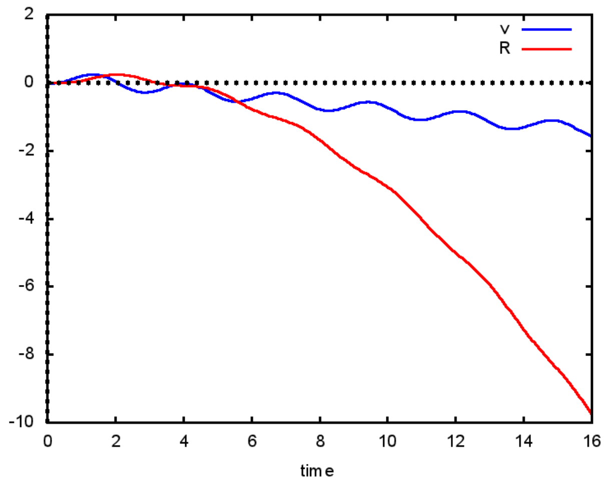


Figure 7.11: Example curve of vertical velocity $v(t)$ and vertical position $R(t)$ for the Laithwaite experiment.

8. Unified fluid dynamics

8.1 Classical fluid dynamics

We begin this chapter with a short overview of classical fluid dynamics, also called fluid mechanics. After that, the fluid dynamics structures of electromagnetism and dynamics will be described, and a unified view will be given.

8.1.1 Navier-Stokes equation

In the preceding chapter, we described the motion of particles by using Lagrangian mechanics. Particle dynamics is characterized by trajectories, i.e., the path of each particle in space, parametrized by time. When an ensemble of particles is to be described as a whole, a more appropriate approach is to consider all particles at the same instant of time. Fluids are thought to consist of small volume elements, which behave continuously. They are handled mathematically using a continuum model. The motion of each element is described by a velocity vector, and the vectors of all elements at one instant of time define the vector field of the velocity, which is dependent on space and time coordinates, as shown graphically in Fig. 8.1. Trajectories of single volume elements can be indicated by streamlines, which are also shown in Fig. 8.1.

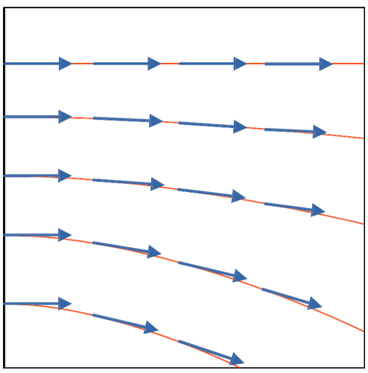


Figure 8.1: Velocity field.

For the mathematical description [82], consider a vector field $\mathbf{u}(X,\,Y,\,Z,\,t)$ with components

$$\mathbf{u} = \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix}. \tag{8.1}$$

Since this is a vector-valued function of several variables, the total time derivative is

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial X}\frac{\partial X}{\partial t} + \frac{\partial \mathbf{u}}{\partial Y}\frac{\partial Y}{\partial t} + \frac{\partial \mathbf{u}}{\partial Z}\frac{\partial Z}{\partial t}. \tag{8.2}$$

The components of the velocity field are

$$\mathbf{v} = \begin{bmatrix} \frac{\partial X}{\partial t} \\ \frac{\partial Y}{\partial t} \\ \frac{\partial Z}{\partial t} \end{bmatrix}. \tag{8.3}$$

Therefore, the total derivative of **u** can be written in operator form as

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \left(v_X \frac{\partial}{\partial X} + v_Y \frac{\partial}{\partial Y} + v_Z \frac{\partial}{\partial Z} \right) \mathbf{u} \tag{8.4}$$

or, in abbreviated form, as

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{u}. \tag{8.5}$$

The total time derivative, also known as the *material* or *convective derivative*, is defined by

$$\frac{D\mathbf{u}}{Dt} := \frac{d\mathbf{u}}{dt}. \tag{8.6}$$

When **u** is the velocity field itself, the material derivative becomes

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v}. \tag{8.7}$$

The equation of motion includes the mass of the moving body. In the case of a fluid, the mass is distributed over the definition volume V, and the total mass is

$$m = \int_V \rho dV. \tag{8.8}$$

The relevant component for the description of the fluid is therefore the local mass density $\rho(X,\,Y,\,Z,\,t)$.

The equation of motion for fluids can generally be written as

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \mathbf{f}, \tag{8.9}$$

where $\boldsymbol{\tau}$ is the stress tensor, and **f** represents external "volume forces". The stress tensor describes the interaction of single volume elements. By using elasticity theory, we can express the stress tensor through deformations. For practical purposes, a number of simplifications have to be made, for example, the fluid is assumed to be isotropic so that a viscosity constant μ can be introduced, as well as a scalar pressure p [82]. The resulting equation is called the *Navier-Stokes equation*:

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} p + \mu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{v}) \right) + \mathbf{f}. \tag{8.10}$$

The volume forces usually consist of the gravitational acceleration $\mathbf{g}$ multiplied by the fluid density:

$$\mathbf{f} = \rho \mathbf{g}. \tag{8.11}$$

By inserting this into the Navier-Stokes equation and splitting the material derivative into its constituents, we obtain

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} - \boldsymbol{\nabla} p + \mu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{v}) \right) + \rho \mathbf{g}. \tag{8.12}$$

The current density of mass transport is

$$\mathbf{J} = \rho \mathbf{v}, \tag{8.13}$$

and therefore the mechanical continuity equation (7.41) takes the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0. \tag{8.14}$$

If the mass density ρ is constant, it follows that

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{8.15}$$

i.e., the velocity field is divergenceless. Then, the Navier-Stokes equation (8.12) takes the simplified form

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} - \boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}. \tag{8.16}$$

This is the *incompressible Navier-Stokes equation*. By using the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) = -\nabla^2 \mathbf{v} + \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{v}), \tag{8.17}$$

it can be rewritten as

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} - \boldsymbol{\nabla} p - \mu \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \rho \mathbf{g}. \tag{8.18}$$

The term $\rho \mathbf{g}$ is constant. The double-curl term describes conservation of angular momentum, which becomes predominant in turbulent flows. Disregarding the time derivative leads to an equation for a stationary flow:

$$\rho (\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v} = -\boldsymbol{\nabla} p - \mu \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \rho \mathbf{g}. \tag{8.19}$$

The ratio between the material derivative and the double-curl term can be roughly estimated by considering their orders of magnitude,

$$|(\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{v}| \sim \frac{v^2}{a}, \tag{8.20}$$

$$|\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v})| \sim \frac{v}{a^2}, \tag{8.21}$$

where v is the modulus of the velocity and a is the linear dimension of an obstacle in the flow. Then, the ratio of the material derivative term to the rotational term in Eq. (8.19) is

$$\mathscr{R} := \frac{\rho a v}{\mu}, \tag{8.22}$$

which is dimensionless and is called the *Reynolds number*. In principle, the Reynolds number is a result of the solutions of the Navier-Stokes equation, but it has also been defined in slightly different ways. For high Reynolds numbers, the flow becomes turbulent and more difficult to predict. In this case, numerical calculations often have convergence problems and require an enormous effort.

8.1.2 Euler's equation

In a perfect fluid, there is no viscosity. This simplifies the stress tensor, and the equation of motion (8.9) then takes the form

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} p + \mathbf{f}, \tag{8.23}$$

which is called *Euler's equation*. This equation follows from the Navier-Stokes equation (8.10), when the viscosity is set to zero. This leads to high Reynolds numbers, so the described flow may become turbulent very easily. On the other hand, this frictionless model of fluids allows a wide analytical mathematical treatment of fluid dynamics, especially when it is restricted to two dimensions.

The material derivative term can be rewritten by using the identity

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) \tag{8.24}$$

(for a proof of the identity, see computer algebra code [148]). The material derivative then becomes

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla} v^2 - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}), \tag{8.25}$$

and Eq. (8.23) takes the form

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) - \frac{1}{2}\boldsymbol{\nabla} v^2 - \frac{1}{\rho}(\boldsymbol{\nabla} p - \mathbf{f})\,. \tag{8.26}$$

Based on this equation, it makes sense to define the *vorticity* $\mathbf{w}$ by

$$\mathbf{w} = \boldsymbol{\nabla}\times\mathbf{v}. \tag{8.27}$$

We can assume that the external force $\mathbf{f}$ is the gradient of a potential. Then the curl of the three last terms in Eq. (8.26) vanishes. Taking the curl of the complete equation gives

$$\boldsymbol{\nabla}\times\frac{\partial \mathbf{v}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})) \tag{8.28}$$

or, written with the vorticity,

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{w}). \tag{8.29}$$

This is called the *vorticity equation*. It can be shown [82, 148] that this equation, in the case of an incompressible flow, can be rewritten as

$$\frac{D\mathbf{w}}{Dt} = (\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{v}. \tag{8.30}$$

It is possible to reinstate the viscosity into the vorticity equation (8.29). The viscosity term $\mu\nabla^2\mathbf{v}$ from Eq. (8.16) can be added to Euler's equation (8.23). Then, Eq. (8.28) takes the expanded form

$$\boldsymbol{\nabla}\times\frac{\partial \mathbf{v}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})) + \frac{\mu}{\rho}\boldsymbol{\nabla}\times(\nabla^2\mathbf{v}). \tag{8.31}$$

The ratio μ/ρ can be replaced with the Reynolds number $\mathscr{R}$ through a rescaling of variables [83]. By assuming an incompressible fluid, we obtain the vorticity equation with turbulence:

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{w}) + \frac{1}{\mathscr{R}}\nabla^2\mathbf{w}. \tag{8.32}$$

The *circulation* is defined as a line integral over the velocity for a closed path in the fluid:

$$\Gamma = \oint \mathbf{v}\, d\mathbf{s}. \tag{8.33}$$

For a perfect fluid with potential forces **f**, it can be shown that

$$\frac{D\Gamma}{Dt} = 0, \tag{8.34}$$

i.e., the circulation is a constant of motion. According Stokes' theorem, the circulation can be written as an integral over the enclosed surface (spanned by the line integral):

$$\Gamma = \int (\boldsymbol{\nabla} \times \mathbf{v})_n\, dA = \int w_n\, dA, \tag{8.35}$$

where the curl is taken over the surface and dA is the infinitesimal area element. The index n indicates the projection of the corresponding vectors to the surface normal. A non-vanishing circulation means that the vector field **v** is non-conservative.

8.1.3 Potential flow

The following holds for an incompressible and vortex-free fluid:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{8.36}$$

$$\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}. \tag{8.37}$$

From the second condition, it follows that **v** can be written as a gradient of a scalar potential Φ (called the *velocity potential*):

$$\mathbf{v} = \boldsymbol{\nabla}\Phi. \tag{8.38}$$

Then, from (8.36), it follows directly that

$$\nabla^2\Phi = 0, \tag{8.39}$$

which is the Laplace equation. At the boundaries of the fluid, the flow must be tangential, i.e., the normal components have to vanish:

$$v_n = \frac{\partial \Phi}{\partial n} = 0. \tag{8.40}$$

Often, the only boundaries possible are $\Phi = \text{const.}$, i.e.,

$$\mathbf{v} = 0. \tag{8.41}$$

In two dimensions, in cartesian coordinates, the Laplace equation reads:

$$\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial Y^2} = 0. \tag{8.42}$$

Then, condition (8.37) is reduced to one equation:

$$\frac{\partial v_Y}{\partial X} - \frac{\partial v_X}{\partial Y} = 0, \tag{8.43}$$

and from (8.36) we obtain

$$\frac{\partial v_X}{\partial X}+\frac{\partial v_Y}{\partial Y}=0. \tag{8.44}$$

The existence of the velocity potential Φ follows from Eq. (8.43):

$$v_X=\frac{\partial\Phi}{\partial X},\quad v_Y=\frac{\partial\Phi}{\partial Y}. \tag{8.45}$$

Inserting these definitions into (8.44) gives the Laplace equation (8.42).

Eq. (8.43) allows the definition of a second potential, the *stream function* Ψ:

$$v_X=\frac{\partial\Psi}{\partial Y},\quad v_Y=-\frac{\partial\Psi}{\partial X}. \tag{8.46}$$

Inserting these definitions into Eq. (8.43) results in the Laplace equation for Ψ:

$$\frac{\partial^2\Psi}{\partial X^2}+\frac{\partial^2\Psi}{\partial Y^2}=0. \tag{8.47}$$

When both definitions of v_X, v_Y (8.45 and 8.46) are inserted into the gradients of Φ and Ψ, it follows that

$$\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Psi=0. \tag{8.48}$$

Since the gradients of these functions are perpendicular, the lines for $\Phi=$ const. and $\Psi=$ const. are perpendicular to each other. The lines with $\Phi=$ const. are called the *equipotential lines*, and the lines with $\Psi=$ const. are called the *stream lines*.

Simple application cases can be handled analytically in two dimensions by this theory. In three dimensions, this is nearly impossible and numerical calculations have to be performed. This field is called *Computational Fluid Dynamics* (CFD).

■ **Example 8.1** We compute a simple example of a two-dimensional potential flow (see computer algebra code [149]). The potential is defined by

$$\Phi=-X^2+Y^2-2XY, \tag{8.49}$$

resulting in

$$v_X=-2X-2Y,\quad v_Y=-2X+2Y. \tag{8.50}$$

We have verified that the Laplace equation for this potential is satisfied and that the divergence and curl of **v** vanish. The velocity field is graphed in Fig. 8.2, together with some stream lines. We see that the potential describes a fourfold symmetric flow, rotated against the coordinate axes. Near the coordinate origin, the flow goes to zero and there is no divergence. In this example, for simplicity, no boundary conditions have been defined. In Fig. 8.3, the equipotential lines are shown, together with the stream lines. It can be seen clearly that they are perpendicular to each other.

We include a second computation to demonstrate a turbulent flow (see computer algebra code [149]). We define the velocity field directly as

$$v_X=X+\frac{Y^2}{2},\quad v_Y=X^2-Y. \tag{8.51}$$

This field is divergenceless, but has a curl of $2X-Y$. In Fig. 8.4, the graph of the field lines shows that there is a center of rotation in the second quadrant. For such a velocity field, no scalar potential can be defined. ■

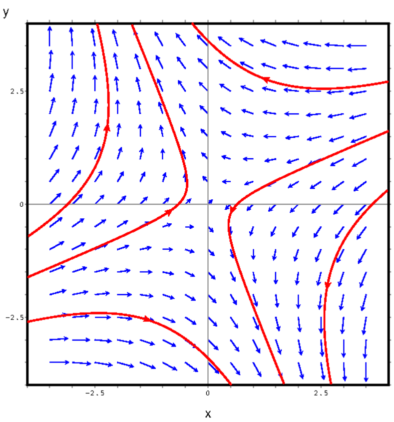


Figure 8.2: Velocity field and stream lines.

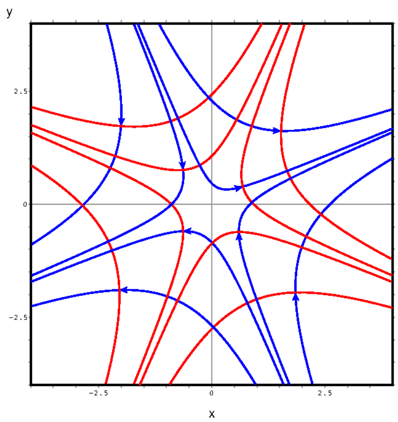


Figure 8.3: Stream lines[2] (blue) and equipotential lines (red).

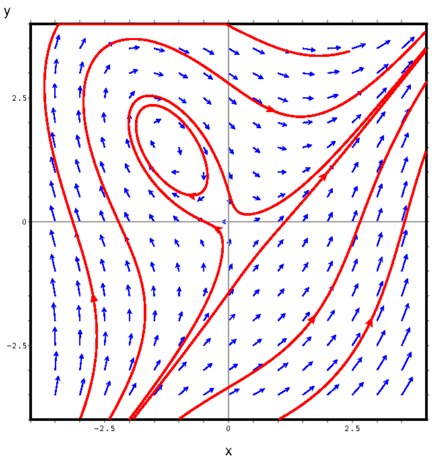


Figure 8.4: Velocity field of a vortex flow with stream lines.

8.2 Fluid electrodynamics

In this section, the field equations of fluid electrodynamics are introduced. This is a new subject area that is derived from Cartan geometry, using equations that were originally derived by Tsutomu Kambe. They have been extended by including the effects of viscosity and turbulence. The unification of electromagnetism and fluid dynamics is achieved, which allows new insights into the structure of spacetime itself and its interaction with matter. The foundational ECE papers for this section are listed in [84]. The impact of the fluid aspects of spacetime on dynamics and gravitation will be discussed in the next sections.

8.2.1 Kambe equations

In this section, the Kambe formulation of the equations of fluid dynamics [85–87] has been translated into equations of ECE2 electrodynamics. This unification of two very large subject areas, which we are calling “fluid electrodynamics”, further demonstrates that ECE2 is a unified field theory.

[2]The direction of stream lines is indicated in inverse direction to the velocity field of Fig. 8.2, because the graphics software defines the velocity field with a negative sign: $\mathbf{v} = -\boldsymbol{\nabla}\Phi$.

Original Kambe equations

The starting points of Kambe's development are Euler's equation (8.23) without external force, and the vorticity equation (8.29), respectively:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p \tag{8.52}$$

and

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{w}), \tag{8.53}$$

with vorticity

$$\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{v}. \tag{8.54}$$

The pressure term can be replaced with the enthalpy per unit mass h. In thermodynamics, the equation

$$dh = \frac{1}{\rho} dp + T ds \tag{8.55}$$

holds, where s is the entropy and T is the temperature. Initially, the entropy in the fluid field is uniform, and it is assumed that the fluid is *isentropic*, meaning that entropy is conserved in the fluid, i.e., $ds = 0$. Then, we have

$$dh = \frac{1}{\rho} dp \tag{8.56}$$

or, written in differential vector form,

$$\boldsymbol{\nabla} h = \frac{1}{\rho}\boldsymbol{\nabla} p. \tag{8.57}$$

Thus, the Euler equation takes the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} = -\boldsymbol{\nabla} h. \tag{8.58}$$

Next, we define the counterparts of the electric field and magnetic induction, named $\mathbf{E}_F$ and $\mathbf{H}_F$, and show that these obey Maxwell's equations. The index F stands for "Fluid". We identify the vector potential with the velocity field $\mathbf{v}$ and the scalar potential with the enthalpy per unit mass h so that, analogously to the electric case, we have

$$\mathbf{E}_F = -\frac{\partial \mathbf{v}}{\partial t} - \boldsymbol{\nabla} h. \tag{8.59}$$

We identify the magnetic induction with the vorticity:

$$\mathbf{H}_F = \mathbf{w} = \boldsymbol{\nabla} \times \mathbf{v}. \tag{8.60}$$

From these definitions, the Gauss law follows directly:

$$\boldsymbol{\nabla} \cdot \mathbf{H}_F = 0. \tag{8.61}$$

We compute the terms

$$\nabla \times \mathbf{E}_F = -\nabla \times \frac{\partial \mathbf{v}}{\partial t}, \tag{8.62}$$

$$\frac{\partial \mathbf{H}_F}{\partial t} = \nabla \times \frac{\partial \mathbf{v}}{\partial t} \tag{8.63}$$

and, by summing both equations, immediately obtain the Faraday law:

$$\nabla \times \mathbf{E}_F + \frac{\partial \mathbf{H}_F}{\partial t} = \mathbf{0}. \tag{8.64}$$

From Euler's equation (8.58) we find that

$$(\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{\partial \mathbf{v}}{\partial t} - \nabla h. \tag{8.65}$$

The right side is exactly the definition of $\mathbf{E}_F$; therefore, we have an alternative equation for the electric field,

$$\mathbf{E}_F = (\mathbf{v} \cdot \nabla)\,\mathbf{v}, \tag{8.66}$$

which depends only on the velocity field, as does the magnetic field. Then, the Coulomb law can be written as

$$\nabla \cdot \mathbf{E}_F = \nabla \cdot ((\mathbf{v} \cdot \nabla)\,\mathbf{v}). \tag{8.67}$$

From this, we define the fluid charge density q_F by

$$q_F = \nabla \cdot ((\mathbf{v} \cdot \nabla)\,\mathbf{v}), \tag{8.68}$$

so that the Coulomb law can be expressed in the well-known form:

$$\nabla \cdot \mathbf{E}_F = q_F. \tag{8.69}$$

Finally, we have to derive the Ampère-Maxwell law. Differentiating Eq. (8.59) by time, we obtain

$$-\frac{\partial \mathbf{E}_F}{\partial t} = \frac{\partial^2 \mathbf{v}}{\partial t^2} + \nabla \frac{\partial h}{\partial t}. \tag{8.70}$$

Adding the term $a_0^2\, \nabla \times \mathbf{H}_F = a_0^2\, \nabla \times (\nabla \times \mathbf{v})$ to both sides (a_0 is a constant) and rearranging gives:

$$\nabla \times \mathbf{H}_F - \frac{1}{a_0^2} \frac{\partial \mathbf{E}_F}{\partial t} = \frac{1}{a_0^2} \mathbf{J} \tag{8.71}$$

with

$$\mathbf{J} = \frac{\partial^2 \mathbf{v}}{\partial t^2} + \nabla \frac{\partial h}{\partial t} + a_0^2\, \nabla \times (\nabla \times \mathbf{v}). \tag{8.72}$$

From comparison with the electromagnetic Ampère-Maxwell law, we see that a_0 is the expansion velocity of the fields, which in this case is the speed of sound. A more thorough definition of the sound velocity follows from thermodynamics where, for a fixed entropy s, we obtain for changes of pressure and density:

$$\Delta p = \left(\frac{\partial p}{\partial \rho} \right)_{s=\text{const}} \Delta \rho. \tag{8.73}$$

If the pressure changes in proportion to the density, the partial derivative is constant, and the ratio of these quantities is the squared sound velocity:

$$\frac{\Delta p}{\Delta \rho} = a_0^2. \tag{8.74}$$

For convenience, the Maxwell-like field definitions and equations are listed below:

$$\mathbf{E}_F = -\frac{\partial \mathbf{v}}{\partial t} - \nabla h = (\mathbf{v}\cdot\nabla)\,\mathbf{v}, \tag{8.75}$$

$$\mathbf{H}_F = \mathbf{w} = \nabla \times \mathbf{v}, \tag{8.76}$$

$$q_F = \nabla \cdot ((\mathbf{v}\cdot\nabla)\,\mathbf{v}), \tag{8.77}$$

$$\nabla \cdot \mathbf{H}_F = 0, \tag{8.78}$$

$$\nabla \times \mathbf{E}_F + \frac{\partial \mathbf{H}_F}{\partial t} = \mathbf{0}, \tag{8.79}$$

$$\nabla \cdot \mathbf{E}_F = q_F, \tag{8.80}$$

$$\nabla \times \mathbf{H}_F - \frac{1}{a_0^2}\frac{\partial \mathbf{E}_F}{\partial t} = \frac{1}{a_0^2}\mathbf{J}_F, \tag{8.81}$$

$$\mathbf{J}_F = \frac{\partial^2 \mathbf{v}}{\partial t^2} + \nabla \frac{\partial h}{\partial t} + a_0^2\,\nabla \times (\nabla \times \mathbf{v}). \tag{8.82}$$

In Table 8.1, an update of Table 7.1, the units of electromagnetism, mechanics and fluid dynamics are compared. Notice that the units of components in the mechanics and fluid dynamics sectors are identical, with the exception of the material quantities (mass, charge and current densities).

	Electromagnetism		Mechanics		Fluid Dynamics	
Field	Symbol	Units	Symbol	Units	Symbol	Units
Electric field	$\mathbf{E}$	V/m	$\mathbf{g}$	m/s^2	$\mathbf{E}_F$	m/s^2
Magnetic induction	$\mathbf{B}$	$T=Vs/m^2$	$\mathbf{\Omega}$	1/s	$\mathbf{H}_F$	1/s
Scalar potential	ϕ	V	Φ	m^2/s^2	h	m^2/s^2
Vector potential	$\mathbf{A}$	Vs/m	$\mathbf{Q}$	m/s	$\mathbf{v}$	m/s
Charge density	ρ	C/m^3	ρ_m	kg/m^3	ρ_F	$1/s^2$
Current density	$\mathbf{J}$	$C/(m^2 s)$	$\mathbf{J}_m$	$kg/(m^2 s)$	$\mathbf{J}_F$	m/s^3
Pressure			p	N/m^2	p	N/m^2

Table 8.1: Comparison among components of electromagnetism, mechanics and fluid dynamics.

Sound wave equations

We will now derive wave equations for $\mathbf{E}_F$ and $\mathbf{H}_F$ analogously to the electromagnetic case. Proceeding as in Section 6.2.4, we compute the curl of Eq. (8.79):

$$\nabla \times (\nabla \times \mathbf{E}_F) + \nabla \times \frac{\partial \mathbf{H}_F}{\partial t} = \mathbf{0}. \tag{8.83}$$

Taking the time derivative of Eq. (8.81) gives

$$\nabla \times \frac{\partial \mathbf{H}_F}{\partial t} - \frac{1}{a_0^2}\frac{\partial^2 \mathbf{E}_F}{\partial t^2} = \frac{1}{a_0^2}\frac{\partial \mathbf{J}_F}{\partial t}. \tag{8.84}$$

By using the identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}_F) = -\nabla^2 \mathbf{E}_F + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}_F), \tag{8.85}$$

and inserting it into (8.83), we obtain

$$-\nabla^2 \mathbf{E}_F + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}_F) + \boldsymbol{\nabla} \times \frac{\partial \mathbf{H}_F}{\partial t} = \mathbf{0}. \tag{8.86}$$

Now, solving Eq. (8.84) for $\boldsymbol{\nabla} \times \frac{\partial \mathbf{H}_F}{\partial t}$ and inserting the result into (8.86) gives the wave equation

$$\frac{1}{a_0^2} \frac{\partial^2 \mathbf{E}_F}{\partial t^2} - \nabla^2 \mathbf{E}_F = -\boldsymbol{\nabla} q_F - \frac{1}{a_0^2} \frac{\partial \mathbf{J}_F}{\partial t}, \tag{8.87}$$

where we have replaced the divergence of $\mathbf{E}_F$ with the fluid charge q_F, in accordance with Eq. (8.80). The source term (right side) of this wave equation contains the gradient of the fluid source. This is a new term that does not appear in the corresponding electrodynamic wave equation. A non-constant fluid charge produces sound waves.

The second wave equation follows by proceeding as in the electromagnetic case. We take the curl of (8.81) and the time derivative of (8.79). This gives us the wave equation for $\mathbf{H}_F$:

$$\frac{1}{a_0^2} \frac{\partial^2 \mathbf{H}_F}{\partial t^2} - \nabla^2 \mathbf{H}_F = \frac{1}{a_0^2} \boldsymbol{\nabla} \times \mathbf{J}_F. \tag{8.88}$$

Extension to viscosity and turbulence effects

Eq. (8.82) for the current contains a double-curl describing an angular momentum, which is a rotation within the current. Kambe has shown that this is a source for a vortex sound wave. The same holds for the current density q_F.

Kambe's derivation does not include the effects of viscosity. He only mentions them in an example. Nevertheless, viscosity is the cause of turbulence, as we know from the Navier-Stokes equation and the vorticity equation (8.32):

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{w}) + \frac{1}{\mathscr{R}} \nabla^2 \mathbf{w}. \tag{8.89}$$

The momentum in the "minimal prescription" of quantum mechanics in ECE and ECE2 theory is

$$\mathbf{p} = m\mathbf{v} = e\mathbf{A} = e\mathbf{W}, \tag{8.90}$$

for a moving particle with mass m and charge e. $\mathbf{A}$ and $\mathbf{W}$ are the vector potentials of ECE and ECE2 theory. It follows from this equation that the aether velocity field can be expressed by the ECE2 potential:

$$\mathbf{v} = \frac{e}{m} \mathbf{W}, \tag{8.91}$$

and that the hydrodynamic vorticity field (using Eq. (8.32)) is

$$\mathbf{w} = \frac{e}{m} \boldsymbol{\nabla} \times \mathbf{W} = \frac{e}{m} \mathbf{B}. \tag{8.92}$$

In this way, we obtain a unification of the aether, or background velocity field, with electromagnetic induction. Inserting this into Eq. (8.32) gives the vorticity equation for electrodynamics:

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mathscr{R}} \nabla^2 \mathbf{B}. \tag{8.93}$$

This will be developed further, below. The Reynolds number is that of the aether, and the magnetic flux density is induced in a circuit. The fluid velocity becomes turbulent at a certain Reynolds number.

Fluid dynamics effects in electrodynamics

We have seen that Kambe's equations (8.75 - 8.82) are formally identical to those of electrodynamics, with some extensions (for example, the definition of the fluid source (8.77)). We will now proceed in reverse and reformulate Kambe's equations to obtain even more information in ECE2 electrodynamics. To accomplish this, we need a relation that connects both fields of physics. This relation is the force density. The electric field of a charge q originates a mechanical force:

$$\mathbf{F} = q\mathbf{E}. \tag{8.94}$$

The fluid dynamics field $\mathbf{E}_F$ is a mechanical acceleration field that creates, according to Newton's law, the force action on a mass m of

$$\mathbf{F} = m\mathbf{E}_F. \tag{8.95}$$

Since we are working with continuous fields, we have to replace charge and mass with their corresponding volume densities ρ and ρ_m. The above equations then read:

$$\left(\frac{\mathbf{F}}{V}\right)_e = \rho\mathbf{E}, \tag{8.96}$$

$$\left(\frac{\mathbf{F}}{V}\right)_m = \rho_m\mathbf{E}_F, \tag{8.97}$$

where the forces have now become force densities. The Navier-Stokes equation (8.10), for example, is an equation of force density, as is Euler's equation (8.23). By equating the force densities, we obtain

$$\mathbf{E} = \frac{\rho_m}{\rho}\mathbf{E}_F \tag{8.98}$$

for the electric field and, similarly,

$$\mathbf{B} = \frac{\rho_m}{\rho}\mathbf{H}_F \tag{8.99}$$

for the magnetic induction field.

■ **Example 8.2** As a small example, we show that the units of Eq. (8.99) are the same on both sides. We start by writing this equation in the form of a force density:

$$\rho\mathbf{B} = \rho_m\mathbf{H}_F. \tag{8.100}$$

As can be seen directly, the right side has units of

$$[\rho_m\mathbf{H}_F] = \frac{\text{kg}}{\text{m}^3}\cdot\frac{1}{\text{s}}. \tag{8.101}$$

This is a kind of frequency density. For the left side, we use the unit relations 1 T = 1 Vs/m^2 and 1 V = 1 J/C = 1 Nm/C. These unit relations give us

$$[\rho\mathbf{B}] = \frac{\text{C}}{\text{m}^3}\cdot\frac{\text{V s}}{\text{m}^2} = \frac{\text{C}}{\text{m}^3}\cdot\frac{\text{N m s}}{\text{C m}^2} = \frac{\text{C}}{\text{m}^3}\cdot\frac{\text{kg m}^2\,\text{s}}{\text{s}^2\,\text{C m}^2} = \frac{\text{kg}}{\text{m}^3}\cdot\frac{\text{m}^2\,\text{s}}{\text{s}^2\,\text{m}^2} = \frac{\text{kg}}{\text{m}^3}\cdot\frac{1}{\text{s}}. \tag{8.102}$$

As we can see, the units on both sides of Eq. (8.100) are (kg/m^3)·(1/s). ■

Returning to our discussion of fluid dynamics effects, we now compare the definitions of electric fields:

$$\mathbf{E} = -\frac{\partial \mathbf{W}}{\partial t} - \boldsymbol{\nabla}\phi_W, \tag{8.103}$$

$$\mathbf{E}_F = -\frac{\partial \mathbf{v}}{\partial t} - \boldsymbol{\nabla} h. \tag{8.104}$$

According to Eq. (8.98), the terms on the right side can be multiplied by the ratio of densities. In the most general case, they have to be included in the arguments of the differential operators. This leads to the equivalence of potentials:

$$\mathbf{W} = \frac{\rho_m}{\rho}\mathbf{v}, \tag{8.105}$$

$$\phi_W = \frac{\rho_m}{\rho} h. \tag{8.106}$$

From the equivalence of Coulomb laws:

$$\rho\boldsymbol{\nabla}\cdot\mathbf{E} = \rho_m\boldsymbol{\nabla}\cdot\mathbf{E}_F, \tag{8.107}$$

we see that

$$\rho = \varepsilon_0\frac{\rho_m}{\rho} q_F \tag{8.108}$$

or

$$\frac{\rho^2}{\varepsilon_0} = \rho_m q_F. \tag{8.109}$$

The continuity equation of Kambe fields can be derived by taking the divergence of Eq. (8.81):

$$-\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{E}_F = \boldsymbol{\nabla}\cdot\mathbf{J}_F. \tag{8.110}$$

From the Coulomb law (8.80), it follows that

$$\frac{\partial q_F}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}_F = 0. \tag{8.111}$$

The continuity equation of electrodynamics is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0. \tag{8.112}$$

By replacing ρ with q_F via (8.108), we obtain a consistent equation by setting

$$\mathbf{J} = \varepsilon_0\frac{\rho_m}{\rho}\mathbf{J}_F. \tag{8.113}$$

In summary, the fields of fluid dynamics are coupled to the fields of electrodynamics by

$$\mathbf{v} = \left(\frac{\rho}{\rho_m}\right)\mathbf{W}, \tag{8.114}$$

$$h = \left(\frac{\rho}{\rho_m}\right)\phi_W, \tag{8.115}$$

$$\mathbf{B}_F = \mathbf{w} = \left(\frac{\rho}{\rho_m}\right)\mathbf{B}, \tag{8.116}$$

$$\mathbf{E}_F = \left(\frac{\rho}{\rho_m}\right)\mathbf{E}, \tag{8.117}$$

$$q_F = \frac{1}{\varepsilon_0}\frac{\rho^2}{\rho_m}. \tag{8.118}$$

Inserting these into the Kambe equations (8.78 - 8.81) gives us the ECE2 field equations in fluid dynamics form:

$$\boldsymbol{\nabla}\cdot\left(\frac{\rho}{\rho_m}\mathbf{B}\right)=0, \tag{8.119}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_m}\mathbf{B}\right)+\boldsymbol{\nabla}\times\left(\frac{\rho}{\rho_m}\mathbf{E}\right)=\mathbf{0}, \tag{8.120}$$

$$\boldsymbol{\nabla}\cdot\left(\frac{\rho}{\rho_m}\mathbf{E}\right)=\frac{1}{\varepsilon_0}\frac{\rho^2}{\rho_m}, \tag{8.121}$$

$$-\frac{1}{a_0^2}\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_m}\mathbf{E}\right)+\boldsymbol{\nabla}\times\left(\frac{\rho}{\rho_m}\mathbf{B}\right)=\frac{1}{a_0^2\varepsilon_0}\frac{\rho}{\rho_m}\mathbf{J}. \tag{8.122}$$

The Gauss law (8.119) can then be written as

$$\frac{\rho}{\rho_m}\boldsymbol{\nabla}\cdot\mathbf{B}+\boldsymbol{\nabla}\left(\frac{\rho}{\rho_m}\right)\cdot\mathbf{B}=0 \tag{8.123}$$

or

$$\boldsymbol{\nabla}\cdot\mathbf{B}=-\frac{\rho_m}{\rho}\boldsymbol{\nabla}\left(\frac{\rho}{\rho_m}\right)\cdot\mathbf{B}, \tag{8.124}$$

which allows for magnetic monopoles, if the charge densities are varying in space. The factor in front of $\mathbf{J}$ in the Ampère-Maxwell law can be written in the following way. Analogously to electrodynamics, we define an acoustic permeability μ by the relation

$$a_0^2=\frac{1}{\varepsilon_0\mu}. \tag{8.125}$$

Then, the factor in front of $\mathbf{J}$ in the Ampère-Maxwell law becomes $a_0^2\varepsilon_0=1/\mu$, leading to

$$-\frac{1}{a_0^2}\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_m}\mathbf{E}\right)+\boldsymbol{\nabla}\times\left(\frac{\rho}{\rho_m}\mathbf{B}\right)=\mu\frac{\rho}{\rho_m}\mathbf{J}, \tag{8.126}$$

which is similar to how it is in electrodynamics. The equations of fluid electrodynamics contain more information than those of both classical and ECE2 electrodynamics. In the case of constant electrical and mechanical densities, Eqs. (8.119 - 8.122) revert to those of ECE2 electrodynamics.

8.2.2 Additional equations with turbulence, and wave equations

In Section 8.2.1, we derived the vorticity equation for electrodynamics (8.93) (in which turbulence is expressed by the Reynolds number) from Eq. (8.89) using only conventional expressions of charge and mass. We will repeat this using the fully unified theory, through the relations (8.114 - 8.118). From the vorticity equation, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_m}\mathbf{B}\right)=\boldsymbol{\nabla}\times\left(\frac{\rho}{\rho_m}\mathbf{v}\times\mathbf{B}\right)+\frac{1}{\mathscr{R}}\nabla^2\left(\frac{\rho}{\rho_m}\mathbf{B}\right). \tag{8.127}$$

This equation, which contains the Lorentz force term $\mathbf{v}\times\mathbf{B}$, turns into the original equation (8.93), when charge densities are constant. In a more precise derivation [88], it has been shown that a baroclinic term proportional to the gradients of ρ and p has to be added to the vorticity equation:

$$\frac{\partial\mathbf{w}}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{w})+\frac{1}{\rho_m^2}\boldsymbol{\nabla}\rho_m\times\boldsymbol{\nabla}p+\frac{1}{\mathscr{R}}\nabla^2\mathbf{w}. \tag{8.128}$$

The baroclinic term can be rewritten as

$$\frac{1}{\rho_m^2}\boldsymbol{\nabla}\rho_m \times \boldsymbol{\nabla} p = \frac{1}{\rho_m}\boldsymbol{\nabla}\rho_m \times \boldsymbol{\nabla} h \tag{8.129}$$

with

$$\boldsymbol{\nabla} h = (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} - \frac{\partial \mathbf{v}}{\partial t}. \tag{8.130}$$

Further development of fluid electrodynamics is possible at a very detailed level.

■ **Example 8.3** As an example of further development, in this case without the inclusion of turbulence, we derive an alternative expression for the field $\mathbf{E}_F$. We combine the curl of Eq. (8.75):

$$\boldsymbol{\nabla}\times\mathbf{E}_F = -\frac{\partial(\boldsymbol{\nabla}\times\mathbf{v})}{\partial t}, \tag{8.131}$$

with the vorticity equation without turbulence (8.29):

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{w}). \tag{8.132}$$

The right side of (8.131) is the time derivative of $\mathbf{w}$. By inserting this into Eq. (8.132), we obtain

$$\boldsymbol{\nabla}\times\mathbf{E}_F = -\boldsymbol{\nabla}\times(\mathbf{w}\times\mathbf{v}). \tag{8.133}$$

One solution of this equation is

$$\mathbf{E}_F = -\mathbf{w}\times\mathbf{v}. \tag{8.134}$$

However, by using this approach we lose the information contained in the scalar potential h. ■

Beltrami flows

For a Beltrami flow, the curl of the fluid velocity is parallel to the fluid field itself:

$$\boldsymbol{\nabla}\times\mathbf{v} = k\mathbf{v} \tag{8.135}$$

with a wave number k. It follows from (8.134), with $\mathbf{w} = k\mathbf{v}$, that

$$\mathbf{E}_F = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \mathbf{0}. \tag{8.136}$$

From Eqs. (8.75 - 8.82), we obtain the following additional properties for a Beltrami flow:

$$\mathbf{H}_F = k\mathbf{v}, \tag{8.137}$$

$$q_F = 0, \tag{8.138}$$

$$\boldsymbol{\nabla} h = 0, \tag{8.139}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \tag{8.140}$$

$$\frac{\partial \mathbf{v}}{\partial t} = 0, \tag{8.141}$$

$$\boldsymbol{\nabla}\times\mathbf{v} = \frac{1}{a_0^2 k}\mathbf{J}_F, \tag{8.142}$$

$$\mathbf{J}_F = a_0^2 k^2\,\mathbf{v}. \tag{8.143}$$

Furthermore, for a Beltrami flow,

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t}. \tag{8.144}$$

From all of these properties combined, it follows that a Beltrami flow is incompressible and inviscid. These properties are depicted in Fig. 6.9, where it can be seen that the flow is source-free and divergence-free, and that the direction of the flow current is identical to that of the velocity field. All non-vanishing fields are parallel to each other, as was explained in Section 6.2.

Wave equations

It is possible to derive wave equations of fluid electrodynamics in 4-vector form [88]. However, we will present only the results here, because the calculation is complex and beyond the scope of this book. We can define a 4-current as

$$J_F^\mu = (a_0 q_F, \mathbf{J}_F)\,, \tag{8.145}$$

and a 4-derivative as

$$\partial_\mu = \left(\frac{1}{a_0}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right). \tag{8.146}$$

The continuity equation (8.111) can be written as

$$\partial_\mu J_F^\mu = 0. \tag{8.147}$$

We now define the velocity four vector as

$$v^\mu = \left(\frac{\Phi}{a_0}, \mathbf{v}\right), \tag{8.148}$$

where Φ is a sum of the enthalpy per unit mass and additional terms derived from viscosity. We assume that

$$\partial_\mu v^\mu = \frac{1}{a_0^2}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{v} = 0. \tag{8.149}$$

This is the *Lorenz gauge assumption of fluid electrodynamics*. With this assumption, wave equations for Φ and $\mathbf{v}$ can be derived:

$$\Box\Phi = q_F, \tag{8.150}$$

$$\Box\mathbf{v} = \frac{1}{a_0^2}\mathbf{J}_F, \tag{8.151}$$

and they can be combined into the single wave equation:

$$\Box v^\mu = \frac{1}{a_0^2} J_F^\mu. \tag{8.152}$$

From the continuity equation and (8.150, 8.151), we get the equation:

$$\Box\left(\frac{1}{a_0^2}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{v}\right) = 0. \tag{8.153}$$

The Lorenz condition (8.149) is a possible solution of Eq. (8.153), which shows that the analysis is rigorously self-consistent. It has been shown here that the entire subject of fluid dynamics can be reduced to a single wave equation that, like all wave equations of physics, is an instance of the ECE wave equation of Cartan geometry:

$$(\Box + R)\, v^\mu = 0, \tag{8.154}$$

provided that the scalar curvature is defined by

$$R\, v^\mu := -\frac{1}{a_0^2} J_F^\mu. \tag{8.155}$$

In the case of Beltrami flows, it follows directly from Eq. (8.143) that R is constant:

$$R = -k^2, \tag{8.156}$$

and thus Eq. (8.154) is transformed into a conventional eigenvalue equation.

8.2.3 Connection to Cartan geometry

The convective derivative is an example of the covariant derivative of Cartan geometry. For a vector V^a, the covariant derivative is defined by

$$\frac{DV^a}{dx^\mu} = \frac{\partial V^a}{\partial x^\mu} + \omega^a{}_{\mu b} V^b \tag{8.157}$$

with indices of tangent space a, b. In cartesian coordinates,

$$(x^\mu) = (ct, X, Y, Z), \tag{8.158}$$

the convective derivative is

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \left(v_X \frac{\partial}{\partial X} + v_Y \frac{\partial}{\partial Y} + v_Z \frac{\partial}{\partial Z} \right) \mathbf{v}, \tag{8.159}$$

and its X component is

$$\frac{Dv_X}{Dt} = \frac{\partial v_X}{\partial t} + v_X \frac{\partial v_X}{\partial X} + v_Y \frac{\partial v_X}{\partial Y} + v_Z \frac{\partial v_X}{\partial Z}. \tag{8.160}$$

For $\mu = 0$, Eq. (8.157) reads:

$$\frac{DV^a}{dt} = \frac{\partial V^a}{\partial t} + c\,\omega^a{}_{0b} V^b. \tag{8.161}$$

We can now define a contravariant velocity 4-vector of Cartan geometry $v^{\mu a}$ with polarization index a of tangent space. We choose the coordinate system of tangent space to be identical to that of the base manifold for each point with cartesian coordinates x^μ, so that the coordinate components of $v^{\mu a}$ are the same for each $\mu = a$, and we can write the 4-vector simply as v^a. According to the definition (8.148), we have:

$$(v^a) = \left(\frac{\Phi}{a_0}, v_X, v_Y, v_Z \right) = \left(\frac{\Phi}{a_0}, \mathbf{v} \right), \tag{8.162}$$

where the 0-component contains the flow potential Φ and sound velocity a_0.

By comparing (8.160) with (8.161) for $V^a = v^a$, we see that the spin connection for $\mu = 0$ is

$$c\,\omega^a{}_{0b} = \frac{\partial v^a}{\partial x^b}. \tag{8.163}$$

This expression is equivalent to the Jacobian $\mathbf{J} = (J_{ab})$ defined by Eq. (2.62) earlier in this book. $\omega^a{}_{0b}$ is the scalar spin connection. The factor c appears in Eq. (8.161) because the original coordinate is $x^0 = ct$. The expression $c\omega^a{}_{0b}$ is a time frequency, as required for dimensional reasons.

In addition to the scalar spin connection, a vector spin connection appears in fluid dynamics when a scalar function is used instead of a vector V^a in Eq. (8.157), which is then called the *Stokes derivative* (see [84], Notes 6-7 for Paper 351).

8.2.4 Vacuum fluid and energy from spacetime

We will show that energy from spacetime is a direct consequence of fluid electrodynamics, and that spacetime acts as a richly structured fluid, which has also been called the “aether” or “vacuum”. In conventional electrodynamics, the vacuum is defined by

$$\rho = 0, \quad \mathbf{J} = \mathbf{0}, \tag{8.164}$$

but in fluid electrodynamics the vacuum is not empty, and has the intrinsic ability to create electromagnetic fields. These fields can be intercepted and used by electric circuits.

Eq. (8.98) shows us that the Kambe field $\mathbf{E}_F$ is equivalent to a vacuum electric field $\mathbf{E}$, and these fields are connected by a mass density ρ_m and charge density ρ:

$$\rho \mathbf{E} = \rho_m \mathbf{E}_F. \tag{8.165}$$

Both of these densities are properties of the vacuum. A non-vanishing electric vacuum field and charge density may appear unusual at a first glance, but they are familiar from the quantum mechanical vacuum, which is different from the classical vacuum described by Eqs. (8.164).

When a circuit is placed into the vacuum, which is automatically the case since the vacuum or aether is everywhere, this electric field appears in the circuit (making it detectable by instruments):

$$\mathbf{E}_{\text{(electr)}} = \frac{\rho_{m\text{(vac)}}}{\rho_{\text{(vac)}}} \mathbf{E}_{F\text{(vac)}}. \tag{8.166}$$

The random noise in a circuit can be considered to be a signal originating from such a field. In the same way, a magnetic field[3]

$$\mathbf{B}_{\text{(electr)}} = \frac{\rho_{m\text{(vac)}}}{\rho_{\text{(vac)}}} \mathbf{B}_{F\text{(vac)}} \tag{8.167}$$

is induced in the circuit. Both are generated by the aether flow $\mathbf{v}$:

$$\mathbf{E}_{\text{(electr)}} = \left(\frac{\rho_m}{\rho}\right)_{\text{(vac)}} (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \left(\frac{\rho_m}{\rho}\right)_{\text{(vac)}} \left(-\frac{\partial \mathbf{v}}{\partial t} - \boldsymbol{\nabla}\Phi\right)_{\text{(vac)}}, \tag{8.168}$$

$$\mathbf{B}_{\text{(electr)}} = \left(\frac{\rho_m}{\rho}\right)_{\text{(vac)}} \boldsymbol{\nabla}\times\mathbf{v}, \tag{8.169}$$

where Φ is the potential of fluid dynamics that was introduced in (8.148).

Coulomb law

The Coulomb law can be formulated so that the electric field is that which is induced in the circuit, but the generating charge density is that of spacetime or the vacuum:

$$(\boldsymbol{\nabla}\cdot\mathbf{E})_{\text{(electr)}} = \frac{\rho_{\text{(vac)}}}{\varepsilon_0}. \tag{8.170}$$

Inserting the definitions of the circuit electric field (8.103) and the vacuum charge density (8.109),

$$\mathbf{E}_{\text{(electr)}} = -\frac{\partial \mathbf{W}}{\partial t} - \boldsymbol{\nabla}\phi_W, \tag{8.171}$$

$$\rho_{\text{(vac)}} = \sqrt{\varepsilon_0\,\rho_m\,q_F}, \tag{8.172}$$

leads to the form of the Coulomb law

$$\left(\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{W} + \nabla^2\Phi_W\right)_{\text{(electr)}} = -\left(\sqrt{\frac{\rho_m}{\varepsilon_0}q_F}\right)_{\text{(vac)}} \tag{8.173}$$

that includes the fluid charge density q_F, as it was defined by (8.68):

$$q_F = \boldsymbol{\nabla}\cdot((\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}). \tag{8.174}$$

[3] The identifier "(electr)" is used to denote electromagnetic effects induced in the electric circuit.

Obviously, the square root expression gives real-valued results only for $q_F \geq 0$. The last equation shows that, if **v** changes sign, the result for q_F remains the same and should be positive for reasons of consistency. When the velocity field is given, the right side of Eq. (8.173) is defined. Only the vacuum matter density remains a parameter, which has to be determined experimentally.

To simplify this equation, we can assume the *Lorenz gauge*:

$$\frac{1}{c^2}\frac{\partial \Phi_W}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{W} = 0. \tag{8.175}$$

Then Eq. (8.173) becomes the wave equation

$$\left(\frac{\partial^2 \Phi_W}{\partial t^2} - \nabla^2 \Phi_W\right)_{\text{(electr)}} = \left(\sqrt{\frac{\rho_m}{\varepsilon_0}} q_F\right)_{\text{(vac)}}. \tag{8.176}$$

By using the d'Alembert operator,

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2, \tag{8.177}$$

this can be written as

$$(\Box \Phi_W)_{\text{(electr)}} = \left(\sqrt{\frac{\rho_m}{\varepsilon_0}} q_F\right)_{\text{(vac)}}. \tag{8.178}$$

Φ_W can be determined by solving this equation. The vacuum electric field $\mathbf{E}_{F\text{(vac)}}$ is computed from the given velocity field **v** by using

$$\mathbf{E}_{F\text{(vac)}} = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}. \tag{8.179}$$

Then, the ratio of vacuum matter density to vacuum charge density is obtained from Eq. (8.166):

$$\frac{\rho_{m\text{(vac)}}}{\rho_{\text{(vac)}}} = \frac{|\mathbf{E}_{F\text{(electr)}}|}{|\mathbf{E}_{F\text{(vac)}}|}. \tag{8.180}$$

Turbulence in spacetime can be explored by solving the vorticity equation (8.32) (starting with the simplest case) and using the resulting velocity field in the equations of this section.

Wave equations

Eq. (8.178) is already a wave equation for energy from spacetime. From the Ampère-Maxwell law of electrodynamics,

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{B} = \mu_0 \mathbf{J}, \tag{8.181}$$

by inserting the potentials for **E** and **B**,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi_W - \frac{\partial \mathbf{W}}{\partial t}, \tag{8.182}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{W}, \tag{8.183}$$

we obtain the equation

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\boldsymbol{\nabla}\phi_W - \frac{\partial \mathbf{W}}{\partial t}\right) + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{W}) = \mu_0 \mathbf{J}. \tag{8.184}$$

Replacing the double-curl gives

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\phi_W + \frac{\partial \mathbf{W}}{\partial t}\right) + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{W}) - \nabla^2\mathbf{W} = \mu_0\mathbf{J}. \tag{8.185}$$

As before, we can use the Lorenz gauge to simplify this equation. Taking the gradient of the Lorenz gauge (8.175), we obtain

$$\frac{1}{c^2}\frac{\partial \boldsymbol{\nabla}\Phi_W}{\partial t} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{W}) = \mathbf{0}. \tag{8.186}$$

Inserting this into (8.185) gives

$$\frac{1}{c^2}\frac{\partial \mathbf{W}^2}{\partial t^2} - \nabla^2\mathbf{W} = \mu_0\mathbf{J}, \tag{8.187}$$

which can be written in short form as in (8.178):

$$\Box\mathbf{W} = \mu_0\mathbf{J}. \tag{8.188}$$

The wave equation for the ECE2 potential Φ_W is (Eq. (6.295)):

$$\Box\Phi_W = \frac{\rho}{\varepsilon_0}. \tag{8.189}$$

We define 4-vectors

$$W^\mu = (\frac{\Phi_W}{c}, \mathbf{W}), \tag{8.190}$$

$$J^\mu = (c\rho, \mathbf{J}). \tag{8.191}$$

Then, the Lorenz condition reads:

$$\partial_\mu W^\mu = 0 \tag{8.192}$$

and, after applying this condition, the wave equation for W^μ becomes

$$\Box W^\mu = \mu_0 J^\mu. \tag{8.193}$$

This is an inhomogeneous equation and has a source term on the right side, as does Eq. (8.178).

By applying the same interpretation that we used for the Coulomb law, we see that the right side is the vacuum current, and that the fields on the left side are fields induced in the circuit. This gives us

$$(\Box W^\mu)_{\text{(electr)}} = (\mu_0 J^\mu)_{\text{(vac)}}. \tag{8.194}$$

The equation for the 0-component ($\mu = 0$) has already been derived and is Eq. (8.178). Writing Eq. (8.113) in this notation gives:

$$\mathbf{J}_{\text{(electr)}} = \varepsilon_0\left(\frac{\rho_m}{\rho}\right)_{\text{(vac)}}\mathbf{J}_F. \tag{8.195}$$

Therefore, the wave equation for the vector potential $\mathbf{W}$ created by spacetime flow reads:

$$(\Box\mathbf{W})_{\text{(electr)}} = \frac{1}{c^2}\left(\frac{\rho_m}{\rho}\right)_{\text{(vac)}}\mathbf{J}_F. \tag{8.196}$$

So far, we have considered electromagnetic effects as being directly equivalent to vacuum structures. These electromagnetic effects can be detected in devices that serve as measuring instruments. The inverse effect is also present. An electromagnetic structure created by standard technical methods produces a corresponding vacuum structure. Electromagnetic fields generate a vacuum flow. This inverse effect is a re-interpretation of Eq. (8.165):

$$\rho \mathbf{E} = \rho_m \mathbf{E}_F. \tag{8.197}$$

Now, the charge density is that of the circuit, as is the electric field. Therefore, as a variation of (8.166), we have

$$\mathbf{E}_{\text{(circuit)}} = \frac{\rho_{m\text{(vac)}}}{\rho_{\text{(circuit)}}} \mathbf{E}_{F\text{(vac)}}. \tag{8.198}$$

This raises the question of what the aether flux looks like when an electromagnetic device is in operation. Inserting the Kambe fields gives us

$$\mathbf{E}_{\text{(circuit)}} = \frac{\rho_{m\text{(vac)}}}{\rho_{\text{(circuit)}}} (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}, \tag{8.199}$$

$$\mathbf{B}_{\text{(circuit)}} = \frac{\rho_{m\text{(vac)}}}{\rho_{\text{(circuit)}}} \boldsymbol{\nabla} \times \mathbf{v}. \tag{8.200}$$

For the potentials, we have from Eqs. (8.114, 8.115):

$$\mathbf{W}_{\text{(circuit)}} = \frac{\rho_{m\text{(vac)}}}{\rho_{\text{(circuit)}}} \mathbf{v}, \tag{8.201}$$

$$\phi_{W\,\text{(circuit)}} = \frac{\rho_{m\text{(vac)}}}{\rho_{\text{(circuit)}}} h. \tag{8.202}$$

Thus, we can replace every solution for electromagnetic properties with the corresponding spacetime properties. Since differential operators appear on the right sides of the above equations, we will obtain a differential equation in general. As an example, we consider the Coulomb field or, more precisely, the attractive force field between two point-like particles with charge, which is

$$\mathbf{E}_{\text{(circuit)}} = \frac{q}{4\pi\varepsilon_0 r^2} \mathbf{e}_r, \tag{8.203}$$

where $\mathbf{e}_r$ is the radial unit vector in spherical polar coordinates. Therefore, Eq. (8.199) has to be solved in spherical polar coordinates. The operator $(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$ is listed in polar and cylindrical coordinates in Example 8.4.

■ **Example 8.4** We present the vector operator $(\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{b}$ in cartesian, cylindrical and spherical coordinates for arbitrary vector functions **a** and **b** [89]:

$$(\mathbf{a}_{\text{cart}} \cdot \boldsymbol{\nabla})\mathbf{b}_{\text{cart}} = \begin{bmatrix} a_X \frac{\partial b_X}{\partial X} + a_Y \frac{\partial b_X}{\partial Y} + a_Z \frac{\partial b_X}{\partial Z} \\ a_X \frac{\partial b_Y}{\partial X} + a_Y \frac{\partial b_Y}{\partial Y} + a_Z \frac{\partial b_Y}{\partial Z} \\ a_X \frac{\partial b_Z}{\partial X} + a_Y \frac{\partial b_Z}{\partial Y} + a_Z \frac{\partial b_Z}{\partial Z} \end{bmatrix}, \tag{8.204}$$

$$(\mathbf{a}_{\text{cyl}} \cdot \boldsymbol{\nabla})\mathbf{b}_{\text{cyl}} = \begin{bmatrix} a_r \frac{\partial b_r}{\partial r} + \frac{a_\theta}{r} \frac{\partial b_r}{\partial \theta} + a_Z \frac{\partial b_r}{\partial Z} - \frac{a_\theta b_\theta}{r} \\ a_r \frac{\partial b_\theta}{\partial r} + \frac{a_\theta}{r} \frac{\partial b_\theta}{\partial \theta} + a_Z \frac{\partial b_\theta}{\partial Z} + \frac{a_\theta b_r}{r} \\ a_r \frac{\partial b_Z}{\partial r} + \frac{a_\theta}{r} \frac{\partial b_Z}{\partial \theta} + a_Z \frac{\partial b_Z}{\partial Z} \end{bmatrix}, \tag{8.205}$$

$$(\mathbf{a}_{\text{sph}} \cdot \boldsymbol{\nabla})\mathbf{b}_{\text{sph}} = \begin{bmatrix} a_r \frac{\partial b_r}{\partial r} + \frac{a_\theta}{r} \frac{\partial b_r}{\partial \theta} + \frac{a_\phi}{r\sin\theta} \frac{\partial b_r}{\partial \phi} - \frac{a_\theta b_\theta + a_\phi b_\phi}{r} \\ a_r \frac{\partial b_\theta}{\partial r} + \frac{a_\theta}{r} \frac{\partial b_\theta}{\partial \theta} + \frac{a_\phi}{r\sin\theta} \frac{\partial b_\theta}{\partial \phi} + \frac{a_\theta b_r}{r} - \frac{a_\phi b_\phi \cot\theta}{r} \\ a_r \frac{\partial b_\phi}{\partial r} + \frac{a_\theta}{r} \frac{\partial b_\phi}{\partial \theta} + \frac{a_\phi}{r\sin\theta} \frac{\partial b_\phi}{\partial \phi} + \frac{a_\phi b_r}{r} + \frac{a_\phi b_\theta \cot\theta}{r} \end{bmatrix}. \tag{8.206}$$

■

For the Coulomb field, there is no angular dependence, so only the radial part of the velocity field has to be considered. Computer algebra gives the following result (see computer code [150]):

$$v_r = \pm\sqrt{\frac{q}{2\pi\varepsilon_0 x}}\sqrt{\frac{1}{r} - c} \tag{8.207}$$

with

$$x = \frac{\rho_{m(\mathrm{vac})}}{\rho_{(\mathrm{circuit})}} \tag{8.208}$$

and an integration constant c. If q is negative, then the factor x also changes sign, so that the solution remains real-valued. To obtain the right asymptotic behavior of v_r for $r \to \infty$, we have to set $c = 0$.

For the Coulomb law, we have to note that the charge density is a δ function, which is different from zero only for $r = 0$. Therefore, there is no charge density ρ in the Coulomb field for $r > 0$, and the factor x is

$$x = \frac{\rho_{m(\mathrm{vac})}}{\rho_{(\mathrm{vac})}}, \tag{8.209}$$

like in the regions outside of a circuit, as discussed at the beginning of Section 8.2.4. In these regions, according to the ECE2 Coulomb law (6.153), we have

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} = -2\left(\frac{1}{W^{(0)}}\mathbf{A} - \boldsymbol{\omega}\right)\cdot\mathbf{E} = 0, \tag{8.210}$$

where $\boldsymbol{\omega}$ is the vector spin connection. It follows that

$$\frac{1}{W^{(0)}}\mathbf{A} - \boldsymbol{\omega} = \mathbf{0}. \tag{8.211}$$

The vector potential of ECE2 theory is defined by

$$\mathbf{W} = W^{(0)}\boldsymbol{\omega}, \tag{8.212}$$

and therefore it is identical to the vector potential $\mathbf{A}$ in free space:

$$\mathbf{W} = \mathbf{A}. \tag{8.213}$$

This property allows us to rewrite the $\mathbf{E}$ field with the vector potential $\mathbf{A}$, in the following way. The antisymmetry law of electrodynamics, Eq. (5.24), with polarization indices omitted, has the form:

$$\mathbf{E} = -2\left(\frac{\partial\mathbf{A}}{\partial t} + c\omega_0\mathbf{A}\right) = -2\left(\boldsymbol{\nabla}\phi - \boldsymbol{\omega}\phi\right). \tag{8.214}$$

ω_0 is the scalar spin connection and $\boldsymbol{\omega}$ is the vector spin connection. Using the first part of the identity, and taking into account that $\mathbf{E}$ is a static field, we obtain

$$\mathbf{E} = -2\,c\,\omega_0\mathbf{A}, \tag{8.215}$$

and after applying (8.114), this becomes

$$\mathbf{E} = -2\,c\,x\,\omega_0\,\mathbf{v}. \tag{8.216}$$

It follows that the static electric field of a charge is a velocity field of aether or vacuum flow. To the best of our knowledge, Tom Bearden [90] was the first to come to this conclusion. This result of ECE theory is the first theoretical foundation for this statement. It cannot be obtained from standard physics, and is a fundamentally new insight into the nature of electromagnetism.

8.2.5 Graphical examples

A number of examples, in particular the fluid fields of vector potentials of given material fields, will be described and graphed in this section. These examples will also further develop the methodology for describing spacetime fluid effects.

Examples of Kambe Fields

■ **Example 8.5** We investigate the dynamic charge density q_F derived from the velocity field $\mathbf{v}$ by Kambe (Eq. (8.77)):

$$q_F = \boldsymbol{\nabla}\cdot(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}. \tag{8.217}$$

For an incompressible fluid, it is required that the velocity field is divergence-free:

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0. \tag{8.218}$$

We will inspect several velocity models by specifying $\mathbf{v}$ analytically. We use plane polar coordinates that are identical to cylindrical coordinates with $Z = 0$. Therefore, we can use the differential operators of cylindrical coordinates (r, θ, Z):

$$\boldsymbol{\nabla}\psi = \begin{bmatrix} \frac{\partial\psi}{\partial r} \\ \frac{1}{r}\frac{\partial\psi}{\partial\theta} \\ \frac{\partial\psi}{\partial Z} \end{bmatrix}, \tag{8.219}$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{1}{r}\frac{\partial(r\,v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_Z}{\partial Z}, \tag{8.220}$$

$$\boldsymbol{\nabla}\times\mathbf{v} = \begin{bmatrix} \frac{1}{r}\frac{\partial v_Z}{\partial\theta} - \frac{\partial v_\theta}{\partial Z} \\ \frac{\partial v_r}{\partial Z} - \frac{\partial v_Z}{\partial r} \\ \frac{1}{r}\frac{\partial(r\,v_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\theta} \end{bmatrix}, \tag{8.221}$$

for a scalar function ψ and a vector $\mathbf{v}$.

We choose an example where the divergence vanishes, although this is not obvious from the velocity field:

$$\mathbf{v}_1 = \begin{bmatrix} \frac{a\cos\theta}{r^2} \\ \frac{a\sin\theta}{r^2} + b \\ 0 \end{bmatrix} \tag{8.222}$$

with constants a and b. Computer algebra gives the results:

$$\boldsymbol{\nabla}\cdot\mathbf{v}_1 = 0, \tag{8.223}$$

$$\mathbf{E}_F = (\mathbf{v}_1\cdot\boldsymbol{\nabla})\,\mathbf{v}_1 = -\frac{a}{r^5}\begin{bmatrix} a\sin^2\theta + b\,r^2\sin\theta + 2\,a\cos^2\theta \\ \cos\theta\left(a\sin\theta - b\,r^2\right) \\ 0 \end{bmatrix}, \tag{8.224}$$

$$\mathbf{W} = \boldsymbol{\nabla}\times\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{b}{r} \end{bmatrix}, \tag{8.225}$$

$$q_1 = \frac{a}{r^6}\left(5\,a\sin^2\theta + b\,r^2\sin\theta + 7\,a\cos^2\theta\right). \tag{8.226}$$

The vector field (8.222) is shown in Fig. 8.5. There is a center of rotation below the coordinate center. The velocities are much higher above the center than below. This leads to partially asymmetric electric field components E_r and E_θ in Eq. (8.224). The field $\mathbf{E}_F$ has been converted to

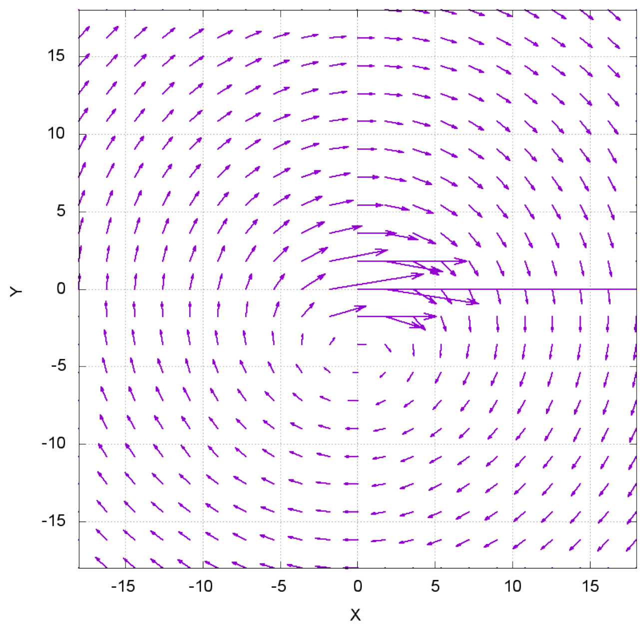


Figure 8.5: Velocity model $\mathbf{v}_1$.

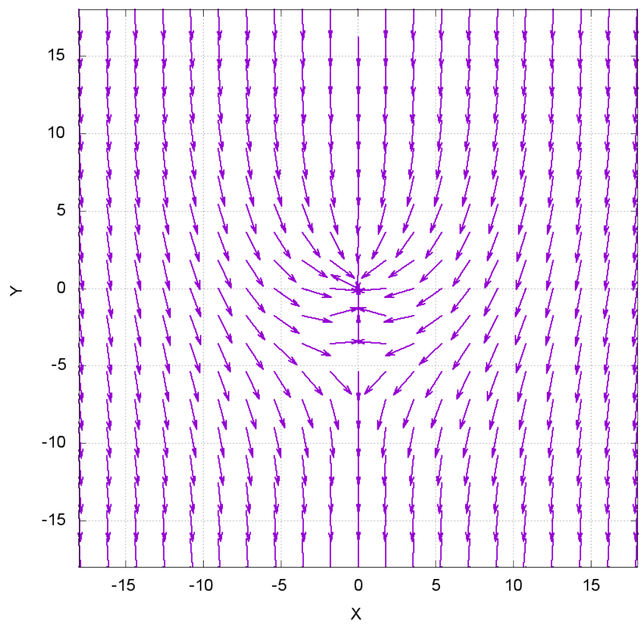


Figure 8.6: Velocity model $\mathbf{v}_1$, directional vectors of field $\mathbf{E}_F$.

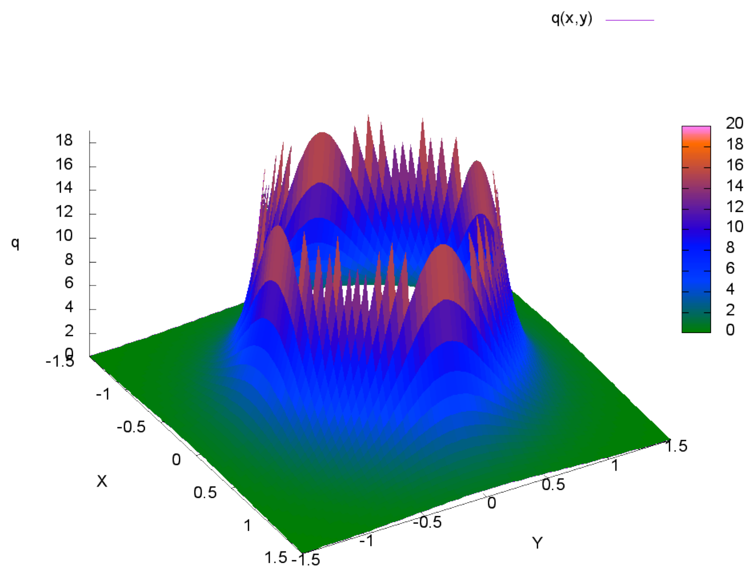


Figure 8.7: Velocity model $\mathbf{v}_1$, charge distribution q_1.

vector form in the XY plane and its (normalized) directional vectors are graphed in Fig. 8.6. In the lower center of Fig. 8.5 we see a "hole" in the electric field, and there is a kind of flow along the Y axis that would not be expected from the form of the velocity field. Despite these asymmetries, the charge distribution of this velocity model is mainly centrally symmetric, as can be seen in Fig. 8.7. This result was not obvious from the formulas. ■

■ **Example 8.6** A more general example can be constructed by using

$$\mathbf{v}_2 = \begin{bmatrix} \frac{a}{r^n} \\ f(r,\theta) \\ 0 \end{bmatrix} \tag{8.227}$$

with a general function $f(r,\theta)$. It then follows that

$$\nabla\cdot\mathbf{v}_2 = r^{-n-1}\left(r^n \frac{\partial}{\partial\theta} \mathrm{f}(r,\theta) - a(1-n)\right), \tag{8.228}$$

$$(\mathbf{v}_2\cdot\nabla)\,\mathbf{v}_2 = \begin{bmatrix} -a^2\, n\, r^{-2n-1} \\ r^{-n-1}\left(r^n \mathrm{f}(r,\theta)\, \frac{\partial}{\partial\theta}\mathrm{f}(r,\theta) + a\, r\, \frac{\partial}{\partial r}\mathrm{f}(r,\theta)\right) \\ 0 \end{bmatrix}, \tag{8.229}$$

$$\nabla\times\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{\partial}{\partial r}\mathrm{f}(r,\theta) + \frac{1}{r}\mathrm{f}(r,\theta) \end{bmatrix}, \tag{8.230}$$

and the charge distribution takes the form

$$q_2 = r^{-2n-2}\left(r^{2n}\,\mathrm{f}(r,\theta)\,\frac{\partial^2}{\partial\theta^2}\mathrm{f}(r,\theta) + r^{2n}\left(\frac{\partial}{\partial\theta}\mathrm{f}(r,\theta)\right)^2 + a\, r^{n+1}\frac{\partial^2}{\partial r\,\partial\theta}\mathrm{f}(r,\theta) + 2a^2 n^2\right). \tag{8.231}$$

The divergence of this velocity model vanishes, if

$$r^{-n-1}\left(r^{n}\left(\frac{\partial}{\partial \theta}\mathrm{f}(r,\theta)\right)-a\,(1-n)\right)=0, \tag{8.232}$$

which is a differential equation for $\mathrm{f}(r,\theta)$ with the solution

$$\mathrm{f}(r,\theta)=\frac{a\,(n-1)\,\theta}{r^{n}}+c, \tag{8.233}$$

where c is an integration constant. For $n=1$, f reduces to a constant function and the divergence (8.228) vanishes. ■

Solutions of fluid dynamics equations

■ **Example 8.7** Four equations of fluid dynamics have been solved numerically by the finite element program FlexPDE [91]. The 3D volume that was chosen is typical for Navier-Stokes applications: a plenum box with a circular inlet at the bottom and an offset circular outlet at the top (see Fig. 8.8). The boundary conditions were set to $\mathbf{v}=\mathbf{0}$ at the borders of the box, and a directional derivative perpendicular to the opening areas was assumed. This allows for a free-floating solution of the velocity field. As a test, a solution for the static Navier-Stokes equation (8.16):

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}+\boldsymbol{\nabla}p-\mu\,\nabla^{2}\mathbf{v}=\mathbf{0} \tag{8.234}$$

with viscosity μ was computed. The pressure term was added because the equation is otherwise homogeneous, which means that there is no source term, leading to a solution that does not guarantee conservation of mass. The divergence of the pressure gradient is assumed to be in proportion to the divergence of the velocity field:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}p=P\,\boldsymbol{\nabla}\cdot\mathbf{v} \tag{8.235}$$

with a constant P for "penalty pressure". This represents an additional equation for determining the pressure. The result for the velocity is graphed in Fig. 8.9, which shows a straight flow through the box. The flow is perpendicular to the inlet and outlet surfaces as required by the boundary conditions.

Next, the vorticity equation (8.29) was solved twice, once in a static form derived in [92], and again with the pressure term included to guarantee solutions:

$$\nabla^{2}\mathbf{w}+\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{w})+\boldsymbol{\nabla}p=\mathbf{0}. \tag{8.236}$$

It is difficult to define meaningful boundary conditions, because this is a pure flow equation for the vorticity $\mathbf{w}$. We used the same boundary conditions that we had used for the Navier-Stokes equations, and the result is graphed in Fig. 8.10. There is a flow-like structure with a divergence at the left, where the flow is not symmetric. There should not be a divergence because the vorticity is divergenceless by definition. We conclude that these boundary conditions are not adequate for this type of equation.

This approach is more meaningful for the vorticity equation with turbulence, which can be written as a static equation in the form that was discussed in [92]:

$$\boldsymbol{\nabla}\times\mathbf{w}+R\,((\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v}-\mathbf{v}\times\mathbf{w})+\boldsymbol{\nabla}p=\mathbf{0}. \tag{8.237}$$

The solution for $R=1$ gives an inclined input and output flow (see Fig. 8.11). At a medium height in the box, the flow drifts more over the sides; therefore, the intensity of the velocity is low in the middle plane, as shown. The divergence (not graphed) is practically zero in this region. Fig. 8.12

shows a divergent and convergent flow in the XY plane at $Z=0$; the flow runs over the full width of the box. Results for higher Reynolds numbers do not reveal any significant difference.

Finally, we solved an equation that holds for a Beltrami flow [92]:

$$\nabla^2 \mathbf{v} - R\,(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} - \boldsymbol{\nabla}\,(\boldsymbol{\nabla}\cdot\mathbf{v}) + \boldsymbol{\nabla} p = \mathbf{0}. \tag{8.238}$$

Here, the flow is strongly enhanced in the middle region (see Fig. 8.13). In the perpendicular plane, a similar effect can be seen (Fig. 8.14). The field is not divergence-free there. For a Beltrami field, we should have

$$\mathbf{w}\times\mathbf{v} = k\mathbf{v}\times\mathbf{v} = \mathbf{0}. \tag{8.239}$$

The vorticity $\mathbf{w}$ corresponding to Fig. 8.14 has been graphed in Fig. 8.15. There are indeed large regions where both $\mathbf{w}$ and $\mathbf{v}$ are parallel or antiparallel; the factor k seems to be location dependent, and we did not constrain the Beltrami property by further means. Therefore, the result is satisfactory. For larger R values the results remain similar. ■

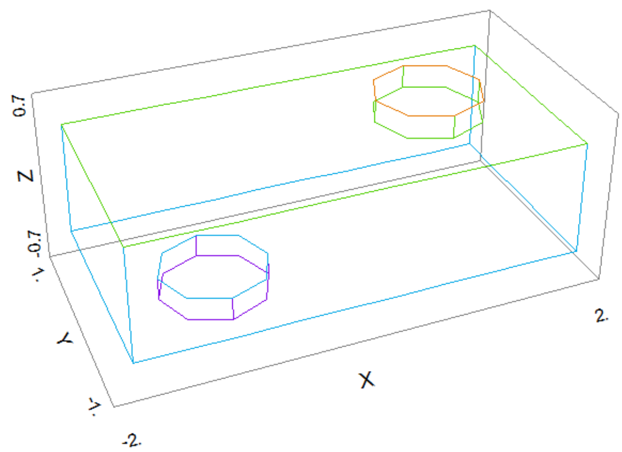


Figure 8.8: Geometry of FEM calculations.

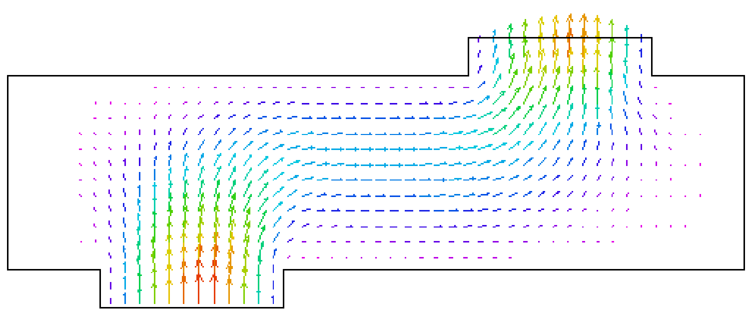


Figure 8.9: Velocity solution of the Navier-Stokes equation (8.234), plane $Y=0$.

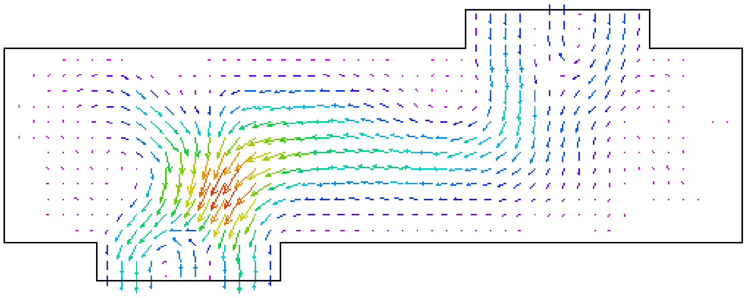


Figure 8.10: Vorticity solution of Eq. (8.236), plane $Y = 0$.

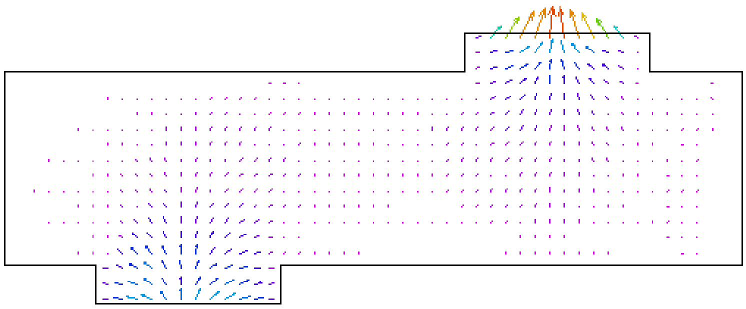


Figure 8.11: Velocity solution of Eq. (8.237) for $R = 1$, plane $Y = 0$.

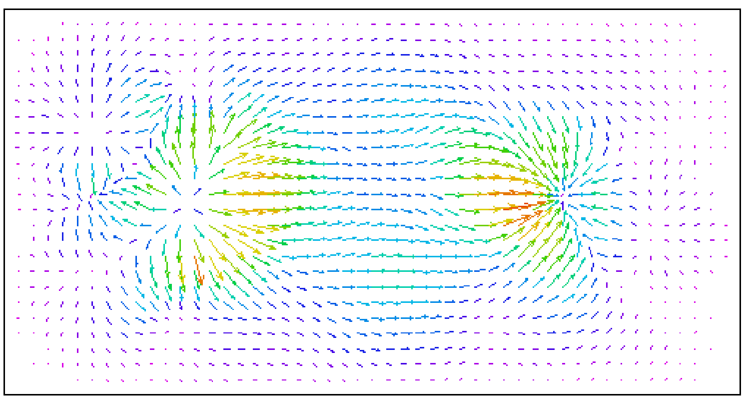


Figure 8.12: Velocity solution of Eq. (8.237) for $R = 1$, plane $Z = 0$.

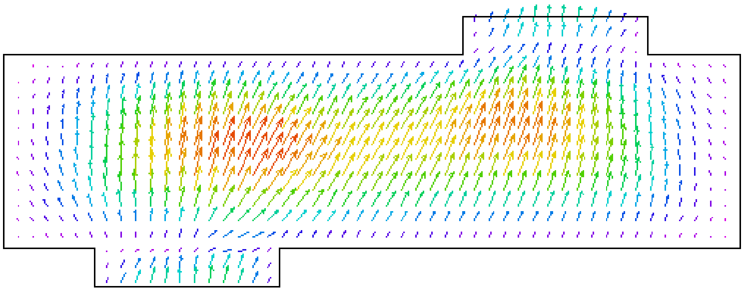


Figure 8.13: Beltrami solution of Eq. (8.238) for $R = 1$, plane $Y = 0$.

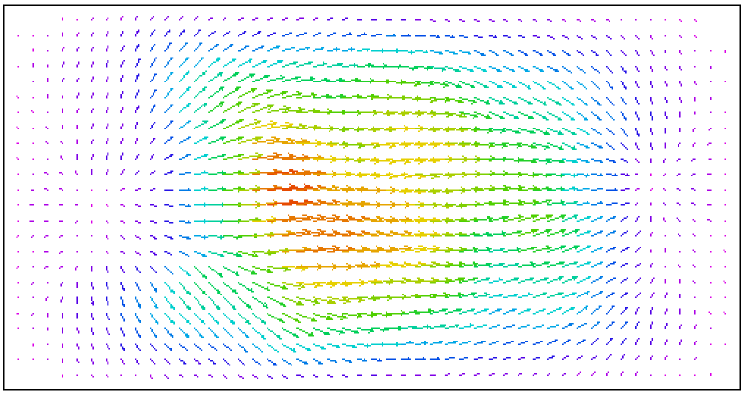


Figure 8.14: Beltrami solution of Eq. (8.238) for $R = 1$, plane $Z = 0$.

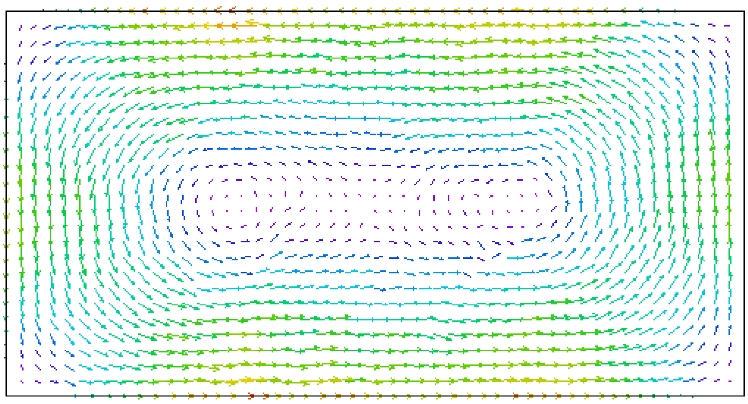


Figure 8.15: Vorticity of the Beltrami solution of Eq. (8.238) for $R = 1$, plane $Z = 0$.

Wave equations of fluid electrodynamics

■ **Example 8.8** In this example, the wave equation of the fluid electrodynamics velocity (8.151) is developed further. In the presence of a current density $\mathbf{J}_F$, which is an external component in standard physics, this wave equation reads:

$$\frac{1}{a_0^2}\frac{\partial^2 \mathbf{v}}{\partial t^2} - \nabla^2 \mathbf{v} = \frac{1}{a_0^2}\mathbf{J}_F. \tag{8.240}$$

Assuming a harmonic time dependence, we define

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_S(\mathbf{r})\exp(i\omega t) \tag{8.241}$$

and

$$\mathbf{J}_F(\mathbf{r},t) = \mathbf{J}_S(\mathbf{r})\exp(i\omega t) \tag{8.242}$$

with a time frequency ω, and only space-dependent velocity $\mathbf{v}_S$ and current density $\mathbf{J}_S$. Then, Eq. (8.240) reads:

$$-\frac{\omega^2}{a_0^2}\mathbf{v}_S - \nabla^2 \mathbf{v}_S = \frac{1}{a_0^2}\mathbf{J}_S, \tag{8.243}$$

which is an eigenvalue equation. For vanishing current density, it can be written in the standard form:

$$\nabla^2 \mathbf{v}_S + \lambda \mathbf{v}_S = \mathbf{0} \tag{8.244}$$

with positive eigenvalues

$$\lambda := \frac{\omega^2}{a_0^2}, \tag{8.245}$$

which correspond to acoustic eigenfrequencies, for example. This equation can be solved numerically by the finite element method. We will reuse the 3D flow box from the preceding section, but with boundary conditions that correspond to this example. The first six eigenvalues (with unspecified units) are listed in Table 8.2. There is a degeneracy between the first and second eigenvalues, and between the fifth and sixth eigenvalues, due to the internal symmetry of the flow box. For each pair, there is a third value that is close. The modulus of the first and sixth velocity eigenstates has been graphed in Figs. 8.16 - 8.19 for two planes of symmetry ($Z = 0$ and $Y = 0$). The sixth eigenstate has a node in the middle plane of symmetry. This symmetry is also present in the vorticity vectors (see Figs. 8.20 and 8.21).

For a correct treatment of the wave equation within fluid electrodynamics, we have to include the current density (8.82), which can be written [92] as

$$\mathbf{J}_F = a_0^2 \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) - \frac{\partial}{\partial t}\left((\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right). \tag{8.246}$$

The second term is not linear in $\mathbf{v}$, so the time-harmonic approach is only possible for the first term, and using Eq. (8.243) leads to the more general eigenvalue equation:

$$\nabla^2 \mathbf{v}_S + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}_S) + \lambda \mathbf{v}_S = \mathbf{0}. \tag{8.247}$$

As a result, the eigenvalues are very small, compared to Eq. (8.244), and there is much more turbulence. The numerical calculation takes half an hour on a standard PC, but converges. The numerical precision, however, is not satisfactory; therefore, these results can only show a tendency. The first six eigenvalues are listed in Table 8.3, and we see that there is no longer any degeneracy. In Figs. 8.22 and 8.23, the vorticity in the plane $Y=0$ has been graphed. If we compare those figures to Figs. 8.20 and 8.21, we see that Eq. (8.247) leads to structures that are much more turbulent, and that eigenstate n possesses $n+1$ vortices (all of which seem to be a peculiarity of Eq. (8.247)).

We also tried a time-dependent calculation by assuming that the second-order time derivative in (8.240) can be neglected against the first-order time derivative in the current density:

$$\nabla^2 \mathbf{v}_S = -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \frac{\partial}{\partial t}\left((\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}\right). \tag{8.248}$$

Adding a pressure term $\boldsymbol{\nabla} p$ as described in the previous example gives a non-singular equation but no time solution. The nonlinearity prevents a solution – at least for this special category of boundary values.

Coming back to the solution of Eq. (8.247), this seems to be the first time that an ECE2 wave equation of type

$$(\Box + R)\mathbf{v} = \mathbf{0} \tag{8.249}$$

(see Eq. (8.154)) has been solved for a curvature R that in turn depends on the variable $\mathbf{v}$. This is certainly a step beyond contemporary standard equations of physics, e.g., the Dirac equation, where a constant curvature has always been assumed. The numerical problems, however, are complex and a lot of work will be required to develop this field of ECE2 physics.

No.	Eigenvalue
1	12.1031274
2	12.1031274
3	12.1919561
4	13.2402655
5	13.3685992
6	13.3685992

Table 8.2: Eigenvalues of Eq. (8.244).

No.	Eigenvalue
1	2.56351677e-3
2	2.68244759e-3
3	4.08141046e-3
4	6.27378404e-3
5	7.79542935e-3
6	8.34876355e-3

Table 8.3: Eigenvalues of Eq. (8.247).

■

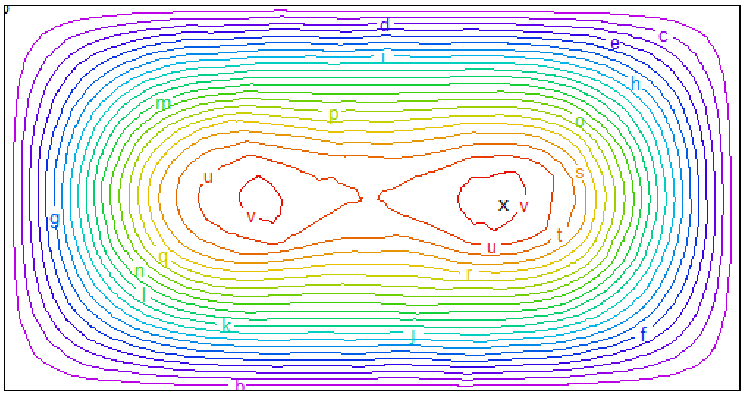


Figure 8.16: Velocity modulus of Eq. (8.244) on Z=0, eigenstate 1.

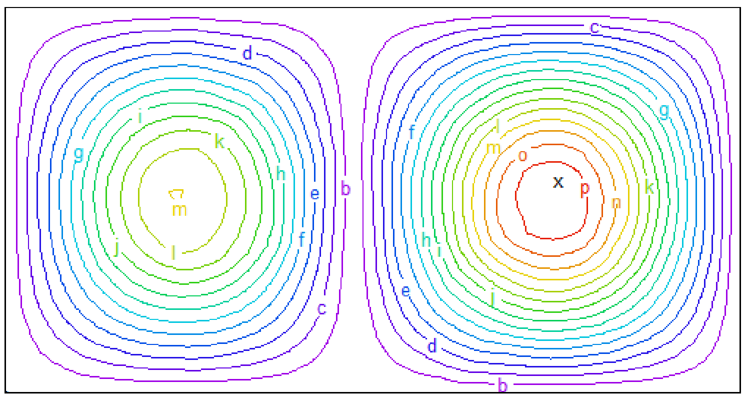


Figure 8.17: Velocity modulus of Eq. (8.244) on Z=0, eigenstate 6.

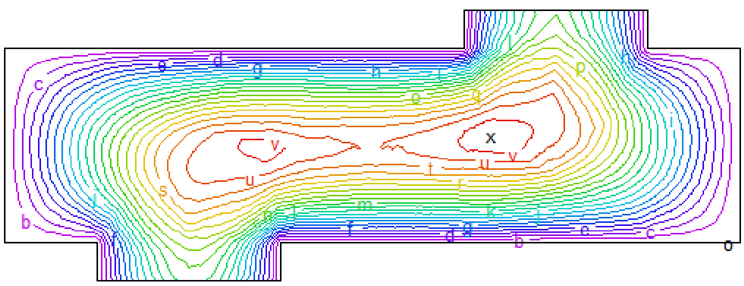


Figure 8.18: Velocity modulus of Eq. (8.244) on Y=0, eigenstate 1.

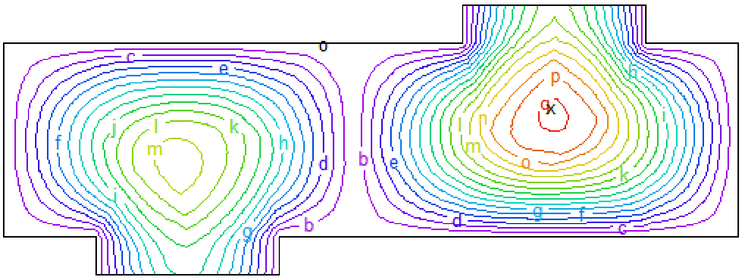


Figure 8.19: Velocity modulus of Eq. (8.244) on Y=0, eigenstate 6.

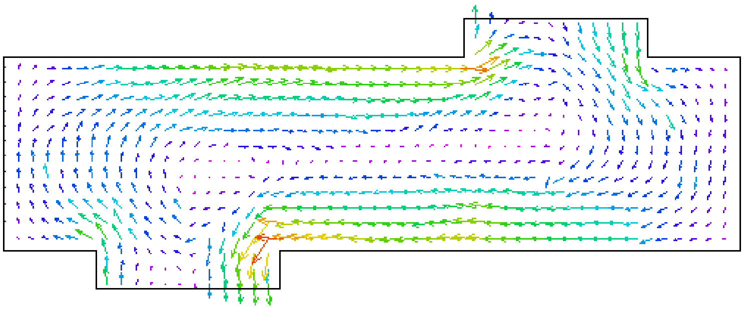


Figure 8.20: Vorticity of Eq. (8.244) on Y=0, eigenstate 1.

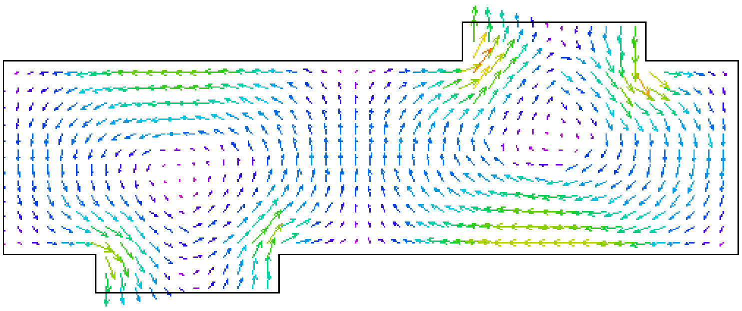


Figure 8.21: Vorticity of Eq. (8.244) on Y=0, eigenstate 6.

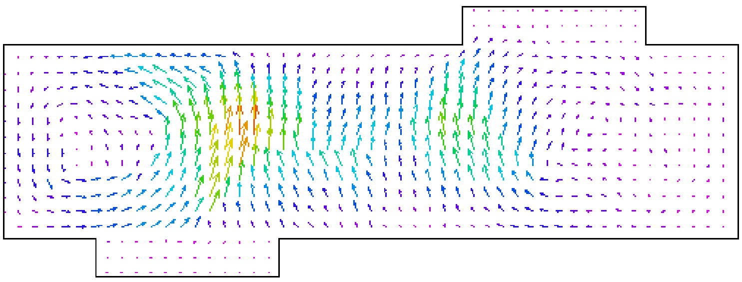


Figure 8.22: Vorticity of Eq. (8.247) on Y=0, eigenstate 1.

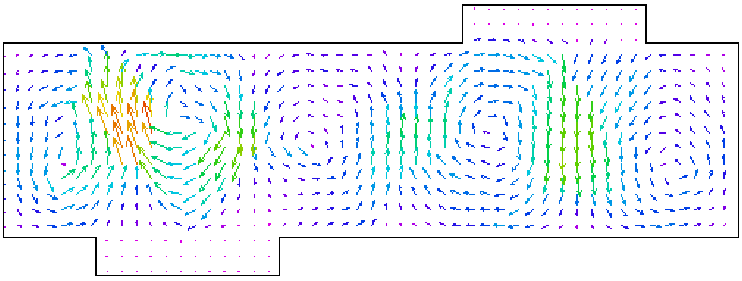


Figure 8.23: Vorticity of Eq. (8.247) on Y=0, eigenstate 4.

Examples of applied fluid electrodynamics

In the following examples, we will consider given material fields that give rise to spacetime fluid effects as described in Section 8.2.4, and present graphics for selected cases.

Electric Coulomb field

■ **Example 8.9** We start by revisiting and expanding Example 8.4. The velocity field of an electric Coulomb field is the solution of the equation

$$\frac{q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r = \frac{\rho_{m(\mathrm{vac})}}{\rho_{(\mathrm{circuit})}}(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} \tag{8.250}$$

(see computer algebra code [150]). For spherical symmetry, the operator (8.206) has to be evaluated and the differential equation for the remaining radial velocity component v_r has to be solved. The result is Eq. (8.207):

$$v_r = \pm\frac{q}{\sqrt{2\pi\varepsilon_0 x}}\sqrt{\frac{1}{r}-c} \tag{8.251}$$

with an integration constant c. This is a function of type $1/\sqrt{r}$ and has been graphed in Fig. 8.24 for three values of c. Setting $c>0$ gives imaginary solutions, and $c<0$ gives asymptotes different from zero for $r\to\infty$; therefore, $c=0$ is the most physically meaningful choice. For comparison, the Coulomb field $-1/(4\pi r^2)$ is also graphed in the figure, and we see that it is descending much more steeply than the velocity fields. ■

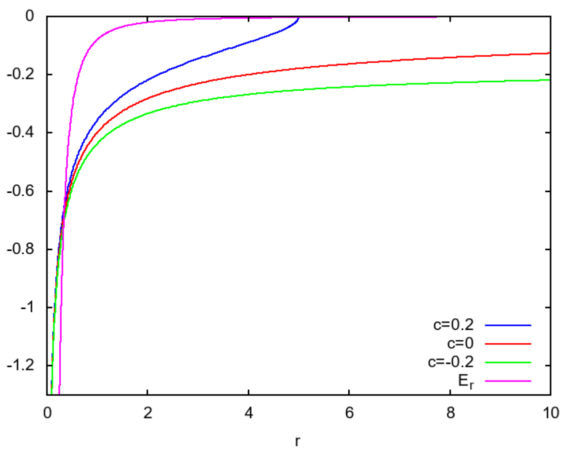


Figure 8.24: Radial velocity component (8.251) of the solution for Eq. (8.250), and Coulomb field E_r.

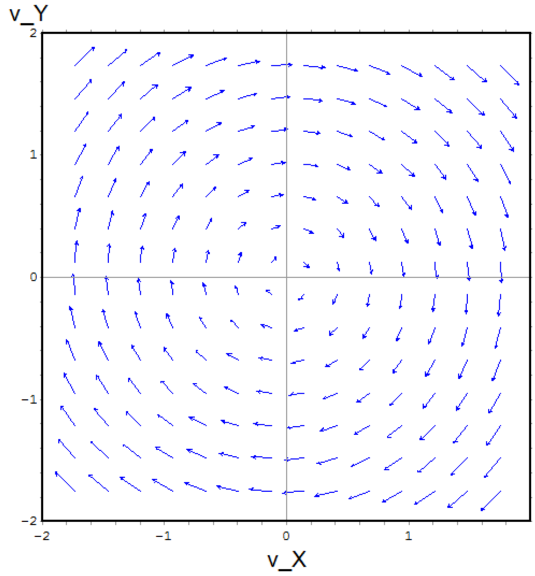


Figure 8.25: Simple rotating velocity field (8.253).

Simple rotating field

■ **Example 8.10** We consider a rotating vector potential in cartesian coordinates:

$$\mathbf{W} = \frac{B^{(0)}}{2}\begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix}. \tag{8.252}$$

This gives a spacetime velocity field:

$$\mathbf{v}_F = \frac{\rho}{\rho_m}\mathbf{W} = \frac{B^{(0)}\rho}{2\rho_m}\begin{bmatrix} Y \\ -X \\ 0 \end{bmatrix}, \tag{8.253}$$

and a resulting vacuum electric field:

$$\mathbf{E}_F = (\mathbf{v}_F \cdot \boldsymbol{\nabla})\mathbf{v}_F = \frac{(B^{(0)})^2\rho^2}{4\rho_m^2}\begin{bmatrix} -X \\ -Y \\ 0 \end{bmatrix}. \tag{8.254}$$

Eq. (8.253) describes a rigid mechanical rotation, since the rotation velocity rises linearly with the radius (see Fig. 8.25). The total derivative operator transforms this into a central electric field, which is also increasing linearly with radial distance (see Fig. 8.26). The velocity field is that of a rigid body, but there is no classical counterpart for the induced electric field. The spacetime velocity further induces both a magnetic field that is constant everywhere:

$$\mathbf{B}_F = \boldsymbol{\nabla} \times \mathbf{v}_F = \frac{B^{(0)}\rho}{\rho_m}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \tag{8.255}$$

and a constant Kambe charge density:

$$q_F = \boldsymbol{\nabla} \cdot \mathbf{E}_F = -\frac{(B^{(0)})^2 \rho^2}{2\rho_m^2}. \tag{8.256}$$

The stationary part of the fluid electric current vanishes:

$$\mathbf{J}_F = a_0^2 \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}_F) = 0. \tag{8.257}$$

■

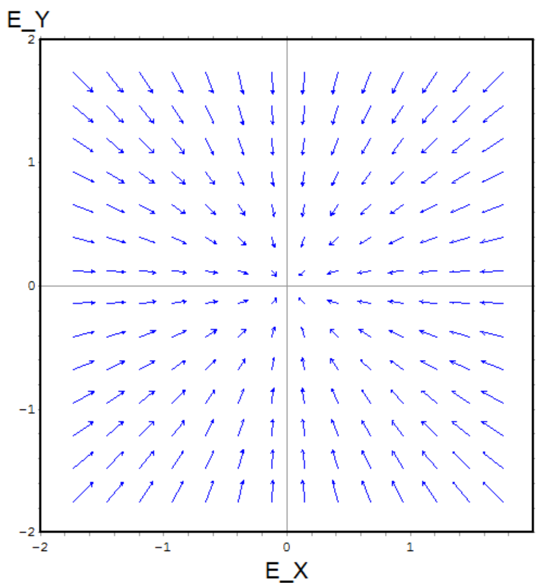


Figure 8.26: Central electric field (8.254), derived from (8.253).

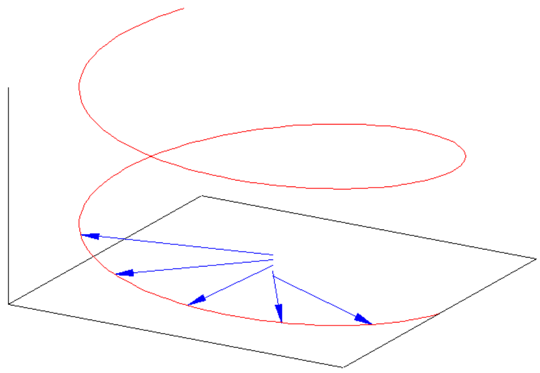


Figure 8.27: Vectors $\mathbf{v}_F$, $\mathbf{B}_F$, and $\mathbf{J}_F$ of the plane wave potential (8.258).

Plane wave potential

■ **Example 8.11** A potential for plane waves in the circular cartesian basis is given by Eqs. (4.124, 4.130):

$$\mathbf{W} = \frac{W^{(0)}}{\sqrt{2}} \exp(i\omega t - \kappa_Z Z) \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \tag{8.258}$$

where ω is the time frequency and κ_Z is the wave vector component in the Z direction. The derived spacetime components are

$$\mathbf{v}_F = \frac{W^{(0)}}{\sqrt{2}}\frac{\rho}{\rho_m}\exp(i\omega t - \kappa_Z Z)\begin{bmatrix}1\\-i\\0\end{bmatrix}, \tag{8.259}$$

$$\mathbf{E}_F = 0, \tag{8.260}$$

$$\mathbf{B}_F = \kappa_Z\frac{W^{(0)}}{\sqrt{2}}\frac{\rho}{\rho_m}\exp(i\omega t - \kappa_Z Z)\begin{bmatrix}1\\-i\\0\end{bmatrix}, \tag{8.261}$$

$$q_F = 0, \tag{8.262}$$

$$\mathbf{J}_F = a_0^2\kappa_Z^2\frac{W^{(0)}}{\sqrt{2}}\frac{\rho}{\rho_m}\exp(i\omega t - \kappa_Z Z)\begin{bmatrix}1\\-i\\0\end{bmatrix}. \tag{8.263}$$

In contrast to the simple rotating field, the derived fluid electric field and charge density disappear. Velocity, magnetic field and current density are all parallel, since they have no Z component. The real part is sketched in Fig. 8.27 for five instants of time t. The tops of the vector arrows describe a helix in space. ■

Magnetostatic current loop

■ **Example 8.12** The field of a circular current loop is best described in spherical polar coordinates (r,θ,ϕ). The vector potential $\mathbf{W}$ of a loop with radius a and current I has only a ϕ component, and is given by

$$\mathbf{W} = \begin{bmatrix}0\\0\\ \dfrac{\mu_0\, a^2\, r\sin(\theta)\, I\left(\frac{15a^2r^2\sin(\theta)^2}{8\left(r^2+a^2\right)^2}+1\right)}{4\left(r^2+a^2\right)^{\frac{3}{2}}}\end{bmatrix}. \tag{8.264}$$

This leads to the velocity field

$$\mathbf{v}_F = \frac{\rho}{\rho_m}\mathbf{W} \neq \mathbf{0}, \tag{8.265}$$

and an electric field perpendicular to $\mathbf{v}_F$ in the (r,θ) plane:

$$\mathbf{E}_F = \frac{\mu_0^2\, a^4\, r\,\rho^2\, I^2\left(\frac{15a^2r^2\sin(\theta)^2}{8\left(r^2+a^2\right)^2}+1\right)^2}{16\rho_m^2\left(r^2+a^2\right)^3}\begin{bmatrix}-\sin(\theta)^2\\ -\cot(\theta)\sin(\theta)^2\\ 0\end{bmatrix}. \tag{8.266}$$

The other fields $\mathbf{B}_F$, q_F, $\mathbf{J}_F$ are also different from zero and highly complicated. $\mathbf{B}_F$ has components in the r and θ directions, and $\mathbf{J}_F$ in the ϕ direction.

The θ dependence of the component v_ϕ is graphed in Fig. 8.28 for $a = 1$. This is largest in the XY plane ($\theta = \pi/2$) and vanishes at the poles. The absolute strength decreases with distance from $r = a$, as expected. The angular distribution of electric field components E_r and E_θ is shown in Fig. 8.29 in a 3D plot. ■

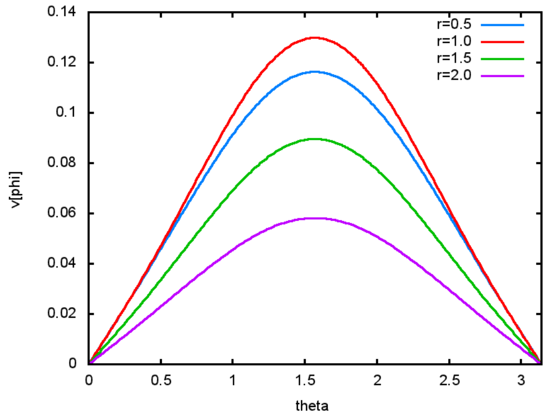


Figure 8.28: Velocity component v_ϕ of the magnetostatic current loop (8.264).

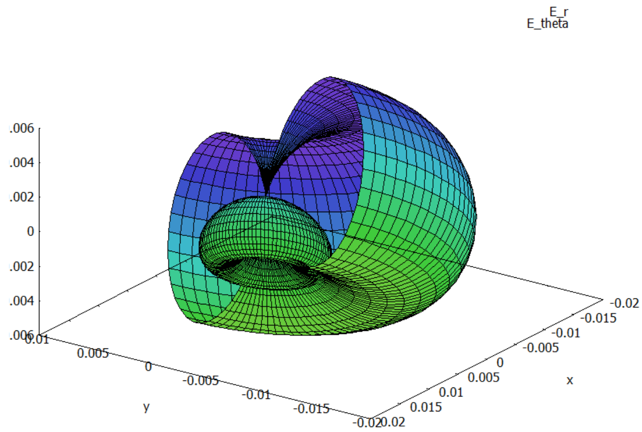


Figure 8.29: Components E_r (outer) and E_θ (inner) of the electric field (8.266) for the magnetostatic current loop, spherical distribution.

8.3 Fluid gravitation

The unification of the field equations of fluid dynamics and gravitation through ECE2 theory, which we call *fluid gravitation*, describes the effect of the fluid vacuum or aether on gravitation. We will now extend the methodology from the preceding section that we used to unify fluid dynamics and electromagnetism and apply it to gravitation, which will also unify it with electromagnetism and thus achieve *triple unification*.

8.3.1 Triple unification

As a starting point, we adapt the equality of force densities used in Eq. (8.165):

$$\rho \mathbf{g} = \rho_m \mathbf{E}_F\,, \tag{8.267}$$

where ρ is the matter density, ρ_m is the vacuum density and $\mathbf{g}$ is the gravitational acceleration field. The densities ρ and ρ_m are different only in matter, where ρ contributes to the force. Outside of matter, ρ is the mechanical aether density and is identical to ρ_m, by definition. Therefore, we can identify Newtonian gravitation with the Kambe electric field directly:

$$\mathbf{g} = -\frac{MG}{r^3}\mathbf{r} = (\mathbf{v}_g \cdot \boldsymbol{\nabla})\,\mathbf{v}_g. \tag{8.268}$$

Newtonian gravitation is the action of acceleration by a mass M of a test mass at distance vector $\mathbf{r}$, and ρ is a mass density concentrated in the gravitational center, as is well known from classical mechanics. $\mathbf{v}_g$ is the aether velocity. A subscript g has been appended to show that it is used here in the context of gravitation. The above equation shows directly that gravitation is connected with the aether. Eq. (8.267) connects the force fields of both realms. $\mathbf{v}_g$ should not be confused with the velocity of the test mass, which is a local velocity and not a velocity field.

The following list shows the gravitational field equations and field definitions of ECE2 theory, as developed in Section 7.1.1, together with the special ECE2 representation of mass and current density by wave vectors, as defined in Eqs. (6.166 - 6.169). (For simplicity, we assume that there are no magnetic densities, and κ_0 has been omitted. The vector potential $\mathbf{Q}$ is a velocity, as discussed in Chapter 7, and it is directly associated with the aether velocity $\mathbf{v}_g$.)

$$\nabla \cdot \boldsymbol{\Omega} = 0, \tag{8.269}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \times \mathbf{g} = \mathbf{0}, \tag{8.270}$$

$$\nabla \cdot \mathbf{g} = -4\pi G \rho = \boldsymbol{\kappa} \cdot \mathbf{g}, \tag{8.271}$$

$$-\frac{1}{c^2}\frac{\partial \mathbf{g}}{\partial t} + \nabla \times \boldsymbol{\Omega} = -\frac{4\pi G}{c^2}\mathbf{J} = \boldsymbol{\kappa} \times \boldsymbol{\Omega}, \tag{8.272}$$

$$\mathbf{g} = -\nabla \phi_g - \frac{\partial \mathbf{v}_g}{\partial t}, \tag{8.273}$$

$$\boldsymbol{\Omega} = \nabla \times \mathbf{v}_g. \tag{8.274}$$

The fluid dynamics equations from the previous sections are as follows:

$$\nabla \cdot \mathbf{H}_F = 0, \tag{8.275}$$

$$\frac{\partial \mathbf{H}_F}{\partial t} + \nabla \times \mathbf{E}_F = \mathbf{0}, \tag{8.276}$$

$$\nabla \cdot \mathbf{E}_F = q_F, \tag{8.277}$$

$$-\frac{1}{a_0^2}\frac{\partial \mathbf{E}_F}{\partial t} + \nabla \times \mathbf{H}_F = \frac{1}{a_0^2}\mathbf{J}_F, \tag{8.278}$$

with the definitions:

$$\mathbf{E}_F = -\nabla h - \frac{\partial \mathbf{v}_F}{\partial t} = (\mathbf{v}_F \cdot \nabla)\,\mathbf{v}_F, \tag{8.279}$$

$$\mathbf{H}_F = \mathbf{w} = \nabla \times \mathbf{v}_F, \tag{8.280}$$

$$q_F = \nabla \cdot ((\mathbf{v}_F \cdot \nabla)\,\mathbf{v}_F), \tag{8.281}$$

$$\mathbf{J}_F = \frac{\partial^2 \mathbf{v}_F}{\partial t^2} + \nabla \frac{\partial h}{\partial t} + a_0^2 \nabla \times (\nabla \times \mathbf{v}_F). \tag{8.282}$$

It can be seen directly that $\mathbf{g}$ corresponds to $\mathbf{E}_F$ and $\boldsymbol{\Omega}$ to $\mathbf{H}_F$. The mechanical potential ϕ_g is identical to the enthalpy h, and as previously stated, $\mathbf{v}_F = \mathbf{v}_g$. From $\rho = \rho_m$ in free space, it follows that

$$\mathbf{g}_{\text{matter}} = \left(-\nabla \phi_g - \frac{\partial \mathbf{v}_g}{\partial t}\right)_{\text{matter}} = \left(-\nabla h - \frac{\partial \mathbf{v}_g}{\partial t}\right)_{\text{vacuum}} \tag{8.283}$$

and

$$(\boldsymbol{\Omega})_{\text{matter}} = (\nabla \times \mathbf{W})_{\text{matter}} = (\nabla \times \mathbf{v}_F)_{\text{vacuum}}\,. \tag{8.284}$$

From comparing Eq. (8.271) to Eq. (8.277), and from $\mathbf{g} = \mathbf{E}_F$ in free space, we obtain

$$(q_F)_{\text{vacuum}} = -\frac{(\rho_m)_{\text{vacuum}}}{4\pi G} = -\frac{(\rho)_{\text{matter}}}{4\pi G} \tag{8.285}$$

or

$$(\nabla \cdot ((\mathbf{v}_F \cdot \nabla)\,\mathbf{v}_F))_{\text{vacuum}} = -\frac{(\rho)_{\text{matter}}}{4\pi G}. \tag{8.286}$$

If the divergence expression in the above equation does not vanish, then this type of spacetime velocity field gives rise to a mass density that acts like a material density. Conversely, any mass density induces a spacetime velocity field. This is an equivalence between matter and aether structures.

■ **Example 8.13** Newton's law of gravitation (8.268) is formally identical to the Coulomb law. Therefore, the spacetime velocity field is the same as the velocity field of an electric Coulomb field, which was computed and graphed in Example 8.9. The velocity field is a central field of the form $1/\sqrt{r}$ (see computer algebra code [151]):

$$v_r = \pm\sqrt{2MG}\sqrt{\frac{1}{r} - c} \tag{8.287}$$

with $c = 0$. The graph (except for constants) is exactly that of Fig. 8.24. ■

Structure of spiral galaxies

■ **Example 8.14** As another example, we compute the spacetime structure of spiral galaxies. From astronomical observations, it is known that the stars in a whirlpool galaxy move with nearly constant velocity, except those close to the center. This result is in conflict with Newton's theory as well as Einstein's general relativity, both of which predict a significant reduction in velocity (see Fig. 8.30). This difference is explained away by assuming that "dark matter" exists in the outer regions of galaxies and that it holds the stars in their positions. It is also implicitly assumed that gravitation is the only force acting in galactic dimensions.

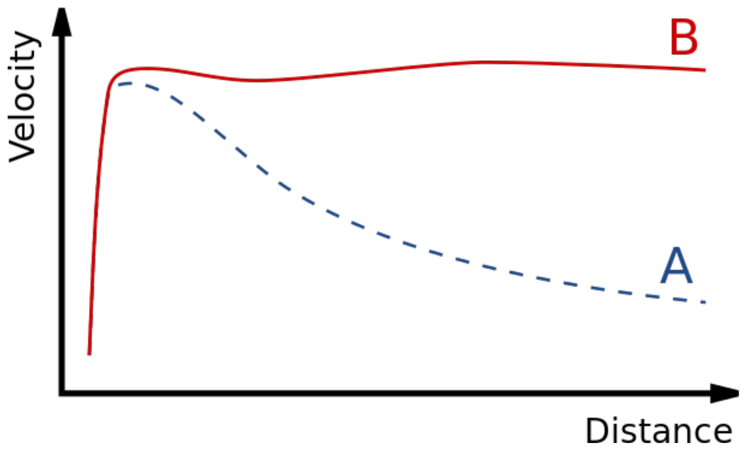


Figure 8.30: Galaxy rotation curve. A: Newtonian theory; B: experimentally observed.

Fluid gravitation gives a much simpler and consistent explanation for the observed structure of whirlpool galaxies. We start by considering the angular momentum of a mass m orbiting a heavy mass M. This is defined by

$$\mathbf{L}_F = m_r \mathbf{r}_F \times \mathbf{v}_F, \tag{8.288}$$

where $\mathbf{r}_F$ is the position vector of mass m taken from the center of mass, and m_r is the reduced mass

$$m_r = \frac{mM}{m+M}. \tag{8.289}$$

In this example, we want to describe the angular momentum of spacetime itself. We do this by taking away the stars, figuratively speaking, and considering only the structure of spacetime. Because there are now no discrete masses, we have to use the angular momentum density instead. We do this by replacing the mass m_r in Eq. (8.288) with the vacuum mass density ρ_m. The equation then reads:

$$\widehat{\mathbf{L}}_F = \rho_m \mathbf{r}_F \times \mathbf{v}_F, \tag{8.290}$$

where $\widehat{\mathbf{L}}_F$ stands for the angular momentum density of the distributed vacuum mass density ρ_m.

By applying the vector function identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \tag{8.291}$$

we obtain from Eq. (8.290):

$$\mathbf{r}_F \times \widehat{\mathbf{L}}_F = \rho_m \mathbf{r}_F \times (\mathbf{r}_F \times \mathbf{v}_F) = \rho_m \left((\mathbf{r}_F \cdot \mathbf{v}_F)\mathbf{r}_F - (\mathbf{r}_F \cdot \mathbf{r}_F)\right)\mathbf{v}_F. \tag{8.292}$$

Since the rotational direction in a spiral galaxy is perpendicular to the distance $\mathbf{r}_F$ from the center, we have

$$\mathbf{r}_F \cdot \mathbf{v}_F = 0, \tag{8.293}$$

and Eq. (8.292) simplifies to

$$\mathbf{v}_F = \frac{1}{\rho_m r_F^2} \widehat{\mathbf{L}}_F \times \mathbf{r}_F. \tag{8.294}$$

We make the XY plane coincident with the galactic plane, so that the angular momentum density will have only a Z component L_{FZ} in cartesian coordinates:

$$\widehat{\mathbf{L}}_F = \widehat{L}_{FZ} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{8.295}$$

With the radius vector

$$\mathbf{r}_F = \begin{bmatrix} X_F \\ Y_F \\ 0 \end{bmatrix}, \tag{8.296}$$

Eq. (8.294) gives the result

$$\mathbf{v}_F = \frac{\widehat{L}L_{FZ}}{\rho_m r_F^2} \begin{bmatrix} -Y_F \\ X_F \\ 0 \end{bmatrix}. \tag{8.297}$$

This is a velocity rotating in the XY plane, which is similar to Example 8.10 (see Fig. 8.25). It is also divergenceless:

$$\boldsymbol{\nabla} \cdot \mathbf{v}_F = 0. \tag{8.298}$$

The gravitomagnetic field of an orbiting volume element at distance r_F from the center (assuming a constant ρ_m) is

$$\boldsymbol{\Omega} = \boldsymbol{\nabla} \times \mathbf{v}_F = \frac{2}{\rho_m r_F^2} \widehat{\mathbf{L}}_F. \tag{8.299}$$

Using the velocity (8.297) gives us the acceleration field:

$$\mathbf{g}_{\text{vacuum}} = (\mathbf{v}_F \cdot \boldsymbol{\nabla})\,\mathbf{v}_F = \frac{\widehat{L}_{FZ}^2}{\rho_m^2 r_F^4} \begin{bmatrix} -X_F \\ -Y_F \\ 0 \end{bmatrix} = -\frac{\widehat{L}_{FZ}^2}{\rho_m^2 r_F^4} \mathbf{r} = -\frac{\widehat{L}_{FZ}^2}{\rho_m^2 r_F^3} \mathbf{e}_r \tag{8.300}$$

(see computer algebra code [152] for all computations). This is an inverse cube law, in contrast to Newton's inverse square law of gravitation. The dynamics solutions for such a potential are not closed orbits but spiral orbits, as will be shown in the next chapter. In this example, we consider only the asymptotic behavior of the velocity.

When written in plane polar coordinates $(r,\ \theta)$, the modulus of the angular momentum density (8.290) is

$$\widehat{L} = \rho_m r v = \rho_m r^2 \frac{d\theta}{dt}, \tag{8.301}$$

which gives us

$$\frac{d\theta}{dt} = \frac{\widehat{L}}{\rho_m r^2}. \tag{8.302}$$

$\widehat{L}$ is a constant of motion. The squared velocity in plane polar coordinates, in general, is

$$\begin{aligned} v^2 &= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right) \\ &= \left(\frac{\widehat{L}}{\rho_m r^2}\right)^2\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right). \end{aligned} \tag{8.303}$$

For the limit $r \to \infty$, the derivative stays finite for bound orbits, so we can write (with a limiting parameter r_0):

$$v^2 = \left(\frac{\widehat{L}}{\rho_m}\right)^2\left(\frac{1}{r^2} + \frac{1}{r_0^2}\right) \xrightarrow[r\to\infty]{} \left(\frac{\widehat{L}}{\rho_m r_0}\right)^2 = \text{const.} \tag{8.304}$$

This matches the velocity curve observed in astronomy and supports fluid gravitation. The stars of a whirlpool galaxy are "swimming" in the aether or spacetime. Because of their huge distances from the galactic center, Newtonian acceleration is extremely small and vanishes into the aether background. Their motion is determined by the aether flow in space. No dark matter or dark energy is required to explain the structure of spiral galaxies, so these concepts have to be ruled out according to Occam's razor.

From Eqs. (8.298) and (8.149), it follows that

$$\frac{\partial \Phi_F}{\partial t} = \frac{\partial h_F}{\partial t} = 0. \tag{8.305}$$

The scalar potential of a whirlpool galaxy is constant. The vacuum charge of a galaxy diverges for $r \to 0$:

$$q_F = 2\boldsymbol{\nabla}\cdot\mathbf{g}_{\text{vacuum}} = \left(\frac{\widehat{L}_{FZ}}{\rho_m r}\right)^2 \xrightarrow[r\to 0]{} \infty\ , \tag{8.306}$$

which gives rise to real mass accumulations. There is a very large mass at the center of a galaxy, as has been observed. If $\mathbf{E}_F$ is time-independent, the spacetime current, according to (8.282), is

$$\mathbf{J}_F = a_0^2\,\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}_F). \tag{8.307}$$

Inserting the spacetime velocity (8.297) leads to

$$\mathbf{J}_F = \mathbf{0}, \tag{8.308}$$

indicating that, because of the static structure of potentials, there is no spacetime current, although a velocity field is present. We have to discern properly between a spacetime flow and a "condensed" structure that appears to be a moving mass density. During the flow, the mass density of spacetime does not change. A mass density of spacetime appears as a static mass density only at the center of the galaxy.

Please note that this does not apply to the motion of stars, because the fixed stars of a whirlpool galaxy are moving with the aether or spacetime, so there is necessarily a mass current of stars. However, this mass current is a derivative effect that comes from spacetime, but is not included in the rotation of spacetime itself. ■

8.3.2 Non-classical acceleration

We have seen that in fluid dynamics the velocity field depends on time, and on coordinates that in turn depend on time, for example, in cartesian coordinates: $\mathbf{v} = \mathbf{v}(X(t), Y(t), Z(t), t)$. In mass point dynamics, there is no velocity field but only a velocity vector with a time dependence: $\mathbf{v} = \mathbf{v}(t)$. This vector is to be applied at the position of the mass point. Therefore, the convective derivative of the velocity is different from that of a mass point. The convective acceleration field is

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v}, \tag{8.309}$$

while in mass point dynamics the acceleration is simply

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}. \tag{8.310}$$

In Example 7.4 (previous chapter), the velocity and acceleration for mass points was worked out in spherical coordinates. Generally, in non-cartesian coordinates, the above acceleration can be written as a covariant derivative with a spin connection matrix $\boldsymbol{\Omega}$:

$$\mathbf{a} = \frac{D}{Dt}\mathbf{v} = \frac{\partial}{\partial t}\mathbf{v} + \boldsymbol{\Omega}\mathbf{v} \tag{8.311}$$

(see Eqs. (7.174, 7.175)). In Section 8.2.3, it was shown that the convective derivative of fluid dynamics is also an example of a covariant derivative of Cartan geometry.

We will now discuss the differences between fluid dynamics and mass point dynamics in cylindrical and plane polar coordinates, because both play a central role in rotational systems and gravitation. The convective derivative in cylindrical coordinates $(r,\ \theta,\ Z)$ has already been given by Eq. (8.205). With $\mathbf{a} = \mathbf{b} = \mathbf{v}$, it is

$$(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = \begin{bmatrix} v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_Z \frac{\partial v_r}{\partial Z} - \frac{v_\theta^2}{r} \\ v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_Z \frac{\partial v_\theta}{\partial Z} + \frac{v_\theta v_r}{r} \\ v_r \frac{\partial v_Z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_Z}{\partial \theta} + v_Z \frac{\partial v_Z}{\partial Z} \end{bmatrix}. \tag{8.312}$$

Details of its derivation can be found in [93]. The convective derivative is

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial}{\partial t}\begin{bmatrix} v_r \\ v_\theta \\ v_Z \end{bmatrix} + \begin{bmatrix} v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_Z \frac{\partial v_r}{\partial Z} - \frac{v_\theta^2}{r} \\ v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_Z \frac{\partial v_\theta}{\partial Z} + \frac{v_\theta v_r}{r} \\ v_r \frac{\partial v_Z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_Z}{\partial \theta} + v_Z \frac{\partial v_Z}{\partial Z} \end{bmatrix}. \tag{8.313}$$

The right side can be decomposed into a matrix-vector product with two matrices in the following way:

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial}{\partial t}\begin{bmatrix} v_r \\ v_\theta \\ v_Z \end{bmatrix} + \left(\begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial Z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} & \frac{\partial v_\theta}{\partial Z} \\ \frac{\partial v_Z}{\partial r} & \frac{1}{r}\frac{\partial v_Z}{\partial \theta} & \frac{\partial v_Z}{\partial Z} \end{bmatrix} + \begin{bmatrix} 0 & -\frac{v_\theta}{r} & 0 \\ \frac{v_\theta}{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_r \\ v_\theta \\ v_Z \end{bmatrix}. \tag{8.314}$$

The second matrix has the antisymmetric structure of a rotation generator. This equation is a special case of the Cartan covariant derivative:

$$\frac{Dv^a}{Dt} = \frac{\partial v^a}{\partial t} + \omega^a{}_{0b} v^b, \tag{8.315}$$

as was previously discussed in Section 8.2.3. In cylindrical coordinates, the components of the velocity are

$$\begin{bmatrix} v_r \\ v_\theta \\ v_Z \end{bmatrix} = \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ \dot{Z} \end{bmatrix}, \tag{8.316}$$

and therefore Eq. (8.314) can be written as

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial}{\partial t} \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ \dot{Z} \end{bmatrix} + \left(\begin{bmatrix} \frac{\partial \dot{r}}{\partial r} & \frac{1}{r}\frac{\partial \dot{r}}{\partial \theta} & \frac{\partial \dot{r}}{\partial Z} \\ \frac{\partial (r\dot{\theta})}{\partial r} & \frac{1}{r}\frac{\partial (r\dot{\theta})}{\partial \theta} & \frac{\partial (r\dot{\theta})}{\partial Z} \\ \frac{\partial \dot{Z}}{\partial r} & \frac{1}{r}\frac{\partial \dot{Z}}{\partial \theta} & \frac{\partial \dot{Z}}{\partial Z} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ \dot{Z} \end{bmatrix}. \tag{8.317}$$

The second line of the first matrix contains terms like $\partial(r\dot{\theta})/\partial r$. From the independence of coordinates, it follows that

$$\frac{\partial r}{\partial r} = 1, \quad \frac{\partial r}{\partial \theta} = 0, \quad \text{etc.}, \tag{8.318}$$

and for this line we obtain

$$\left(\dot{\theta} + r\frac{\partial \dot{\theta}}{\partial r}, \ \frac{\partial \dot{\theta}}{\partial \theta}, \ r\frac{\partial \dot{\theta}}{\partial Z} \right). \tag{8.319}$$

Computer algebra (see computer code [153]) shows us that the convective acceleration is

$$\mathbf{a} = \frac{D\mathbf{v}}{Dt} = \begin{bmatrix} \ddot{r} + 2\frac{\partial \dot{r}}{\partial Z}\dot{Z} - r\dot{\theta}^2 + 2\frac{\partial \dot{r}}{\partial \theta}\dot{\theta} + 2\dot{r}\frac{\partial \dot{r}}{\partial r} \\ r\ddot{\theta} + 2r\frac{\partial \dot{\theta}}{\partial Z}\dot{Z} + 2r\dot{\theta}\frac{\partial \dot{\theta}}{\partial \theta} + 2r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + 3\dot{r}\dot{\theta} \\ \ddot{Z} + 2\dot{Z}\frac{\partial \dot{Z}}{\partial Z} + 2\dot{\theta}\frac{\partial \dot{Z}}{\partial \theta} + 2\dot{r}\frac{\partial \dot{Z}}{\partial r} \end{bmatrix}. \tag{8.320}$$

Please note that the dotted variables are components of the velocity; therefore, as an example for the general case, we have

$$\frac{\partial \dot{\theta}}{\partial r} = \frac{\partial \dot{\theta}(r,\theta,Z)}{\partial r} \neq 0. \tag{8.321}$$

However, for equal variables in the numerator and denominator, we have

$$\frac{\partial \dot{r}}{\partial r} = \frac{d}{dt}\frac{\partial r(\theta,Z)}{\partial r} = 0, \tag{8.322}$$

$$\frac{\partial \dot{\theta}}{\partial \theta} = \frac{d}{dt}\frac{\partial \theta(r,Z)}{\partial \theta} = 0, \tag{8.323}$$

$$\frac{\partial \dot{Z}}{\partial Z} = \frac{d}{dt}\frac{\partial Z(r,\theta)}{\partial Z} = 0. \tag{8.324}$$

Here, $r(\theta,\ Z)$, etc., are functions. This further simplifies Eq. (8.320):

$$\mathbf{a} = \begin{bmatrix} \ddot{r} + 2\frac{\partial \dot{r}}{\partial Z}\dot{Z} - r\dot{\theta}^2 + 2\frac{\partial \dot{r}}{\partial \theta}\dot{\theta} \\ r\ddot{\theta} + 2r\frac{\partial \dot{\theta}}{\partial Z}\dot{Z} + 2r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + 3\dot{r}\dot{\theta} \\ \ddot{Z} + 2\dot{\theta}\frac{\partial \dot{Z}}{\partial \theta} + 2\dot{r}\frac{\partial \dot{Z}}{\partial r} \end{bmatrix}. \tag{8.325}$$

The Newtonian acceleration (7.161) in plane polar coordinates is

$$\mathbf{a}_N = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta. \tag{8.326}$$

The term $-r\dot{\theta}^2\mathbf{e}_r$ is the centrifugal acceleration, and $(r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$ is the Coriolis acceleration. In cylindrical coordinates, the acceleration is extended by a term $\ddot{Z}\mathbf{e}_Z$, thus the Newtonian acceleration, written in component form, becomes

$$\mathbf{a}_N = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ \ddot{Z} \end{bmatrix}. \tag{8.327}$$

This is only a part of the convective derivative (8.320). Denoting the difference between $\mathbf{a}$ and $\mathbf{a}_N$ by $\mathbf{a}_1$, we have

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}_1 \tag{8.328}$$

with

$$\mathbf{a}_1 = \begin{bmatrix} 2\frac{\partial \dot{r}}{\partial Z}\dot{Z} + 2\frac{\partial \dot{r}}{\partial \theta}\dot{\theta} \\ 2r\frac{\partial \dot{\theta}}{\partial Z}\dot{Z} + 2r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{r}\dot{\theta} \\ 2\dot{\theta}\frac{\partial \dot{Z}}{\partial \theta} + 2\dot{r}\frac{\partial \dot{Z}}{\partial r} \end{bmatrix}. \tag{8.329}$$

We see that $\mathbf{a}_1$ consists mainly of terms arising from the coordinate dependence of the velocity field $\mathbf{v}$, as expected. The structure of the convective acceleration field is much more complicated than the Newtonian acceleration.

In the case of plane polar coordinates, the third component disappears, and thus all Z dependencies disappear as well. The accelerations then become

$$\mathbf{a} = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 + 2\frac{\partial \dot{r}}{\partial \theta}\dot{\theta} \\ r\ddot{\theta} + 2r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + 3\dot{r}\dot{\theta} \end{bmatrix}, \tag{8.330}$$

$$\mathbf{a}_N = \begin{bmatrix} \ddot{r} - r\dot{\theta}^2 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{bmatrix}, \tag{8.331}$$

$$\mathbf{a}_1 = \begin{bmatrix} 2\frac{\partial \dot{r}}{\partial \theta}\dot{\theta} \\ 2r\dot{r}\frac{\partial \dot{\theta}}{\partial r} + \dot{r}\dot{\theta} \end{bmatrix}. \tag{8.332}$$

■ **Example 8.15** As a non-trivial example[4], we consider a three-dimensional vortex field called the *Torkado* [94] (see Fig. 8.31). (This field could also be an explanation for the dynamics of the plasma model of galaxies.) We concentrate on a streamline in the middle of the structure graphed in Fig. 8.32. The flow is slow in the outer region and goes up very quickly in the inner tube. It is a continuous motion that is not caused by an external force, in our consideration. The central streamline can be described by an analytical approach in cylindrical coordinates $(r,\ \theta,\ Z)$:

$$r(\theta) = r_0 + r_1 \cos\left(\frac{\theta}{10}\right)^2, \tag{8.333}$$

$$Z(\theta) = -Z_0 \sin\left(\frac{\theta}{5}\right), \tag{8.334}$$

[4] This is a revision of UFT Paper 361.

with constants $r_0,\ r_1,\ Z_0$. For a plot in cartesian coordinates, we transform the cylindrical orbit using

$$X = r(\theta)\cos(\theta), \tag{8.335}$$
$$Y = r(\theta)\sin(\theta), \tag{8.336}$$
$$Z = Z(\theta). \tag{8.337}$$

We assume conservation of angular momentum around the Z axis:

$$L_Z = mr^2\dot{\theta}, \tag{8.338}$$

so that the angular velocity $\dot{\theta}$ can be expressed by the r coordinate function (8.333):

$$\dot{\theta} = \frac{L_Z}{mr^2}. \tag{8.339}$$

The other time derivatives can be rewritten as follows:

$$\ddot{\theta} = \frac{d}{dt}\dot{\theta} = \frac{dr}{dt}\frac{\partial\dot{\theta}}{\partial r} = -2\dot{r}\frac{L_Z}{mr^3}, \tag{8.340}$$
$$\dot{r} = \frac{d\theta}{dt}\frac{\partial r}{\partial\theta} = \dot{\theta}\frac{\partial r}{\partial\theta}, \tag{8.341}$$
$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(\dot{\theta}\frac{\partial r}{\partial\theta}\right) = \ddot{\theta}\frac{\partial r}{\partial\theta} + \dot{\theta}\frac{\partial\dot{r}}{\partial\theta}, \tag{8.342}$$
$$\dot{Z} = \frac{d\theta}{dt}\frac{\partial Z}{\partial\theta} = \dot{\theta}\frac{\partial Z}{\partial\theta}, \tag{8.343}$$
$$\ddot{Z} = \frac{d}{dt}\dot{Z} = \frac{d}{dt}\left(\dot{\theta}\frac{\partial Z}{\partial\theta}\right) = \ddot{\theta}\frac{\partial Z}{\partial\theta} + \dot{\theta}\frac{\partial\dot{Z}}{\partial\theta}. \tag{8.344}$$

In this way, any explicit time dependence is eliminated, and we obtain dependencies on the orbits $r(\theta)$ and $Z(\theta)$, exclusively.

Now, we can compute the fluid acceleration (8.325), and its constituents (8.327) and (8.329). This will show us how fluid spacetime affects the classical motion of the central Torkado streamline. The calculations are a bit complicated, but can be understood by reviewing the computer algebra code [154].

First, we consider the velocity components (8.316). In the calculation, please note that r and Z are functions of θ, as are $\dot{r}$ and $\dot{\theta}$, and that they are different from the pure coordinates r and Z. This important mathematical distinction is not always presented clearly in physics. The velocity components are graphed in Fig. 8.33. The radial component changes sign, when the streamline is nearest to the Z axis. The angular component is always positive, indicating that there are no turning points, and the Z component seems to be mostly positive. In the regions between the maxima, it is slightly negative, which is not recognizable because of the 20-fold reduction factor in the graph. Please note that this calculation does not produce the time dynamics of the motion. The flow elements spend a very long time in the outer region and only a very short time in the region with high velocities. Therefore, the continuity of motion in the Z direction is guaranteed.

The Newtonian acceleration is graphed in Fig. 8.34. There is no angular acceleration component, and this could be surprising, but it follows from the fact that there is no external potential and obviously no Coriolis-like force. If only a central, radial force is present, Newtonian central motion stays in a plane. Here we have no fixed plane of motion but, in spite of this, we still have no forces that would initiate an angular acceleration. Nevertheless, an angular velocity component is present, and it is imposed by the central acceleration.

If we consider the $\mathbf{a}_1$ deviation from the Newtonian case (see Fig. 8.35), fluid effects add positive contributions to the otherwise negative radial acceleration. They also produce an angular acceleration that is not present in the Newtonian case, and the acceleration in the Z direction is strongly enhanced. The total fluid acceleration $\mathbf{a}$ is graphed in Fig. 8.36. The positive peaks of the radial acceleration lead to indentations in the curve, compared to the Newtonian case. This effect is characteristic of fluid dynamics. ■

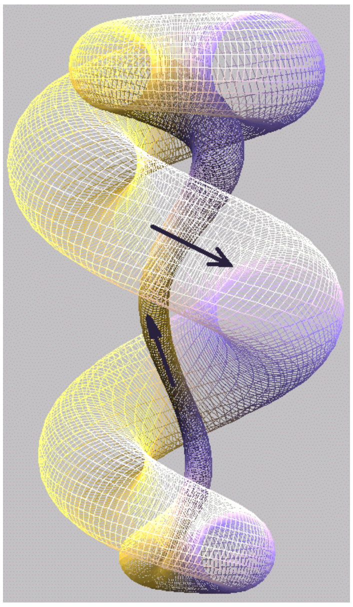


Figure 8.31: Flux structure of the Torkado 3D orbit (vortex) [187].

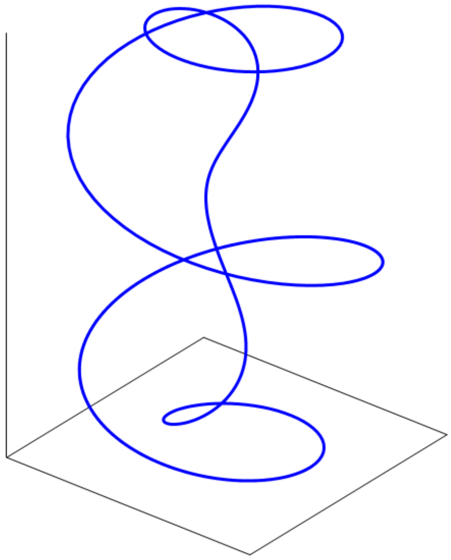


Figure 8.32: Central streamline within the Torkado 3D orbit.

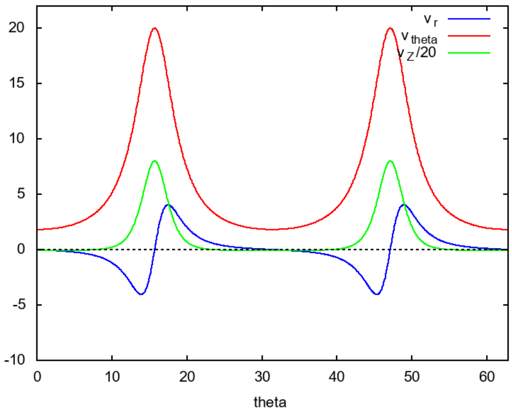


Figure 8.33: Angular dependence of velocity components, Eq. (8.316).

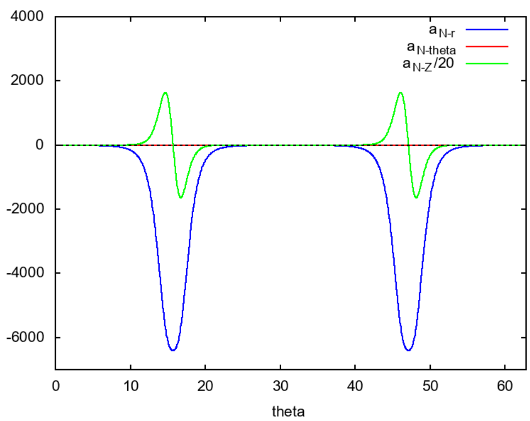


Figure 8.34: Angular dependence of Newtonian acceleration components, Eq. (8.327).

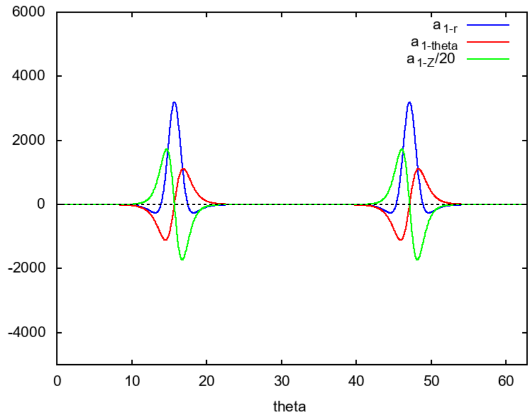


Figure 8.35: Angular dependence of non-Newtonian acceleration components, Eq. (8.329).

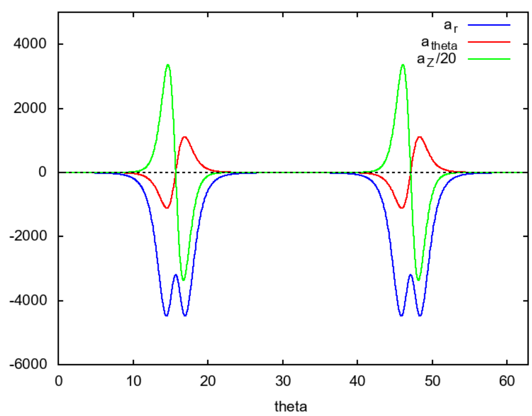


Figure 8.36: Angular dependence of total acceleration components, Eq. (8.325).

8.3.3 Impact of spin connection on gravitation

In classical dynamics the vacuum is a "nothingness", but in fluid gravitation it is richly structured, as discussed earlier in this chapter. We limit this discussion to planar orbits in a plane polar coordinate system. From Eq. (8.314) it follows that the acceleration in fluid spacetime in two dimensions can be written as

$$\frac{D}{Dt}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\end{bmatrix}=\frac{\partial}{\partial t}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\end{bmatrix}+\begin{bmatrix}0 & -\dot{\theta}\\ \dot{\theta} & 0\end{bmatrix}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\end{bmatrix}+\begin{bmatrix}\Omega^1{}_{01v} & \Omega^1{}_{02v}\\ \Omega^2{}_{01v} & \Omega^2{}_{02v}\end{bmatrix}\begin{bmatrix}\dot{r}\\ r\dot{\theta}\end{bmatrix} \tag{8.345}$$

with the spin connection matrix

$$\begin{bmatrix}\Omega^1{}_{01v} & \Omega^1{}_{02v}\\ \Omega^2{}_{01v} & \Omega^2{}_{02v}\end{bmatrix}=\begin{bmatrix}\frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial\theta}\\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\end{bmatrix}. \tag{8.346}$$

The index v indicates that the spin connection relates to the velocity $\mathbf{v}=\left[\dot{r},\ r\dot{\theta}\right]$. In general, the convective derivative of a vector field $\mathbf{F}$ is

$$\frac{D\mathbf{F}}{Dt}=\frac{\partial\mathbf{F}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{F}. \tag{8.347}$$

We can identify $\mathbf{F}$ with the position vector of an element of a fluid:

$$\mathbf{R}=\mathbf{R}(\mathbf{r},t)\,. \tag{8.348}$$

It follows that the velocity field of the fluid is

$$\mathbf{v}(\mathbf{r},t)=\frac{D\mathbf{R}}{Dt}=\frac{\partial\mathbf{R}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{R}. \tag{8.349}$$

This can be written with the spin connection as

$$\frac{D}{Dt}\begin{bmatrix}R_r\\ R_\theta\end{bmatrix}=\frac{\partial}{\partial t}\begin{bmatrix}R_r\\ R_\theta\end{bmatrix}+\begin{bmatrix}\Omega^1{}_{01r} & \Omega^1{}_{02r}\\ \Omega^2{}_{01r} & \Omega^2{}_{02r}\end{bmatrix}\begin{bmatrix}R_r\\ R_\theta\end{bmatrix}. \tag{8.350}$$

This spin connection is different from the one for the velocity; therefore, we use an index r here. The structure of the spin connection matrix can be found by evaluating the term $(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{R}$ in Eq. (8.349). With

$$\mathbf{R}=R_r\mathbf{e}_r+R_\theta\mathbf{e}_\theta\,, \tag{8.351}$$

computer algebra gives

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{R} = \begin{bmatrix} -\frac{R_\theta v_\theta}{r} + \frac{\left(\frac{\partial}{\partial\theta}R_r\right)v_\theta}{r} + \left(\frac{\partial}{\partial r}R_r\right)v_r \\ \frac{\left(\frac{\partial}{\partial\theta}R_\theta\right)v_\theta}{r} + \frac{R_r v_\theta}{r} + \left(\frac{\partial}{\partial r}R_\theta\right)v_r \end{bmatrix} \tag{8.352}$$

(see computer algebra code [155]). Similarly to Eq. (8.345), Eq. (8.350) can also be rewritten with a sum of two matrix-vector products:

$$\frac{D}{Dt}\begin{bmatrix} R_r \\ R_\theta \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} R_r \\ R_\theta \end{bmatrix} + \begin{bmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{bmatrix}\begin{bmatrix} R_r \\ R_\theta \end{bmatrix} + \begin{bmatrix} {\Omega^1}_{01} & {\Omega^1}_{02} \\ {\Omega^2}_{01} & {\Omega^2}_{02} \end{bmatrix}\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \tag{8.353}$$

with

$$\begin{bmatrix} {\Omega^1}_{01} & {\Omega^1}_{02} \\ {\Omega^2}_{01} & {\Omega^2}_{02} \end{bmatrix} = \begin{bmatrix} \frac{\partial R_r}{\partial r} & \frac{1}{r}\frac{\partial R_r}{\partial\theta} \\ \frac{\partial R_\theta}{\partial r} & \frac{1}{r}\frac{\partial R_\theta}{\partial\theta} \end{bmatrix}. \tag{8.354}$$

This is also shown by the computer algebra code. Please note that the second matrix is now multiplied by the velocity vector, and not the position vector. Therefore, the elements of this matrix are dimensionless, in contrast to the usual spin connection.

This calculation can be simplified further. In plane polar coordinates, the position vector has a radial component only:

$$\mathbf{R} = r\mathbf{e}_r, \tag{8.355}$$

thus:

$$R_r = r,\; R_\theta = 0. \tag{8.356}$$

Therefore, Eq. (8.353) takes the form

$$\frac{D}{Dt}\begin{bmatrix} r \\ 0 \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{bmatrix}\begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} {\Omega^1}_{01} & {\Omega^1}_{02r} \\ {\Omega^2}_{01} & {\Omega^2}_{02r} \end{bmatrix}\begin{bmatrix} \dot r \\ r\dot\theta \end{bmatrix}, \tag{8.357}$$

and the applicable spin connection matrix is

$$\begin{bmatrix} {\Omega^1}_{01} & {\Omega^1}_{02r} \\ {\Omega^2}_{01} & {\Omega^2}_{02r} \end{bmatrix} = \begin{bmatrix} \frac{\partial R_r}{\partial r} & \frac{1}{r}\frac{\partial R_r}{\partial\theta} \\ 0 & 0 \end{bmatrix}. \tag{8.358}$$

The velocity field components are

$$v_r = \left(1 + {\Omega^1}_{01}\right)\dot r + {\Omega^1}_{02}r\dot\theta \tag{8.359}$$

and

$$v_\theta = \dot\theta r = \omega r. \tag{8.360}$$

The spin connections affect only the radial part of the velocity.

In contrast to the above analysis, classical dynamics is defined by

$$\mathbf{v}(t) = \frac{D\mathbf{r}(t)}{Dt} = \frac{\partial\mathbf{r}(t)}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{r}(t), \tag{8.361}$$

i.e., by the convective derivative of the position $\mathbf{r}(t)$ of a particle, rather than the position $\mathbf{R}(\mathbf{r},t)$ of a fluid element. Therefore, in classical dynamics, the following holds:

$$\begin{bmatrix} v_r \\ v_\theta \end{bmatrix} = \frac{\partial}{\partial t} \begin{bmatrix} r(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ 0 \end{bmatrix}. \tag{8.362}$$

In component format, the above equation is

$$v_r = \frac{\partial r(t)}{\partial t}, \tag{8.363}$$

$$v_\theta = r(t)\ \dot{\theta}, \tag{8.364}$$

as is well known from orbital dynamics.

■ **Example 8.16** In the following example, we use the fluid dynamics velocities (8.359) and (8.360) to compute the equations of motion. The Hamiltonian and Lagrangian are

$$\mathscr{H} = \frac{1}{2} m \left(v_r^2 + v_\theta^2\right) + U \tag{8.365}$$

and

$$\mathscr{L} = \frac{1}{2} m \left(v_r^2 + v_\theta^2\right) - U, \tag{8.366}$$

where U is the potential energy. Deviations from a closed elliptic orbit due to spin connection terms are expected to result in a precession. It is well known that in the Solar System precession is a very tiny effect, so

$$\Omega^1{}_{01} \sim \Omega^1{}_{02} \ll 1. \tag{8.367}$$

The Euler-Lagrange equations (7.202) of the system are

$$\frac{\partial \mathscr{L}}{\partial r} = \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{r}} \tag{8.368}$$

and

$$\frac{\partial \mathscr{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{\theta}}. \tag{8.369}$$

When

$$U = -\frac{mMG}{r}, \tag{8.370}$$

the Lagrangian is

$$\mathscr{L} = \frac{m}{2} \left(\left(\Omega^1{}_{02} r\dot{\theta} + (\Omega^1{}_{01} + 1)\dot{r}\right)^2 + r^2\dot{\theta}^2 \right) + \frac{mMG}{r}. \tag{8.371}$$

This leads to a new type of equations of motion, which are quite complicated (see computer algebra code [155]). The orbit is changed by $\Omega^1{}_{01}$ and $\Omega^1{}_{02}$, but the equations become much simpler if we assume that

$$\Omega^1{}_{02} \approx 0. \tag{8.372}$$

Then, the Lagrange equations take the form

$$\ddot{r} = \frac{r^3\dot{\theta}^2 - MG}{(\Omega^1{}_{01}+1)^2 r^2}, \tag{8.373}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}. \tag{8.374}$$

The Lagrangian (8.371) does not explicitly depend on θ, so in (8.369) we have

$$\frac{\partial \mathscr{L}}{\partial \theta} = 0, \tag{8.375}$$

and $\partial\mathscr{L}/\partial\dot{\theta}$ is a constant of motion, the angular momentum:

$$L = \frac{\partial \mathscr{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}, \tag{8.376}$$

and it is conserved in the assumed case, where $\Omega^1{}_{02} = 0$.

The equations of motion have been solved numerically by the classical Runge-Kutta method with initial conditions for bound orbits. This gives the trajectories $\theta(t)$ and $r(t)$. We will study such trajectories in detail in the next chapter. The orbit $r(\theta)$ has been graphed in two-dimensional plots. In Fig. 8.37, the orbits for $\Omega^1{}_{01} = 0$ and $\Omega^1{}_{01} = 0.02$ are compared. In the case of a vanishing spin connection, an elliptic orbit is obtained, as expected. A spin connection $\Omega^1{}_{01} > 0$ gives a forward precessing orbit. If the spin connection is negative ($\Omega^1{}_{01} = -0.02$), a backward precession is obtained, as can be seen in Fig. 8.38. As a result, the existence of a single fluid dynamic spin connection term is sufficient to produce non-Newtonian orbits. Alternatively, such precessing ellipses can also be obtained through relativistic effects. This will be shown in the next chapter of this book. ■

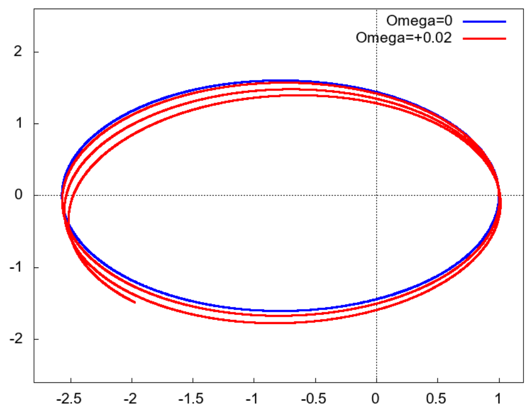


Figure 8.37: Orbital precession for $\Omega^0{}_{01} > 0$ (forward precession).

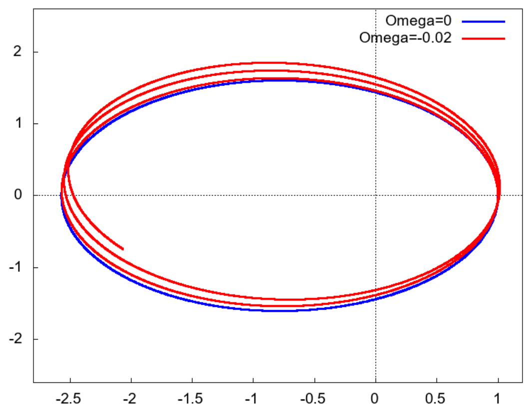


Figure 8.38: Orbital precession for $\Omega^0{}_{01} < 0$ (backward precession).

8.4 Intrinsic structure of fields

In this chapter, we have described the unification of electrodynamics, mechanics and fluid dynamics. In Example 8.13, we saw that Newton's law of gravitation and the Coulomb law are formally identical, and they can both be described by Kambe's divergence equation (8.69):

$$\nabla \cdot \mathbf{E}_F = q_F, \tag{8.377}$$

which is a fluid dynamics interpretation of both laws. $\mathbf{E}_F$ is a flow field, and q_F is a source or sink of the flow. Consequently, electric and gravitational fields should have an internal flow structure. In ECE theory, both fields are defined in equivalent form by the potentials and spin connections: Eq. (4.211),

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} - c\omega_{0e}\mathbf{A} + \boldsymbol{\omega}_e\phi, \tag{8.378}$$

for the electric case, and Eq. (7.38),

$$\mathbf{g} = -\nabla\Phi - \frac{\partial \mathbf{Q}}{\partial t} - c\omega_{0g}\mathbf{Q} + \boldsymbol{\omega}_g\Phi, \tag{8.379}$$

for the gravitational case. We have denoted the spin connections in the respective equations by the indices e and g, and have omitted the polarization indices of ECE theory. In our interpretation of potentials, $\mathbf{A}$ and $\mathbf{Q}$ are aether flows. In particular, $\mathbf{Q}$ has the units of m/s and was handled as a velocity in preceding examples. In the above equations, we see that there are two contributions from the vector potentials: a time derivative and a contribution that is directly proportional to $\mathbf{A}$ and $\mathbf{Q}$. The time derivatives are also used in standard physics, but the direct contributions (multiplied by a spin connection) appear only in ECE theory. In the static case, the Coulomb and gravitational fields read:

$$\mathbf{E} = -\nabla\phi - c\omega_{0e}\mathbf{A} + \boldsymbol{\omega}_e\phi \tag{8.380}$$

and

$$\mathbf{g} = -\nabla\Phi - c\omega_{0g}\mathbf{Q} + \boldsymbol{\omega}_g\Phi, \tag{8.381}$$

in which the scalar potentials ϕ and Φ also contribute through their gradients and linear factors. In the fluid dynamics interpretation of spacetime, the gradients can be considered as terms of aether pressure. We see that both types of potentials (scalar and vector) are present in static fields of electrodynamics and gravitation. This is a result that cannot be obtained from standard physics.

The fields (8.380) and (8.381) can be simplified further by applying the antisymmetry laws (5.14) and (7.59):

$$-\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi - c\omega_{0e}\mathbf{A} - \boldsymbol{\omega}_e\phi = \mathbf{0}, \tag{8.382}$$

$$-\frac{\partial \mathbf{Q}}{\partial t} + \nabla\Phi - c\omega_{0g}\mathbf{Q} - \boldsymbol{\omega}_g\Phi = \mathbf{0}, \tag{8.383}$$

giving for $\mathbf{E}$ and $\mathbf{g}$ (in the static case):

$$\mathbf{E} = -2c\omega_{0e}\mathbf{A}, \tag{8.384}$$

$$\mathbf{g} = -2c\omega_{0g}\mathbf{Q}. \tag{8.385}$$

This is the same formula for the electric field that was derived in Example 8.4, where the vector potential **A** corresponds directly to velocity field **v**, via the ratio x between mass and charge density in the vacuum (see Eqs. (8.209, 8.216)):

$$\mathbf{E} = -2cx\omega_0\mathbf{v}. \tag{8.386}$$

Therefore, both equations (8.384) and (8.385) refer to an aether flow directly.

The interpretation of static fields as flow fields is not new. Nicola Tesla argued in that direction, and Thomas Bearden [90] interpreted the field of an electric charge as an output flux of condensed aether material. If there is a current of aether output flux, there must also be an input flux, otherwise the continuity equation would be violated. We know that charges are always connected with matter (see, for example, the famous ratio e/m for electrons). So, when there is an output flux of the electric field, there must also be an equivalent input flux of aether material (see Fig. 8.39), and a gravitational field is the only available candidate. A gravitational field is always attractive, i.e., it provides the same aether current for both types of charges, and is nothing more than a compensating flow caused by electromagnetic effects. Because these flow types are different, there must be "aether particles" or, more accurately, "aether compounds", with different internal structures, which constitute an electric and a gravitational aether flux.

In ECE theory, "aether particle" and "aether compound" are defined in the following way. An "aether particle" is the smallest (discrete) unit of aether, as well as the basic building-block of micro-macroscopic aether. Electromagnetic and gravitational waves consist of structures that are specific arrangements of these basic particles, and we are calling these structures "aether compounds". This research subject is essentially unexplored, with discussions being primarily on a philosophical level (see, for example, [95]).

The remarkable result that static fields are determined solely by the vector potentials or spacetime flows can be developed further. In Eqs. (8.384, 8.385), the spin connections represent a wave number or, with the factor c included, an angular velocity (or time frequency). This may be a hint that the fields are connected with quantum states, analogously to the quantum energy $\hbar\omega$. The fields may be interpreted as the internal structure of aether compounds that constitute the flows. According to contemporary quantum electrodynamics, photons mediate the electromagnetic interaction, and gravitons mediate the gravitational field. The quantum energy of photons is $\hbar\omega$, which gives us an interpretation of the spin connection ω_{0e} in Eq. (8.384). The quantum energy of the mediating photon then becomes

$$E_e = \hbar c\omega_{0e}, \tag{8.387}$$

and that of the mediating graviton becomes

$$E_g = \hbar c\omega_{0g}. \tag{8.388}$$

These will have very different values, because the electromagnetic and gravitational field energies differ by many orders of magnitude. Both intrinsic structures (photons and gravitons) represent radiation fields that are also present in neutral matter. Because molecules contain internal covalent or ionic bonds, they also contain strong electric fields, as do atoms in their electronic shell structure.

Knowledge about the internal structure of fields could allow us to counteract gravitation. Assume that we know the internal frequency ω_{0g} of the graviton radiation of a body. We overlay this field with electromagnetic radiation having the same frequency. Then, this electromagnetic field would provide aether compounds of the type expressed as gravitation, and the external gravitational field of the Earth, for example, would not be able to couple to the body. This is depicted in Fig. 8.40. Such processes were reported to have been realized experimentally, but it is difficult to find references for these experiments, which were mentioned in journals of aeronautics in the 1950s.

The counter-gravitational effect may be related to metamaterials. These materials have a negative permittivity and permeability, and the energy of the Poynting vector propagates inversely to the phase velocity [96].

The frequency of graviton radiation should be in the spectral range between microwave and infrared radiation, where the penetration depth into solids is largest [97]. This effect is beyond the scope of standard physics. ECE theory, however, provides a possible explanation that does not require quantum electrodynamics or other highly complicated theories.

In summary, physical fields can be described on three logical levels:

1. Force fields
2. Potentials
3. Intrinsic flow quanta

We looked at the third level in this subsection, and arrived at a significant proposition that has refined our understanding of physical fields and, most importantly, is independent of quantum-mechanical methods. As has often been the case throughout the development of ECE theory, we have found that classical and semi-classical methods can be developed in a way that avoids the need for a quantum-mechanical description.

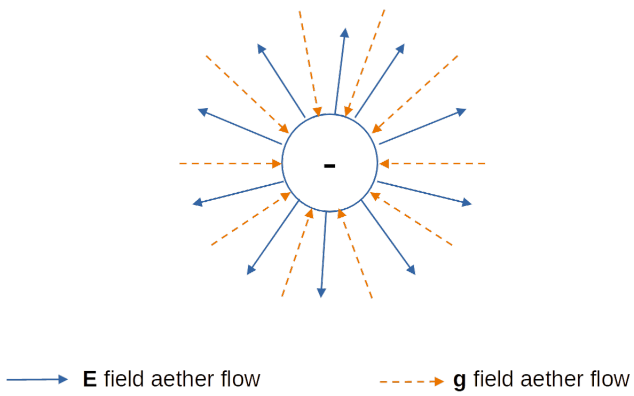


Figure 8.39: Aether fields of a source charge.

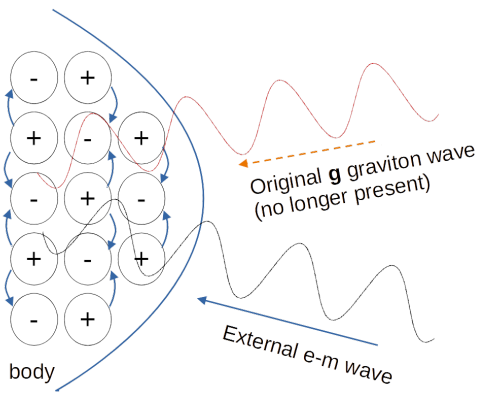


Figure 8.40: Replacement of a graviton wave by an electromagnetic wave (not all **E** fields that are present in the body are shown).

9. Gravitation

Many books have been written about gravitation, and Einsteinian theory is considered to be the definitive model, even though it is complicated and requires great mathematical effort to describe observable effects. Even then, many of them (for example, the velocity curves of galaxies, and the polarization of light by gravitational deflection) cannot be described correctly, or even at all. This textbook and other publications [6] have shown that Einsteinian general relativity is mathematically incorrect. In this chapter, we present ECE theory as an effective alternative. In particular, it is a much simpler approach for generally relativistic effects that also allows us to use the well-known Lagrange theory in its fully relativistic formulation.

9.1 Classical gravitation

Before proceeding to ECE-based gravitation, we will recapitulate the results of non-relativistic, classical theory [98], which is also the basis of ECE gravitation.

9.1.1 Conic sections

Classical gravitating masses in bound states move around each other in elliptic orbits, and perturbations of Newton's law of gravitation lead to precessing ellipses. These types of effects, which we already found when we investigated the influence of fluid spacetime on gravitation (see Section 8.3.3), are related to conic sections. We start by revisiting some basic geometrical facts about conic sections, particularly ellipses. A conic section is the curve obtained by the intersection of a plane, called the cutting plane, with the surface of a cone. As shown in Fig. 9.1, the type of intersection depends on the value of ε, the eccentricity parameter:

$$\begin{aligned} \varepsilon = 0: &\quad \text{circle,} \\ 0 < \varepsilon < 1: &\quad \text{ellipse,} \\ \varepsilon = 1: &\quad \text{parabola,} \\ \varepsilon > 1: &\quad \text{hyperbola.} \end{aligned}$$

Notice that only the circle and the ellipse are closed orbits.

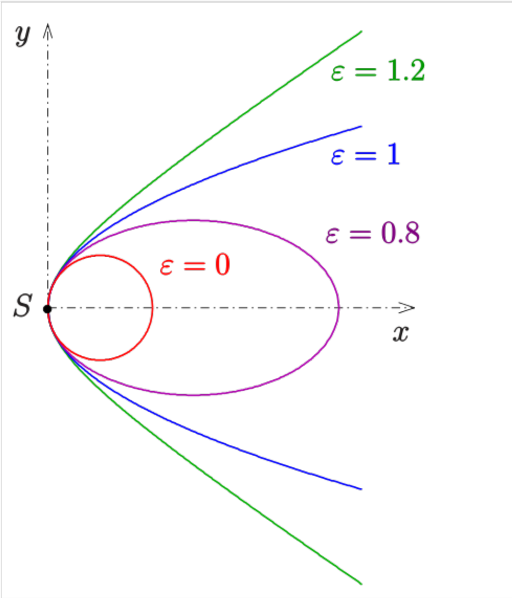


Figure 9.1: Conic sections [188].

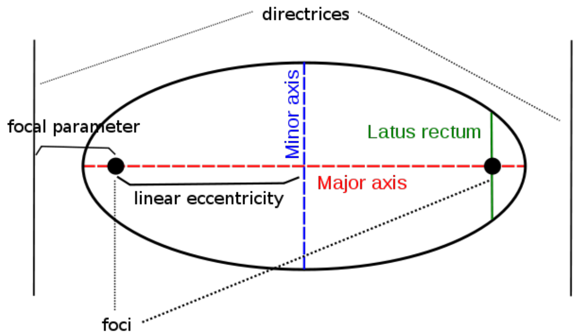


Figure 9.2: Parameters of an ellipse [189].

The symbols that are used for the parameters of an ellipse (see Fig. 9.2) are

α: half right latitude (semi latus rectum),
a: semi major axis,
b: semi minor axis,
ε: eccentricity,
X, Y: cartesian coordinates,
r, θ: radius and polar angle of plane polar coordinates.

They are interrelated in the following ways:

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1, \tag{9.1}$$

$$\varepsilon^2=1-\frac{b^2}{a^2}, \tag{9.2}$$

$$\alpha=a(1-\varepsilon^2), \tag{9.3}$$

$$a=\frac{\alpha}{1-\varepsilon^2}, \tag{9.4}$$

$$b=\frac{\alpha}{\sqrt{1-\varepsilon^2}}, \tag{9.5}$$

$$r_{\min}=\frac{\alpha}{1+\varepsilon}=a(1-\varepsilon), \tag{9.6}$$

$$r_{\max}=\frac{\alpha}{1-\varepsilon}=a(1+\varepsilon). \tag{9.7}$$

9.1.2 Newtonian equations of motion

Newton's gravitational force is of type $1/r^2$:

$$\mathbf{F}=-\frac{mMG}{r^2}\mathbf{e}_r, \tag{9.8}$$

where r is the distance of the orbiting mass m from the central mass M. We use polar coordinates (r, θ), and $\mathbf{e}_r$ is the radial unit vector. The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_r^2 + v_\theta^2\right), \tag{9.9}$$

and the potential energy is

$$U = -\frac{mMG}{r}. \tag{9.10}$$

The total energy E is equal to the Hamiltonian:

$$\mathscr{H} = E = \frac{1}{2}m\left(v_r^2 + v_\theta^2\right) + U. \tag{9.11}$$

The dynamics can be computed by applying Lagrange theory, as we did in Example 8.16. According to Eq. (8.366), the Lagrangian is

$$\mathscr{L} = T - U = \frac{1}{2}m\left(v_r^2 + v_\theta^2\right) - U \tag{9.12}$$

with velocity components

$$v_r = \dot{r}, \tag{9.13}$$
$$v_\theta = r\dot{\theta}, \tag{9.14}$$

in polar coordinates. Then, the Euler-Lagrange equations of motion are

$$\ddot{r} = r\dot{\theta}^2 - \frac{MG}{r^2}, \tag{9.15}$$
$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}. \tag{9.16}$$

The constant of motion is the angular momentum:

$$L = mr^2\dot{\theta}. \tag{9.17}$$

From (9.15) we get the radial *Leibniz equation*:

$$\mathbf{F} = m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r = -\frac{mMG}{r^2}\mathbf{e}_r, \tag{9.18}$$

which can be transformed into the *Binet equation* [98]:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r). \tag{9.19}$$

The Binet equation enables us to derive the force law from any given orbit $r(\theta)$.

The solutions of the equations of motion are the conic sections. For bound states, the total energy is less than zero ($E < 0$), and is related to an ellipse. For unbound ("scattering") states, the total energy is greater than zero ($E > 0$), by convention, and is related to a hyperbola. Finally, $E = 0$ indicates a parabola, which can be regarded as an infinitely large ellipse.

For all conic sections, the orbit $r(\theta)$ is given by

$$r = \frac{\alpha}{1 + \varepsilon\cos(\theta)}, \tag{9.20}$$

and the squared velocity of the orbiting mass m is

$$v^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right). \tag{9.21}$$

There are additional relations among the orbit parameters, the energies, and the angular momentum:

$$a = \frac{MG}{2|E|}, \tag{9.22}$$
$$b = \frac{L}{\sqrt{2m|E|}} = \sqrt{\alpha\, a}\,, \tag{9.23}$$
$$E = -\frac{mMG}{2a}, \tag{9.24}$$
$$\varepsilon = \sqrt{1 + \frac{2EL^2}{mM^2G^2}}, \tag{9.25}$$
$$L^2 = m^2 MG\alpha. \tag{9.26}$$

The radial velocity $\dot{r}$ and the orbit function $\theta(r)$ can be expressed as

$$\dot{r} = \pm\sqrt{\frac{2}{m}(E-U) - \frac{L^2}{m^2r^2}}, \tag{9.27}$$
$$\theta(r) = \int \frac{\pm L/r^2}{\sqrt{2m\left(E - U - \frac{L^2}{2mr^2}\right)}} dr. \tag{9.28}$$

9.1.3 Non-Newtonian force laws

The Lagrange formalism can be used to obtain trajectories from any force law. The force has to be representable by a potential U appearing in the Lagrangian (9.12), i.e., it must be a conservative force. As an example, we will derive the force law for a whirlpool galaxy, in which the stars are assumed to move on spiral arms.

■ **Example 9.1** We assume that the spiral arms of a galaxy are logarithmic spirals, and then derive the corresponding force law. A logarithmic spiral is defined in plane polar coordinates by the orbit

$$r(\theta) = -\frac{r_0}{\theta}, \tag{9.29}$$

where r_0 is a parameter. The minus sign provides motion from the outer part of the spiral arm to the inner, as is most often the case. Stars are "born" at the outermost places of the galaxy, as has been found by astronomical observations. The force law:

$$F(r) = \frac{L^2}{mr^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}, \tag{9.30}$$

is derived from the Binet equation (9.19). Insertion of the orbit (9.29) gives the force:

$$F(r) = \frac{L^2\theta^3}{mr_0^3} = -\frac{L^2}{mr^3}. \tag{9.31}$$

This is a cubic force law. Such a force law was already derived from the spacetime structure of spiral galaxies in Eq. (8.300) of Example 8.14. In that case, it referred to a volume element of the vacuum at position $\mathbf{r}_F$. Here, we apply the force law to a real mass, which is attracted by this force.

Therefore, we can replace the density quantities $\widehat{L}_{FZ}$ and ρ_m by their point mass values L and m. The potential energy of the force (9.31) is

$$U(r) = -\int F(r)dr = -\frac{L^2}{2mr^2}. \tag{9.32}$$

All calculations are available in computer algebra code [156]. ■

■ **Example 9.2** We will now compute the dynamics of a whirlpool galaxy. The Lagrange equations of motion have been solved numerically for two cases, using computer algebra code [157]. In the first one, we use the potential energy (9.32). Then, according to Eq. (9.12), the Lagrangian is

$$\mathscr{L} = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{L^2}{2mr^2}, \tag{9.33}$$

and the resulting equations of motion are

$$\ddot{r} = r\dot{\theta}^2 - \frac{L^2}{m^2r^3}, \tag{9.34}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}. \tag{9.35}$$

For a numerical solution, the initial conditions for r and $\dot{\theta}$ should obey the relation for the angular momentum, which is a constant of motion:

$$L = mr^2\dot{\theta}. \tag{9.36}$$

The orbit described by a numerical solution with illustrative parameters is graphed in Fig. 9.3. The star moves from the outside to the inside on a spiral with only one winding. We could not find a parameter set leading to more windings of the orbit. The time trajectory diagram of the velocity components (Fig. 9.4) shows that, when the radius approaches zero, the angular velocity diverges and the rotational speed accelerates to infinity.

This limit of a hyperbolic spiral is not realistic for a galaxy, because the potential will not be of pure $-1/r^2$ type close to its center. Therefore, we have added a potential of type $-a/r$ with a constant a, which models a Newtonian behavior in this region. The equations of motion then become

$$\ddot{r} = r\dot{\theta}^2 - \frac{L^2}{m^2r^3} - \frac{a}{r^2}, \tag{9.37}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}. \tag{9.38}$$

Only the equation for r is affected by the additional term in the potential. In the numerical solution, this prevents the stars from falling into the center; the direction changes at a minimal radius, so that a loop-like orbit arises (see Fig. 9.5). In order not to describe unphysical effects, one has to stop the simulation at the point of closest approach to the center.

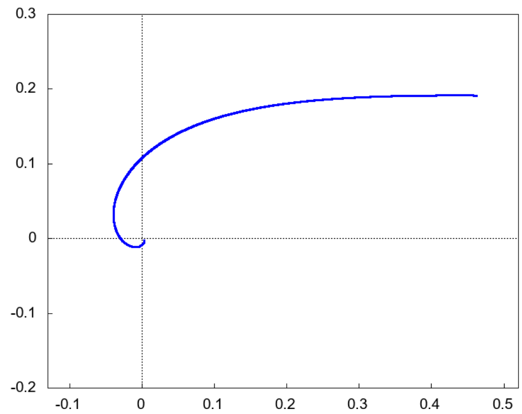


Figure 9.3: Orbit of a star approaching the galactic center, $-1/r^2$ potential.

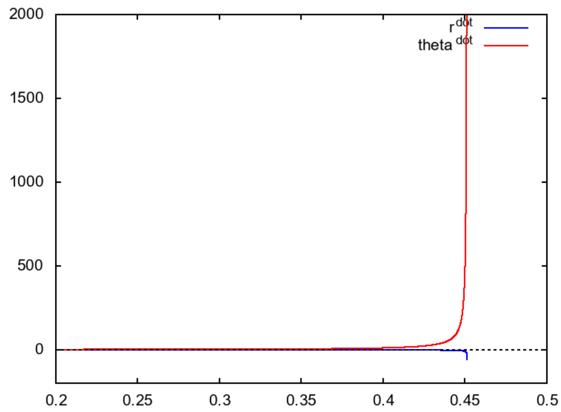


Figure 9.4: Trajectories of $\dot{r}$ and $\dot{\theta}$.

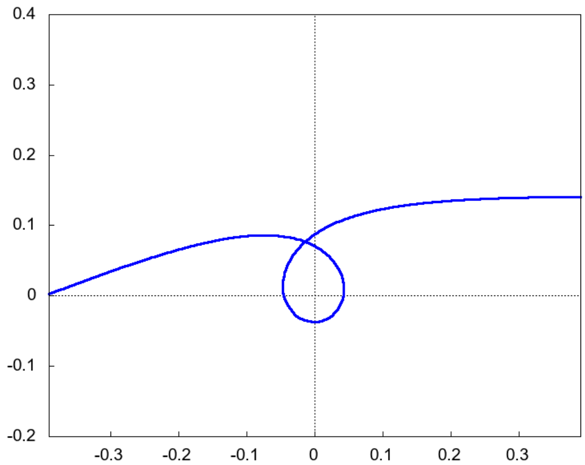


Figure 9.5: Orbit of a star approaching the galactic center, $-1/r^2$ and additional $-1/r$ potential.

To verify the solutions, we inserted the analytical orbit (9.29) into the equations of motion (9.34, 9.35), which gives us a differential equation for $\theta(t)$:

$$2\dot{\theta}^2 - \theta\ddot{\theta} = 0 \tag{9.39}$$

(see computer algebra code [158]). This equation has the solution

$$t = -\frac{1}{\omega_0\,\theta} \quad \text{or} \quad \theta = -\frac{1}{\omega_0\,t} \tag{9.40}$$

with a constant ω_0. This solution is valid for both equations of motion, as required. It means that the rotation accelerates to infinity, when t approaches zero. To obtain motion from the outside to the inside, we have to start at negative time values and stop before $t = 0$ is reached. The velocity components are

$$v_r = \dot{r} = -\frac{d}{dt}\left(\frac{r_0}{\theta}\right) = -r_0\frac{d}{dt}(-\omega_0\, t) = r_0\, \omega_0, \tag{9.41}$$

$$v_\theta = r\dot{\theta} = r_0\omega_0\, t\, \frac{d}{dt}\left(-\frac{1}{\omega_0 t}\right) = \frac{r_0}{t}. \tag{9.42}$$

In the limit $t \to \pm\infty$ we have

$$v_r \to r_0\omega_0 = \text{const.}, \tag{9.43}$$

$$v_\theta \to 0. \tag{9.44}$$

This means that $v \to$ const., which is a second, direct proof for the velocity curve of whirlpool galaxies discussed in Example 8.14. Neither Newtonian nor Einsteinian theory is able to describe the velocity curve. The description provided by ECE theory uses only the concept of a rotating background spacetime, which is a concept of general covariance and therefore of general relativity. As we have seen, the calculations can be performed successfully even within a non-relativistic framework. Our results do not depend on ad hoc concepts like dark matter or dark energy, and these concepts should now be abandoned, according to Occam's razor. ■

9.2 Relativistic gravitation

Having discussed aspects of the non-relativistic theory of gravitation, we will now introduce relativistic approaches to gravitation, all of which are alternatives to Einstein's erroneous field equation of general relativity. We will first introduce the line element and equations of special relativity, and then generalize them.

9.2.1 Relativistic line element

The geometrical distance between two points or "events" in four-dimensional spacetime was introduced in Section 2.1.2. Such a distance is described by the quadratic differential line element of special, as well as general, relativity:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - d\mathbf{r}^2. \tag{9.45}$$

τ is "proper time", defined as being local to the system under consideration, and t is the time scale of an external observer. c is the speed of light without gravitational fields in vacuo, which is a universal constant. In cartesian coordinates, the infinitesimal distance in space is

$$d\mathbf{r} = dX\mathbf{e}_X + dY\mathbf{e}_Y + dZ\mathbf{e}_Z, \tag{9.46}$$

and its squared modulus is

$$d\mathbf{r}^2 = dX^2 + dY^2 + dZ^2. \tag{9.47}$$

The observer moves with a velocity $\mathbf{v}$ relative to the system under consideration:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{9.48}$$

This is the velocity that he measures in his own inertial system. It then follows that

$$d\mathbf{r}^2 = v^2 dt^2, \tag{9.49}$$

and with Eq. (9.45) we obtain

$$c^2 d\tau^2 = (c^2 - v^2)dt^2. \tag{9.50}$$

We can rewrite this equation as

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{c^2}{(c^2 - v^2)} = \left(1 - \frac{v^2}{c^2}\right)^{-1}. \tag{9.51}$$

The derivative $dt/d\tau$ is called the γ *factor of special relativity*:

$$\gamma := \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{9.52}$$

In plane polar coordinates, we have

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{9.53}$$

and

$$d\mathbf{r}^2 = dr^2 + r^2 d\theta^2. \tag{9.54}$$

In 3D polar coordinates we have, according to Eq. (7.170),

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2(\sin\theta)^2\left(\frac{d\phi}{dt}\right)^2 \tag{9.55}$$

and, correspondingly,

$$d\mathbf{r}^2 = dr^2 + r^2 d\theta^2 + r^2(\sin\theta)^2 d\phi^2. \tag{9.56}$$

9.2.2 Hamiltonian and Lagrangian in special relativity

The Lagrangian we have used so far has been

$$\mathscr{L} = T - U, \tag{9.57}$$

where T is the kinetic and U the potential energy, and the total energy E, or Hamiltonian, is

$$\mathscr{H} = E = T + U. \tag{9.58}$$

In the relativistic mechanics based on special relativity, the total energy of a mass m contains the rest mass mc^2 and is

$$E = \gamma mc^2 \tag{9.59}$$

with the γ factor

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{9.60}$$

Therefore, the kinetic energy is a consequence of the γ factor. In contrast to the original Lorentz transformation, where v is a constant velocity difference between frames of reference, v here is assumed to be variable. This version of the Lagrangian is defined by

$$\mathscr{L} = -\frac{mc^2}{\gamma} - U. \tag{9.61}$$

The term with the inverse γ factor may seem strange at first glance, but it is the kinetic energy, as can be seen in the following discussion. With the approximation

$$\sqrt{1-x} \approx 1 - \frac{x}{2} \quad \text{for} \quad x \ll 1, \tag{9.62}$$

where $x = v^2/c^2$, the inverse γ can be approximated for non-relativistic velocities by

$$\frac{1}{\gamma} \approx 1 - \frac{v^2}{2c^2}, \tag{9.63}$$

and therefore:

$$\frac{mc^2}{\gamma} \approx mc^2 - \frac{1}{2}mv^2. \tag{9.64}$$

Consequently, the relativistic definition and the practical approximation of kinetic energy are

$$T_{\text{rel}} := -\frac{mc^2}{\gamma} \approx \frac{1}{2}mv^2 - mc^2, \tag{9.65}$$

so for the non-relativistic kinetic energy we obtain

$$T_{\text{non-rel}} = \frac{1}{2}mv^2 \approx -\frac{mc^2}{\gamma} + mc^2 = \left(1 - \frac{1}{\gamma}\right) mc^2. \tag{9.66}$$

It is more convenient to subtract the rest energy mc^2 from the total energy. This avoids huge numerical values, and results are directly comparable to those of the non-relativistic theory. Therefore, we use the *Sommerfeld Hamiltonian*:

$$\mathscr{H}_0 := (\gamma - 1)mc^2 + U. \tag{9.67}$$

Adding a constant to the Lagrangian has no effect on the Euler-Lagrange equations, and this allows us to define a reduced kinetic relativistic energy:

$$T_{\text{rel},0} := \left(1 - \frac{1}{\gamma}\right) mc^2. \tag{9.68}$$

9.2.3 Generally covariant Hamiltonian and Lagrangian

In Eq. (9.50), which was derived from the relativistic line element (9.45), no restriction was imposed on the velocity v. Therefore, we can identify v with an arbitrary velocity of a mass m moving in the frame of the observer. Since the line element is valid within Cartan geometry, which includes curvature and torsion, the Hamiltonian and Lagrangian derived in the preceding section are also valid in a Cartan framework of general relativity. The differences with respect to special relativity will become visible as soon as we apply differential operations to these equations, for example, when we derive the Euler-Lagrange equations from the Lagrangian. We will now consider three possibilities for generalizing Newton's law of gravitation.

The relativistic Euler-Lagrange equations

We consider the gravitational problem of a mass m moving within the gravitational field of a central mass M. In cartesian coordinates (X, Y, Z), the squared velocity is

$$v^2 = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2. \tag{9.69}$$

Then, when the Lagrangian

$$\mathscr{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r^2} \tag{9.70}$$

is written out in cartesian coordinates it becomes

$$\mathscr{L} = -mc^2\sqrt{1-\frac{\dot{X}^2+\dot{Y}^2+\dot{Z}^2}{c^2}} + \frac{mMG}{X^2+Y^2+Z^2}. \tag{9.71}$$

The Euler-Lagrange equations, in short notation (where $\mathbf{r}$ stands for one of the components X, Y, Z), then become

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{r}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{r}} = 0. \tag{9.72}$$

This calculation is lengthy because of the occurrence of the γ factor. Computer algebra [159] shows that the result can be brought into a quite compact vector form:

$$\boxed{\ddot{\mathbf{r}} = \frac{MG}{\gamma r^3}\left(\frac{\dot{\mathbf{r}}\,(\dot{\mathbf{r}}\cdot\mathbf{r})}{c^2} - \mathbf{r}\right).} \tag{9.73}$$

In the Euler-Lagrange equations (9.72), we have used the observer time t. In the literature, proper time τ is used. This is a question of interpretation. Since we always observe a system on the basis of an observer frame, for example the center of the central mass M, the trajectories of interest are $\mathbf{r}(t)$, etc., and not $\mathbf{r}(\tau)$. Consequently, we prefer to use the Euler-Lagrange equations in the form shown above. When written with τ, they become

$$\frac{d}{d\tau}\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{r}}} - \frac{\partial \mathscr{L}}{\partial \mathbf{r}} = 0. \tag{9.74}$$

Because of Eq. (9.52), we have

$$dt = \gamma\, d\tau. \tag{9.75}$$

Therefore, the Euler-Lagrange equations for τ contain an additional factor $1/\gamma$, and in this second approach Eq. (9.73) takes the form

$$\ddot{\mathbf{r}} = \frac{MG}{\gamma^2 r^3}\left(\frac{\dot{\mathbf{r}}\,(\dot{\mathbf{r}}\cdot\mathbf{r})}{c^2} - \mathbf{r}\right). \tag{9.76}$$

The difference in the results from these two forms will be discussed in Example 9.3. In both cases, the non-relativistic result will be obtained for $\gamma \to 1$, $c \to \infty$:

$$\ddot{\mathbf{r}} = -\frac{MG}{r^3}\mathbf{r}. \tag{9.77}$$

In addition to these two interpretations regarding the formulation of a generally covariant relativistic theory of gravitation, there are two more approaches. In these approaches, the relativistic linear momentum of a mass m moving with velocity $\mathbf{v}$ is defined by

$$\mathbf{p} = \gamma m\mathbf{v}. \tag{9.78}$$

In contrast to the original perspective that there is a “relativistic mass increase” $m \to \gamma m$, we use the more modern view that the momentum is increased, while the mass remains identical to the rest mass. $\mathbf{v}$ is determined by the relativistic dynamics of the system, and is already a relativistic quantity in this sense.

The relativistic Newton equation

In a third approach, using the relativistic momentum (9.78), Newton's law can be generalized to

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m \mathbf{v}). \tag{9.79}$$

We compute the kinetic energy from this force. Reducing **F** to one dimension for convenience gives us the kinetic energy as the line integral

$$T = \int F ds, \tag{9.80}$$

and with $ds = v dt$, it can be written as

$$\int F v dt = \int \frac{d}{dt}(\gamma m v) v dt = m \int v d(\gamma v). \tag{9.81}$$

This expression can be integrated by parts. We apply the formula

$$\int u \, dw = uw - \int w \, du \tag{9.82}$$

with the functions

$$u = v, \quad w = \gamma v. \tag{9.83}$$

uw has to be evaluated at the integration limits. Application of integration by parts to (9.81) gives

$$T = m \int_0^v v \, d(\gamma v) = m\gamma v^2 \big|_0^v - m \int_0^v \gamma v \, dv = m\gamma v^2 - m \int_0^v \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} dv \tag{9.84}$$

$$= m\gamma v^2 + mc^2 \sqrt{1 - \frac{v^2}{c^2}} \Bigg|_0^v = m\gamma v^2 + \frac{1}{\gamma} mc^2 - mc^2.$$

The second term at the end of the chain can be expanded and rewritten as

$$\frac{1}{\gamma} mc^2 = mc^2 \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m (c^2 - v^2). \tag{9.85}$$

Inserting this into (9.84) gives the final result:

$$T = (\gamma - 1) mc^2, \tag{9.86}$$

which is identical to the kinetic energy in the Sommerfeld Hamiltonian (9.67).

This demonstrates the compatibility Newton's generalized law of dynamics (9.79) with the Hamiltonian and Lagrangian. By equating this law with Newton's law of gravitation, we obtain

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m \mathbf{v}) = -\frac{mMG}{r^3}\mathbf{r}. \tag{9.87}$$

We have to evaluate the time derivative of $\gamma \mathbf{v}$ in a way that is similar to the one used for the kinetic energy T, but without integration by parts. The details can be found in [99], with only the results being presented here:

$$\mathbf{F} = m\gamma^3 \frac{d\mathbf{v}}{dt} = m\gamma^3 \ddot{\mathbf{r}}. \tag{9.88}$$

Equating this with Newton's law of gravitation gives the final result:

$$\ddot{\mathbf{r}} = -\frac{MG}{\gamma^3 r^3}\mathbf{r}. \tag{9.89}$$

Relativity introduces a factor of $1/\gamma^3$, but no relativistic terms of the order $1/c^2$. This is different from the gravitational equation (9.73) obtained from Lagrange theory.

The Minkowski force

A fourth approach to generalizing Newton's law of gravitation involves the Minkowski force. This is defined as the 4-vector of force in the local frame of the moving mass, using proper time:

$$F_M^\mu = \frac{dp^\mu}{d\tau} \tag{9.90}$$

with the 4-momentum

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right). \tag{9.91}$$

When we use the definitions

$$E = \gamma mc^2, \quad \mathbf{p} = \gamma m\mathbf{v}, \tag{9.92}$$

as before, the space part of the Minkowski force is

$$\mathbf{F}_M = \gamma \mathbf{F}, \tag{9.93}$$

where F is the relativistic generalization of the Newton force (9.79), which leads to

$$\mathbf{F}_M = m\gamma^4 \frac{d\mathbf{v}}{dt} = m\gamma^4 \ddot{\mathbf{r}}. \tag{9.94}$$

In analogy to (9.89), the gravitational acceleration then is

$$\ddot{\mathbf{r}} = -\frac{MG}{\gamma^4 r^3}\mathbf{r}. \tag{9.95}$$

When related to the observer frame, the Minkowski force is

$$F_M^\mu = \frac{dp^\mu}{dt}, \tag{9.96}$$

and because $d\tau = dt/\gamma$, one factor of γ cancels out in (9.95). The Minkowski force then becomes identical to the relativistic Newtonian version (9.89).

Relativistic motion of the S2 star

■ **Example 9.3** We present the results of relativistic analysis of the motion of the S2 star, and compare them with non-relativistic results. The S2 star is one of several heavy galactic objects that orbit the center of our home galaxy every few years. These stars were investigated by astronomical observation several years ago. The problem is that the galactic center is hidden by gas clouds so that its surrounding region is very difficult to observe. Since the mass of the galactic center (a "black hole" according to an interpretation of Einstein's obsolete theory) is a few million solar masses, we expect strong relativistic effects on star orbits in its vicinity.

We have solved the Euler-Lagrange equation (9.73) (see computer algebra code [160]), and the result shows that the orbit is an ellipse with quite high ellipticity, as can be seen in Fig. 9.6. The exact motion depends strongly on the initial conditions of the calculation. Since these are not known very precisely from experiment, one has to vary them so that best agreement is achieved. Details on this are reported in [100]. In particular, no structural parameters of an elliptic orbit (Eqs. (9.1 - 9.7)) can be obtained from a numerical methodology a priori. These parameters have to be extracted from the numerically generated orbit by algorithms (see [100] for details).

The relativistic γ factor depends on the velocity of the star and is largest at the periastron (closest approach to the center). From Fig. 9.7 it can be seen that, at these maxima, γ deviates from unity by only a few ten-thousandths. The relativistic angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \gamma\, m\, \mathbf{r} \times \mathbf{v}, \tag{9.97}$$

where $\mathbf{p}$ is the relativistic linear momentum. For planar motion in cartesian coordinates (XY plane), the relativistic angular momentum is

$$L_{\text{rel}} = \gamma\, m(X\dot{Y} - Y\dot{X}), \tag{9.98}$$

while the non-relativistic angular momentum is

$$L_{\text{n-r}} = m(X\dot{Y} - Y\dot{X}). \tag{9.99}$$

Both are graphed in Fig. 9.8. The relativistic angular momentum is a constant of motion as expected, while the non-relativistic angular momentum is not. The deviations have peaks at the periastron where the γ factor (see Fig. 9.7) is largest.

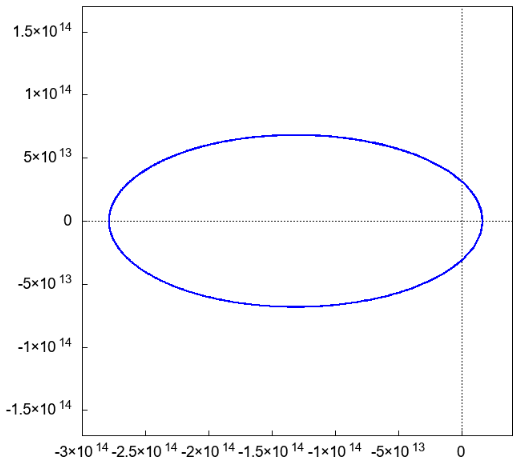


Figure 9.6: Orbit of the S2 star.

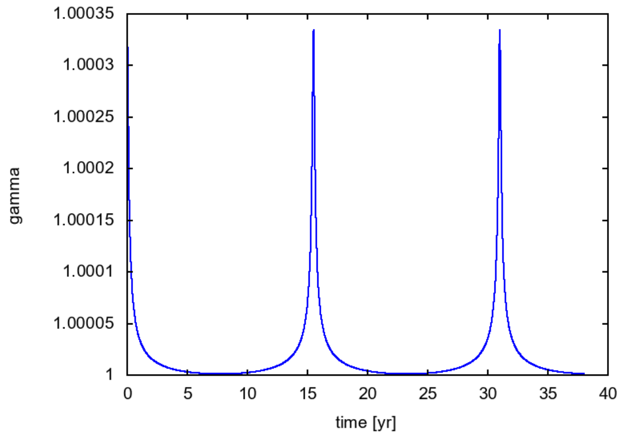


Figure 9.7: γ factor of the S2 star.

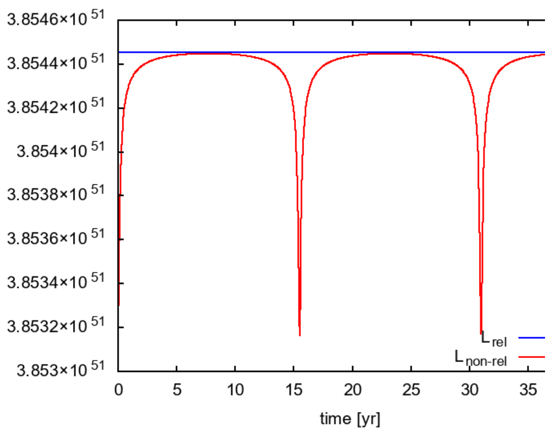


Figure 9.8: Angular momenta of the S2 star.

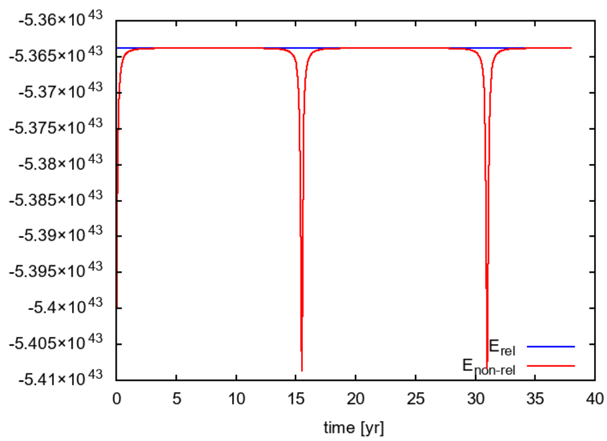


Figure 9.9: Total energies of the S2 star.

We get a similar graph for the total energies in the relativistic and non-relativistic cases:

$$E_{\text{rel}} = (\gamma - 1)mc^2 - \frac{mMG}{\sqrt{X^2+Y^2}}, \tag{9.100}$$

$$E_{\text{n-r}} = \frac{1}{2}mv^2 - \frac{mMG}{\sqrt{X^2+Y^2}}. \tag{9.101}$$

As can be seen in Fig. 9.9, the total energies deviate significantly only at the periastron, which is similar to the behavior of the angular momenta. The orbital velocity of the S2 star reaches between 1.6 and 3 percent (5000 - 7700 km/s) of the velocity of light at these points. This is still far away from an ultra-relativistic case with $v \approx c$. ■

Comparison of force laws

■ **Example 9.4** The numerical results for the four force laws are compared. The following are valid formulations for covariant relativistic gravitational problems:

$$\ddot{\mathbf{r}} = \frac{MG}{\gamma r^3}\left(\frac{\dot{\mathbf{r}}\,(\dot{\mathbf{r}}\cdot\mathbf{r})}{c^2} - \mathbf{r}\right) \quad \text{Lagrange in observer time,} \tag{9.102}$$

$$\ddot{\mathbf{r}} = \frac{MG}{\gamma^2 r^3}\left(\frac{\dot{\mathbf{r}}\,(\dot{\mathbf{r}}\cdot\mathbf{r})}{c^2} - \mathbf{r}\right) \quad \text{Lagrange in proper time,} \tag{9.103}$$

$$\ddot{\mathbf{r}} = -\frac{MG}{\gamma^3 r^3}\mathbf{r} \quad \text{Relativistic Newton,} \tag{9.104}$$

$$\ddot{\mathbf{r}} = -\frac{MG}{\gamma^4 r^3}\mathbf{r} \quad \text{Minkowski in proper time.} \tag{9.105}$$

A comparison of all four theory variants, for the orbit of the S2 star, is shown in Table 9.1. The differences in the maximum radius and eccentricity are marginal. The orbital period T deviates from experiment by half of a year for the Minkowski force, but is well represented by the other calculations.

The angle of precession per orbit is negative (or retrograde) for both the Minkowski and the relativistic Newton forces. When we use Lagrange theory, the precession is positive. However, the precession is extremely small in the τ version of Lagrange theory. It is barely above the numerical precision limit of 10^{-8} rad as determined in [100]. There is also a logical problem for the Lagrange theory based on proper time τ, as stated earlier. The experimental value of the precession is not known very precisely and even differs in sign. Its mean value is positive, so the first Euler-Lagrange calculation seems to be in best accordance with experiments.

In both versions of Lagrange theory, the relativistic angular momentum is conserved, as expected. However, for the relativistic Newton and Minkowski forces, only the non-relativistic angular momentum is conserved (see Fig. 9.10), giving an inconsistent result. This is surprising, because the consistency of all four force laws with relativistic mechanics was shown before. One possible explanation is that the angular momentum should have been defined in these cases by the velocity in the rest frame of the mass, which would have given

$$L_{\text{rel}} = \gamma m\mathbf{r} \times \frac{d\mathbf{r}}{d\tau} = \gamma m\mathbf{r} \times \frac{d\mathbf{r}}{dt}\cdot\frac{1}{\gamma} = m\mathbf{r} \times \frac{d\mathbf{r}}{dt} = L_{\text{n-r}}. \tag{9.106}$$

	T [yr]	r_{max} [10^{14} m]	ε	$\Delta\phi$ [rad]	Const. of Motion
Euler-Lagr. t	15.50	2.78609	0.88712	$5.9033 \cdot 10^{-4}$	L_{rel}
Euler-Lagr. τ	15.57	2.79440	0.88746	$7.5090 \cdot 10^{-7}$	L_{rel}
Rel. Newton t	15.50	2.78621	0.88720	$-1.7697 \cdot 10^{-3}$	$L_{non-rel}$
Minkowski τ	15.06	2.79452	0.88753	$-2.3585 \cdot 10^{-3}$	$L_{non-rel}$
Experiment	15.56	2.68398	0.8831	-0.017… +0.035	

Table 9.1: Parameters of the S2 star orbit ($v_0 = 7.7529648 \cdot 10^6$ m/s, various calculations and experiment).

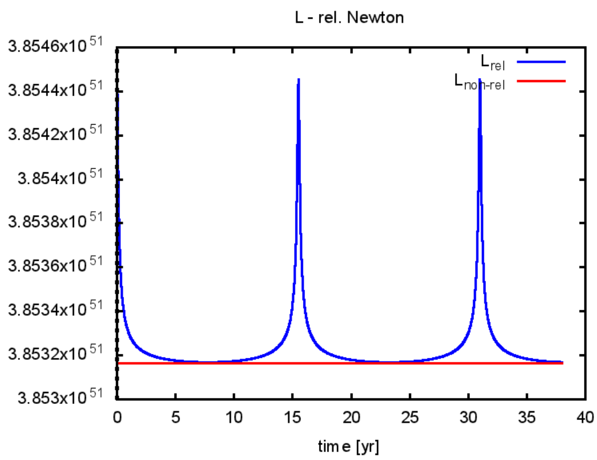


Figure 9.10: Angular momenta of the S2 star for the relativistic Newton force.

Relativistic motion of the Hulse-Taylor pulsar

■ **Example 9.5** Another interesting example is the Hulse-Taylor double-star system, which consists of a pulsar and a neutron star of nearly equal mass. We will present only the results from the detailed discussion in [100] (ECE2 Covariant Precession Versus the Einstein Theory in the S2 Star and Hulse Taylor Binary Pulsar). For the double-star system, r was used as the coordinate for the center of mass. The Euler-Lagrange equations derived from the covariant Lagrangian were solved using the same approach that we used for the S2 star. Since relativistic effects in this double-star system are smaller than for the S2 star, we ran into numerical precision problems. Therefore, we redefined the system of units to facilitate the calculation (see [100]).

The results of the calculations are listed in Table 9.2. As in the case for S2, we had to alter the periastron velocity v_0 significantly to obtain the experimental orbit period of 7.75 hours. This overestimates the maximum radius (apastron) and ellipticity. The experimental precession of 4.226 degrees per Earth year has been converted to a value per single orbit that is on the order of 10^{-5} rad. This is an order of magnitude higher than the values from our calculations, which are quite insensitive to changes of v_0. Perhaps additional fluid gravitation effects have to be taken into account (see discussion below). The orbits of the Hulse-Taylor pulsar and its partner are graphed in Fig. 9.11. The ellipse of the neutron star is a bit larger, because the masses of the two stars differ slightly.

Since the equations of motion are complicated, we tried to simplify them by approximating the γ factor in the Lagrangian:

$$\sqrt{1-u} \approx 1 - \frac{u}{2} - \frac{u^2}{8} + \ldots \tag{9.107}$$

using

$$u = \frac{v^2}{c^2}. \tag{9.108}$$

The results of this quadratic approximation coincide exactly with the fully relativistic calculation (see line five in Table 9.2). When we restrict the calculation to the linear term, we get the non-relativistic result (9.66). Doing a non-relativistic calculation gives practically the same results (see the subsequent line in Table 9.2). This may be surprising, because the Hulse-Taylor pulsar is considered to be a source of gravitational waves. However, when we compare the v_0 value of 450 km/s with that of the S2 star (see the caption for Table 9.1), we see that the v_0 of the Hulse-Taylor pulsar is smaller by an order of magnitude. This leads to a γ factor deviating from unity by only an order of 10^{-6}. Therefore, relativistic effects are very small in the Hulse-Taylor system, despite the fact that two stars comparable to the Sun come quite close to each other. The fast rotation of the pulsar (17/s) does not play a role in this type of gravitational theory, but it may be the reason for the energy loss that is observed. This leads to a decrease in the orbit period of 76.5 μs per year, corresponding to a decrease of the semi major axis by 3.5 m per year. The loss of power is reported to be $7.35 \cdot 10^{24}$ W, which corresponds to about $8 \cdot 10^7$ kg/s. This is far too low to account for the orbit decrease. Since the precession data suggest fluid gravitation effects, these effects may also be a reason for the shrinking of the orbit.

v_0 [m/s]	T [h]	$r_{\max}$ [10^9 m]	ε	$\Delta\phi$ [rad]
450 000	4.75	1.04648	0.51840	$3.1966 \cdot 10^{-6}$
466 863	7.17	1.48350	0.63433	$2.9697 \cdot 10^{-6}$
468 831	7.60	1.55474	0.64814	$2.9447 \cdot 10^{-6}$
469 526	7.76	1.58133	0.65303	$2.9360 \cdot 10^{-6}$
Rel. approx. 2nd order:				
469 526	7.76	1.58133	0.65303	$2.9360 \cdot 10^{-6}$
Non-rel.:				
469 526	7.76	1.58131	0.65303	$9.7865 \cdot 10^{-10}$
Experiment:				
450 000	7.75	1.40201	0.617155	$6.5209 \cdot 10^{-5}$

Table 9.2: Parameters of the Hulse-Taylor double star system (various calculations and experiment).

The shrinking of the orbit can be understood qualitatively in the following way. If a spacetime or aether flow is superimposed on the orbit of the double-star system, we can add such a flow velocity to the orbital velocity in the kinetic energy, i.e., in the γ factor. Using the quadratic approximation (9.107) and assuming an additional X component v_{aX} of the aether, we obtain

$$\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{(v_X + v_{aX})^2 + v_Y^2}{c^2} - \frac{1}{8}\left(\frac{(v_X + v_{aX})^2 + v_Y^2}{c^2}\right)^2. \tag{9.109}$$

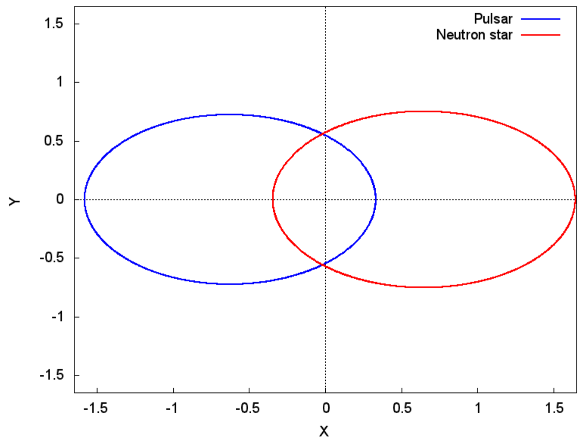


Figure 9.11: Orbit of the Hulse-Taylor pulsar and its neutron star partner (in 10^9 m).

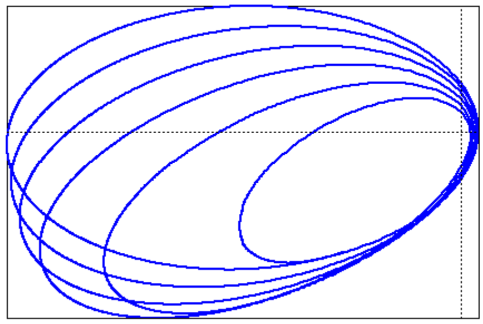


Figure 9.12: Model of a shrinking orbit.

Since the effects have to be related to a single orbit, the reported decrease of period time and radius are so small that they cannot be obtained from our calculation. However, a numerical test showed that the angle of precession $\Delta\phi$ depends sensitively on a change of kinetic energy, as in Eq. (9.109). Another reason could be related to electromagnetism, since the pulsar has a huge magnetic moment.

The decrease of radius has been shown qualitatively by a model calculation that used the γ factor from above (see Fig. 9.12, and computer algebra code [161]). In addition to the shrinking of the radius, there is a precession of the ellipse, both of which are observed in the Hulse-Taylor pulsar system. Einsteinian general relativity cannot provide an adequate explanation because it is mathematically incorrect. ■

■ **Example 9.6** We derive the relativistic Euler-Lagrange equations in plane polar coordinates. In Section 9.2.3, Eq. (9.73), we had derived the relativistic Euler-Lagrange orbital equations in cartesian coordinates, and those equations are

$$\ddot{\mathbf{r}} = \frac{MG}{\gamma r^3}\left(\frac{\dot{\mathbf{r}}\,(\dot{\mathbf{r}}\cdot\mathbf{r})}{c^2} - \mathbf{r}\right). \tag{9.110}$$

To transform them into plane polar coordinates, we use the coordinate transformations

$$X = r\cos(\phi), \tag{9.111}$$
$$Y = r\sin(\phi). \tag{9.112}$$

Then, the velocity components are

$$v_X = \dot{X} = \dot{r}\cos(\phi) - r\dot{\phi}\sin(\phi), \tag{9.113}$$
$$v_Y = \dot{Y} = \dot{r}\sin(\phi) + r\dot{\phi}\cos(\phi), \tag{9.114}$$

and the accelerations are

$$a_X = \dot{v}_X = \ddot{r}\cos(\phi) - 2\dot{\phi}\dot{r}\sin(\phi) - \ddot{\phi}r\sin(\phi) - \dot{\phi}^2 r\cos(\phi), \tag{9.115}$$
$$a_X = \dot{v}_X = \ddot{r}\sin(\phi) + 2\dot{\phi}\dot{r}\cos(\phi) + \ddot{\phi}r\cos(\phi) - \dot{\phi}^2 r\sin(\phi). \tag{9.116}$$

The scalar product $\dot{\mathbf{r}}\cdot\mathbf{r}$ in (9.110) simplifies to

$$\dot{\mathbf{r}}\cdot\mathbf{r} = v_X X + v_Y Y = r\dot{r}. \tag{9.117}$$

Inserting (9.113 - 9.117) into (9.110) gives (after some trigonometric reductions and solving for $\ddot{\phi}$ and $\ddot{r}$):

$$\ddot{\phi}\,\mathbf{e}_X = \left(\frac{GM\,\dot{\phi}\,\dot{r}}{\gamma c^2\,r^2} - \frac{2\dot{\phi}\,\dot{r}}{r}\right)\mathbf{e}_X, \tag{9.118}$$

$$\ddot{r}\,\mathbf{e}_Y = \left(\frac{GM\,\dot{r}^2}{\gamma c^2\,r^2} + \dot{\phi}^2 r - \frac{GM}{\gamma r^2}\right)\mathbf{e}_Y, \tag{9.119}$$

where

$$\mathbf{e}_X = \begin{bmatrix}1\\0\end{bmatrix}, \quad \mathbf{e}_Y = \begin{bmatrix}0\\1\end{bmatrix} \tag{9.120}$$

are the cartesian unit vectors. Because these are the same on both sides of (9.118, 9.119), it follows directly that

$$\ddot{\phi} = \frac{GM\,\dot{\phi}\,\dot{r}}{\gamma c^2\,r^2} - \frac{2\dot{\phi}\,\dot{r}}{r}, \tag{9.121}$$

$$\ddot{r} = \frac{GM\,\dot{r}^2}{\gamma c^2\,r^2} + \dot{\phi}^2 r - \frac{GM}{\gamma r^2}. \tag{9.122}$$

Alternatively, these equations can be derived from the Lagrangian

$$\mathscr{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \tag{9.123}$$

with the γ factor

$$\gamma = \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{-1/2}. \tag{9.124}$$

In both equations of motion, there is a relativistic term proportional to $1/c^2$, which depends on the dynamic terms $\dot{r}$ and $\dot{\phi}$ and contains the gravitational force term GM/r^2. In this sense, we have a coupling between kinetic and potential energy. The constant of motion is the relativistic angular momentum in polar coordinates:

$$L = \gamma m r^2 \dot{\phi}. \tag{9.125}$$

The computer algebra code for both methods of derivation can be found in [162] and [163]. ■

9.2.4 Spin connection vector

Newton's law in the form of the Poisson equation (8.271) reads:

$$\boldsymbol{\nabla}\cdot\mathbf{g} = -4\pi G\,\rho_m = \boldsymbol{\kappa}\cdot\mathbf{g}, \tag{9.126}$$

where $\mathbf{g}$ is the gravitational acceleration, ρ_m is the mass density, and $\boldsymbol{\kappa}$ is a combination of potential and spin connection in ECE2 theory (see Eq. (6.165)). If we identify $\boldsymbol{\kappa}$ with a characteristic spin connection wave number, it becomes a quantity that connects gravitation with ECE2 spacetime. Using the relativistically generalized form of Newton's gravitational law (9.104),

$$\mathbf{g} = -\frac{MG}{\gamma^3 r^3}\mathbf{r}, \tag{9.127}$$

we obtain from

$$\boldsymbol{\kappa}\cdot\mathbf{g}=\boldsymbol{\nabla}\cdot\mathbf{g} \tag{9.128}$$

(in cartesian coordinates) the equation

$$-\frac{GM\,\kappa_Y Y}{(Y^2+X^2)^{\frac{3}{2}}\,\gamma^3}-\frac{GM\,\kappa_X X}{(Y^2+X^2)^{\frac{3}{2}}\,\gamma^3}=-\frac{2GM}{(Y^2+X^2)^{\frac{3}{2}}\,\gamma^3}+\frac{3GM\,Y^2}{(Y^2+X^2)^{\frac{5}{2}}\,\gamma^3}+\frac{3GM\,X^2}{(Y^2+X^2)^{\frac{5}{2}}\,\gamma^3}, \tag{9.129}$$

which can be simplified to

$$\kappa_X X+\kappa_Y Y=-1 \tag{9.130}$$

(see [101] and computer algebra code [164]). Because the gravitational field is a central field,

$$\boldsymbol{\nabla}\times\mathbf{g}=0. \tag{9.131}$$

We will now review the term $\boldsymbol{\kappa}\times\mathbf{g}$. In electrodynamics, $\mathbf{g}$ corresponds to the electric field $\mathbf{E}$, and there is a term $\boldsymbol{\kappa}\times\mathbf{E}$ on the right side of the field equation (6.167), which is the electromagnetic Faraday law, or the gravitomagnetic law in the case of mechanics. Since we are considering gravitostatics, there is no magnetic or gravitomagnetic field. Furthermore, we consider $\boldsymbol{\kappa}_{(\Lambda)}$ to be equal to $\boldsymbol{\kappa}$. Because there is no homogeneous current, we obtain

$$\boldsymbol{\kappa}\times\mathbf{g}=0, \tag{9.132}$$

i.e., $\boldsymbol{\kappa}$ is parallel to $\mathbf{g}$. Therefore, we can write

$$\kappa_X=\frac{g_X}{v_0^2}=-\frac{MG}{v_0^2\gamma^3(X^2+Y^2)}X, \tag{9.133}$$

$$\kappa_Y=\frac{g_Y}{v_0^2}=-\frac{MG}{v_0^2\gamma^3(X^2+Y^2)}Y, \tag{9.134}$$

with a factor v_0^2, which has the dimension of squared velocity. By inserting the above equations into Eq. (9.130), the factor can be determined, and it turns out to be

$$v_0^2=\frac{MG}{\gamma^3(X^2+Y^2)}. \tag{9.135}$$

Inserting v_0^2 into (9.133, 9.134) then gives the simplified equations

$$\kappa_X=-\frac{X}{X^2+Y^2}, \tag{9.136}$$

$$\kappa_Y=-\frac{Y}{X^2+Y^2}. \tag{9.137}$$

The γ factor has canceled out, so these equations hold for the relativistic as well as the non-relativistic cases.

Condition (9.131) leads to the result

$$\kappa_X Y-\kappa_Y X=0. \tag{9.138}$$

By using it together with Eq. (9.130), we obtain two equations for two unknowns X and Y:

$$\kappa_X X+\kappa_Y Y=-1, \tag{9.139}$$

$$\kappa_X Y-\kappa_Y X=0. \tag{9.140}$$

The solution for the variables X, Y is

$$X = -\frac{\kappa_X}{\kappa_X{}^2 + \kappa_Y{}^2}, \tag{9.141}$$

$$Y = -\frac{\kappa_Y}{\kappa_X{}^2 + \kappa_Y{}^2}. \tag{9.142}$$

This result is the inverse of Eqs. (9.136, 9.137). X and Y are completely defined by the spin connections κ_X and κ_Y. Alternatively, we could solve the equation system (9.139, 9.140) for the variables κ_X and κ_Y, to obtain Eqs. (9.136, 9.137) again.

The spin connection vectors for an elliptic orbit have been graphed in Fig. 9.13. It can be seen that they all point to the center of motion, i.e., they are parallel to the central gravitational field. The spin connections have been plotted on an equidistant angular grid, but their respective distances do not represent the timing of the motion of the orbiting mass. ■

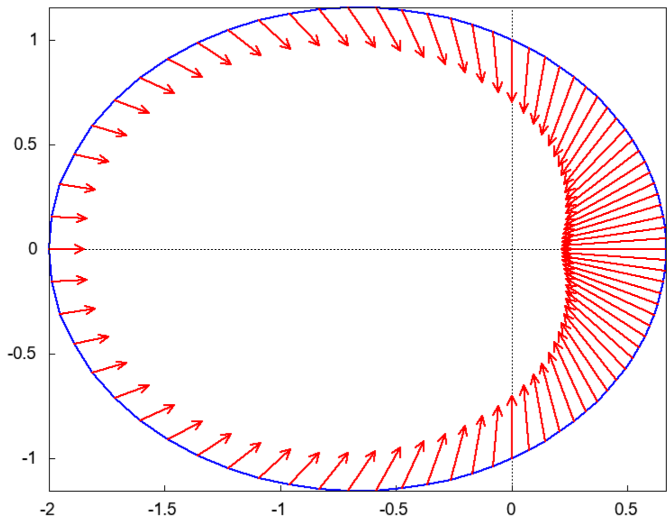


Figure 9.13: Spin connection vectors for an elliptic orbit.

9.2.5 Counter gravitation

We will consider two methods to counteract gravitation. The Biefeld-Brown effect describes an interaction between electricity and gravitation. Another method to achieve counter gravitation involves mechanical means alone.

Biefeld-Brown effect

Consider the Coulomb law:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{9.143}$$

and Newton's law (Poisson equation):

$$\boldsymbol{\nabla} \cdot \mathbf{g} = -4\pi G\, \rho_m, \tag{9.144}$$

where ρ is the electrical charge density and ρ_m is the gravitational mass density. When we use the electric potential ϕ and gravitational potential Φ, the force fields are

$$\mathbf{E} = -\boldsymbol{\nabla}\phi + \omega_e \phi \tag{9.145}$$

and

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi + \boldsymbol{\omega}_m\Phi, \tag{9.146}$$

as derived in earlier chapters. Note that we distinguish between the spin connections of electromagnetism $\boldsymbol{\omega}_e$ and gravitation $\boldsymbol{\omega}_m$. In order to add Eqs. (9.143) and (9.144), we multiply Eq. (9.143) by a factor α, which gives us the same physical units for both equations. By inserting the definitions of $\mathbf{E}$ and $\mathbf{g}$, we obtain

$$-\nabla^2(\alpha\phi + \Phi) + \boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\omega}_e\phi + \boldsymbol{\omega}_m\Phi) = \alpha\frac{\rho}{\varepsilon_0} - 4\pi G\,\rho_m. \tag{9.147}$$

Perfect counter gravitation is achieved if $\mathbf{g} = \mathbf{0}$, which means that

$$\boldsymbol{\nabla}\Phi = \boldsymbol{\omega}_m\Phi. \tag{9.148}$$

Then, the terms with the gravitational potential in Eq. (9.147) cancel out, giving

$$-\nabla^2\phi + \boldsymbol{\nabla}\cdot(\boldsymbol{\omega}_e\phi) = \frac{\rho}{\varepsilon_0} - \frac{4\pi G}{\alpha}\rho_m. \tag{9.149}$$

The right side includes the mass density of the apparatus whose gravity is to be counteracted. To achieve the Biefeld-Brown effect, the electric components on the left side have to be defined in a way that allows the equation to be satisfied. α is a coupling constant between electrostatics and gravitation, which has to be determined experimentally. The divergence term in the above equation can be rewritten to give

$$\nabla^2\phi - \boldsymbol{\omega}_e\cdot\boldsymbol{\nabla}\phi - (\boldsymbol{\nabla}\cdot\boldsymbol{\omega}_e)\phi = -\frac{\rho}{\varepsilon_0} + \frac{4\pi G}{\alpha}\rho_m. \tag{9.150}$$

When the spin connection $\boldsymbol{\omega}_e$ has a negative sign, this becomes an equation for an Euler-Bernoulli resonance, as was discussed in Section 5.4.4. The engineering challenge is to design an electric spin connection that satisfies the resonance condition.

Mechanical counter gravitation

■ **Example 9.7** We compute the spin connections of the Earth's gravitational field, and investigate whether counter gravitation is possible through an interaction between mechanics and gravitation (see computer algebra code [165]). As a consequence of the antisymmetry law, the gravitational field can be expressed in two ways:

$$\mathbf{g} = \frac{1}{2}(-\boldsymbol{\nabla}\Phi + \boldsymbol{\omega}\Phi) \tag{9.151}$$

$$= \frac{1}{2}\left(-\frac{\partial\mathbf{Q}}{\partial t} - c\omega_0\mathbf{Q}\right), \tag{9.152}$$

where Φ is the gravitational potential, $\mathbf{Q}$ is the vector potential, and ω_0 and $\boldsymbol{\omega}$ are the scalar and vector spin connections of gravitation.

Gravitation of the Earth is a gravitostatics problem, and therefore $\partial\mathbf{Q}/\partial t = 0$. The mass and radius of the Earth, and the gravitational constant have the values

$$\begin{aligned} M_E &= 5.97219e24\ \text{kg},\\ r_E &= 6.371009e6\ \text{m},\\ G &= 6.67408e\text{-}11\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}. \end{aligned} \tag{9.153}$$

Therefore, the gravitational potential on the Earth's surface is

$$\Phi_E = -\frac{M_E\ G}{r_E} = -6.256e7\ \mathrm{m}^2\mathrm{s}^{-2}. \tag{9.154}$$

The radial component of the gravitational field is

$$g_r = -\boldsymbol{\nabla}_r \Phi_E = -9.81\ \mathrm{ms}^{-2}. \tag{9.155}$$

The vector spin connection is the same as in the Coulomb potential (see Eq. (4.105) of Example 4.1):

$$\omega_r = \frac{1}{r}, \tag{9.156}$$

which for the Earth's radius is

$$\omega_{rE} = 1.569610e\text{-}7\ \mathrm{m}^{-1} \tag{9.157}$$

and gives

$$\omega_{rE}\Phi = -9.81\ \mathrm{ms}^{-2}. \tag{9.158}$$

Both terms in Eq. (9.151) are of the same size, and we ultimately obtain the well-known absolute value

$$g = 9.81\ \mathrm{ms}^{-2}. \tag{9.159}$$

From Eq. (9.152) it follows that

$$g = -\frac{1}{2} c\omega_0 Q_{rE}, \tag{9.160}$$

where Q_{rE} is the radial part of the gravitational vector potential on the Earth's surface. To completely counteract the force of gravity, the sign of g must be reversed so that the total sum is zero:

$$g \to -g = \frac{1}{2} c\omega_0 Q_{rE}. \tag{9.161}$$

This can be achieved by modifying the spin connection ω_0. Furthermore, to completely counteract the gravitational field, an additional spin connection is required:

$$c\omega_0 = \frac{2g}{Q_{rE}}. \tag{9.162}$$

In this notation, the left side is a time-frequency, so we have to know the value of the vector potential Q_{rE} in order to determine the appropriate ω_0. To ascertain this potential, we have to make a "detour" via the gravitomagnetic field. This field follows from the inverse equation of the mechanical Lorentz force, which is also called the gravitomagnetic equation (see Eq. (7.50)):

$$\boldsymbol{\Omega} = -\frac{1}{c^2}\mathbf{v}\times\mathbf{g}. \tag{9.163}$$

The field $\boldsymbol{\Omega}$ is determined by the gravitational field $\mathbf{g}$ and the linear velocity $\mathbf{v}$ of a point on the Earth's surface. At the equator, it is

$$v = \frac{2\pi}{T} r_E = \frac{2\pi \cdot 6.371009e6\ \mathrm{m}}{24\cdot 3600\ \mathrm{s}} = 463.3\ \mathrm{ms}^{-1}, \tag{9.164}$$

which gives us

$$\Omega = \frac{vg}{c^2} = 5.057e\text{-}14\ \mathrm{s}^{-1}. \tag{9.165}$$

According to a more precise calculation in this book, Eq. (7.71), for the mean gravitomagnetic field on the Earth's surface we obtain

$$\Omega = 1.588e\text{-}14\ \mathrm{s}^{-1}. \tag{9.166}$$

Now, we have to determine the vector potential Q_{rE} at the Earth's surface. The gravitomagnetic field is defined by

$$\mathbf{\Omega} = \mathbf{\nabla} \times \mathbf{Q} - \boldsymbol{\omega} \times \mathbf{Q}. \tag{9.167}$$

If, as in the case of the **g** field, we assume that both contributions are equal, it follows that

$$\Omega = 2\,|-\boldsymbol{\omega} \times \mathbf{Q}| \approx 2\omega Q, \tag{9.168}$$

where $\boldsymbol{\omega} \perp \mathbf{Q}$ has been assumed. At the surface of the Earth, with $\omega = 1/r_E$, the gravitomagnetic field then is

$$\Omega_E \approx \frac{2}{r_E} Q_{rE} \tag{9.169}$$

and, consequently,

$$Q_{rE} \approx \frac{r_E}{2} \Omega_E = 5.059e\text{-}8\ \mathrm{ms}^{-1}. \tag{9.170}$$

From Eq. (9.162) we get a mechanical time-related spin connection of

$$\omega_{t0} := c\omega_0 = \frac{2g}{Q_{rE}} = 3.879e8\ \mathrm{s}^{-1}, \tag{9.171}$$

which corresponds to a time frequency of

$$f_{t0} = \frac{\omega_{t0}}{2\pi} = 6.173e7\ \mathrm{Hz}. \tag{9.172}$$

This value is in the range of what is called "hypersonic", and suggests opportunities to study the effects of very high sound frequencies on gravity. It has been reported [95] that radiation in the wavelength range between 0.3 and 4.3 mm leads to antigravity effects (also see Section 8.4). This corresponds to frequencies between and 10^{10} and 10^{12} Hz, which are even higher than what was computed above. This discussion is the first explanation of a basis for a counter gravitational "hypersonic effect". ■

9.3 Precession and rotation

9.3.1 x theory

We have seen that relativistic effects induce precession of gravitational orbits. In our calculations, precession is a result of relativistic Lagrange theory, which is a first-principles theory, without additional assumptions. On the other hand, the precession angles of planetary motion are quite small and require special numerical effort to get reliable results. In the Solar system, for example, the precession angles are only a few arc seconds per century. Therefore, it would be advantageous to have an analytical method to compute and compare precessions. Then, we could continue to use the orbit parameters of ellipses like eccentricity, half right latitude and angular momentum. In a numerical Lagrange calculation, these parameters are not available a priori, and have to be extracted from the results. The calculation is solely determined by the initial conditions of position and velocity of the planets.

In the following discussion, we will be using a plane polar coordinate system. In *x theory* we assume a modified progression of the orbital angle of motion compared to standard elliptic orbits. First, we assume a deviation by a constant factor. Later, we will discuss a method of extending this procedure to variable progressions. If we denote the deviation factor of the polar angle by x, the deviation of a full orbit of 2π is described by $x \cdot 2\pi$, and the analytical form of the orbit (9.20) then becomes

$$r = \frac{\alpha}{1 + \varepsilon \cos(x\theta)}. \tag{9.173}$$

For $x < 1$, we obtain forward precession, because for $\theta = 2\pi$ a full circle has not yet been completed. For $x > 1$, we have backward precession. The analytical results for conic sections given in Sections 8.1.1 and 8.1.2 can be extended in the following way. (For the detailed calculations, see computer algebra code [166].)

The orbital derivative is

$$\frac{dr}{d\theta} = \frac{\varepsilon r^2 x}{\alpha} \sin(x\theta), \tag{9.174}$$

and the time derivative can be computed by

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{\varepsilon r^2 x \dot{\theta}}{\alpha} \sin(x\theta). \tag{9.175}$$

The angular momentum is a constant of motion, augmented with a factor of x:

$$L = mr^2 x\dot{\theta}. \tag{9.176}$$

Since the orbit $r(\theta)$ is known from Eq. (9.173), using the Binet equation (9.19) allows us to compute the force law:

$$F(r) = \frac{L^2}{m}\left(\frac{x^2 - 1}{r^3} - \frac{x^2}{\alpha r^2}\right). \tag{9.177}$$

For $x = 1$, the relation (9.26) for the angular momentum holds:

$$L^2 = m^2 MG\alpha, \tag{9.178}$$

and the force law (9.177) takes the Newtonian form:

$$F(r) = -\frac{mMG}{r^2}. \tag{9.179}$$

In general, the squared velocity of the orbiting mass m is

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \tag{9.180}$$

and, by using Eq. (9.175), it can be written as

$$v^2 = \dot{\theta}^2\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) = r^2\dot{\theta}^2\left(1 + \left(\frac{rx\varepsilon}{\alpha}\sin(x\theta)\right)^2\right). \tag{9.181}$$

From (9.176), the angular frequency is

$$\dot{\theta} = \frac{L}{xmr^2}, \tag{9.182}$$

and inserting this into the velocity equation gives

$$v = \frac{L}{xmr}\left(1+\left(\frac{rx\varepsilon}{\alpha}\sin(x\theta)\right)^2\right)^{1/2}. \tag{9.183}$$

From the orbit (9.174), it follows that

$$\sin(x\theta)^2 = 1-\frac{1}{\varepsilon^2}\left(\frac{\alpha}{r}-1\right)^2, \tag{9.184}$$

so the velocity takes the final form:

$$v = \frac{L}{xm\alpha}\left(\frac{2x^2\alpha}{r}-x^2(1-\varepsilon^2)+\frac{\alpha^2}{r^2}(1-x^2)\right)^{1/2}, \tag{9.185}$$

which only depends on r. The θ coordinate has been eliminated. For $x = 1$, it turns into the equation

$$v = \frac{L}{m\alpha}\left(\frac{2\alpha}{r}-(1-\varepsilon^2)\right)^{1/2}. \tag{9.186}$$

This can be brought into the form:

$$v^2 = MG\left(\frac{2}{r}-\frac{1}{a}\right), \tag{9.187}$$

which is the well-known equation (9.21) of Newtonian theory.

The x theory approach is depicted in the following graphs, which should clarify the meaning of the precession parameter x for conic section orbits. We start with ellipses. Significant deviations of x from unity lead to changes in the minor axis, which becomes identical to the major axis (see Fig. 9.14). For $x = 2$, the curve is deformed and no longer identifiable as an ellipse. Therefore, we call these types of curves *generalized conic sections*.

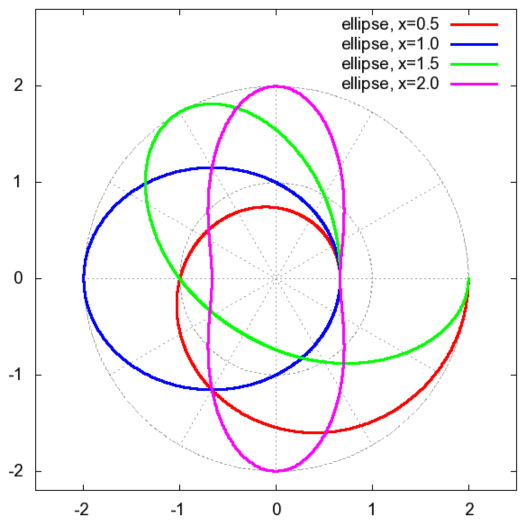


Figure 9.14: Generalized ellipses ($\varepsilon = 0.5$).

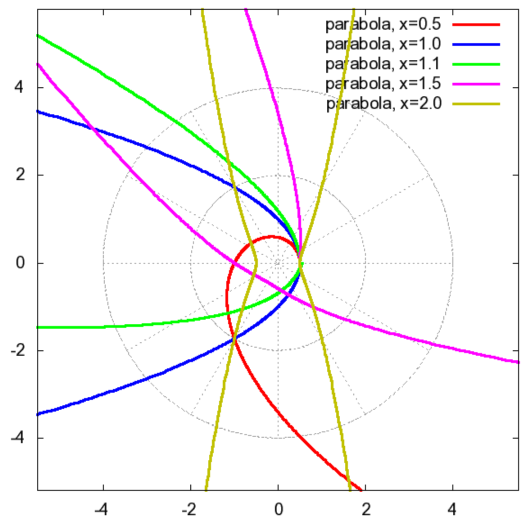


Figure 9.15: Generalized parabolas ($\varepsilon = 1.0$).

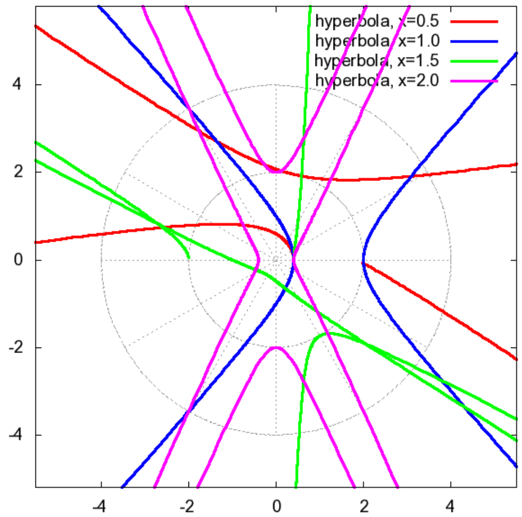


Figure 9.16: Generalized hyperbolas ($\varepsilon = 1.5$).

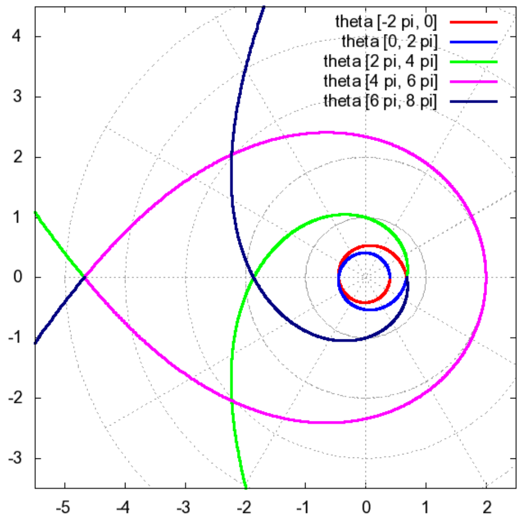


Figure 9.17: Special generalized hyperbolas ($\varepsilon = 1.5,\ x = 0.3$).

The deformation of generalized parabolas can be seen in Fig. 9.15. For values of x that are very different from unity, the curves are more like spirals or hyperbolas than parabolas. For generalized hyperbolas (Fig. 9.16), the curves are distorted in a similar way. In most plots, only the angular range of θ between 0 and 2π is shown, in order not to overload the diagrams, with the exception being hyperbolas with $x = 0.3$ (graphed in Fig. 9.17), which are shown with orbits for a broader range of angles. Between $-\pi$ and π, the orbit is a circle, showing that even closed orbits for $\varepsilon > 1$ are possible for generalized conic sections. For larger angles, several kinds of loops are observable.

All this is reminiscent of the alleged behavior of "black holes" of the superseded Einstein theory. If the central gravitational mass is enormous, as reported for the centers of galaxies by astronomers, we can assume significant deviations of x from unity, leading to the orbits in Fig. 9.17, for example. Even deflection of light can be described by Eq. (9.173). Therefore, we can conceive of light in the vicinity of such massive stars being bent completely around the center and trapped. This would result in invisibility from the outside, and the massive star would behave like a "black hole". All of these conclusions follow directly from classical physics.

The Lagrangian of x theory is

$$\mathscr{L} = T - U, \tag{9.188}$$

as usual. The kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right). \tag{9.189}$$

By integrating the force law (9.177) computed from the Binet equation, we get the potential energy

$$U = -\int F(r)dr = -mMG\left(\frac{x^2}{r} + \frac{\alpha(1-x^2)}{2r^2}\right). \tag{9.190}$$

Then, the resulting Lagrangian of x theory is

$$\mathscr{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mMG\left(\frac{x^2}{r} + \frac{\alpha(1-x^2)}{2r^2}\right), \tag{9.191}$$

and the equations of motion are

$$\ddot{r} = MG\left(-\frac{x^2}{r^2} - \frac{\alpha\left(1-x^2\right)}{r^3}\right) + r\,\dot{\theta}^2, \tag{9.192}$$

$$\ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r}. \tag{9.193}$$

Obviously, only the radial equation is affected by the x factor. In the limit $x \to 1$ we obtain the original Lagrange equations for Newtonian motion in a plane (9.15 - 9.16). Eq. (9.192) is identical to the force law (9.177). Here, in the equation of dynamics, the centrifugal acceleration $r\dot{\theta}^2$ additionally appears.

So far, we have considered a constant value of x. We can extend the theory to an x that is variable and depends on the coordinates, by writing $x(r,\theta)$. This makes sense, since the relativistic effects of precession depend on the velocity, and are largest at the periastron of orbits (point of closest approach to the central mass). For a varying x, the orbit (9.173) takes the form

$$r = \frac{\alpha}{1+\varepsilon\cos(x(r,\theta)\,\theta)}. \tag{9.194}$$

The dependence on r makes this a transcendental equation, and the Binet equation can no longer be used to determine the force law. As an alternative, we will use Lagrange theory directly. Assuming that the potential energy (9.190) remains approximately valid (which it will, if x varies only slowly), we obtain the Euler-Lagrange equations:

$$\ddot{r} = MG\left(\frac{2r-\alpha}{r^2}x\frac{\partial x}{\partial r} - \frac{x^2}{r^2} - \frac{\alpha\left(1-x^2\right)}{r^3}\right) + r\,\dot{\theta}^2, \tag{9.195}$$

$$\ddot{\theta} = MG\frac{2r-\alpha}{r^4}x\frac{\partial x}{\partial \theta} - 2\frac{\dot{r}\dot{\theta}}{r}. \tag{9.196}$$

These equations contain the derivatives of x as expected, leading to an additional term in both equations. If x depends only on r, the angular acceleration is the same as before, and the angular momentum (9.176) is a constant of motion. If x depends on θ, this is not the case anymore. For $x = \text{const.}$, we obtain Eqs. (9.192, 9.193) again.

■ **Example 9.8** We compute the x factor for the precession of the planet Mercury (see computer algebra code [167]). The angle of precession per orbit (which matches both experiment, and Einstein theory approximation [102]), is given by

$$\Delta\theta = \frac{6\pi MG}{c^2\alpha}, \tag{9.197}$$

where M is the mass of the Sun and α is the half right latitude of Mercury's orbit:

$$M = 1.98855e30\text{ kg}, \tag{9.198}$$
$$\alpha = 5.7909050e10\text{ m}.$$

This gives a precession angle of

$$\Delta\theta = 4.80664e\text{-}7 \tag{9.199}$$

per orbit. From x theory, we have

$$\Delta\theta = 2\pi(1-x), \tag{9.200}$$

from which we get

$$x = 1 - \frac{\Delta\theta}{2\pi} = 1 - \frac{3MG}{c^2\alpha}, \tag{9.201}$$

which then gives a numerical value of

$$x = 1 - 7.649998e\text{-}8. \tag{9.202}$$

This value is very close to unity, as expected. The computed value of $\Delta\theta$ amounts to 41.16 arc seconds per (Earth) century, while NASA reports 42.98 arc seconds per century. For the conversion factors see [103]. ■

9.3.2 Precession by rotating spacetime

In the preceding sections, we have seen that gravitational precession is induced by any kind of perturbation of Newton's equations. The causes of these perturbations include relativistic effects, and influences of spacetime structures. We will now study the effect of the gravitational vector potential, which leads to precession by the gravitational Lorentz force.

The vector potential of mechanics in ECE2 theory is a velocity; therefore, we denote it by a vector $\mathbf{v}_g$ for clarity:

$$\mathbf{W} = \mathbf{v}_g. \tag{9.203}$$

The gravitomagnetic field is defined by

$$\boldsymbol{\Omega}_g = \boldsymbol{\nabla} \times \mathbf{v}_g. \tag{9.204}$$

The vector potential gives rise to its own momentum field, in addition to the mechanical momentum

$$\mathbf{p}_m = m\mathbf{v} \tag{9.205}$$

or, relativistically,

$$\mathbf{p}_{mr} = \gamma m\mathbf{v}. \tag{9.206}$$

In electrodynamics, we know that the field momentum is

$$\mathbf{p}_A = q\mathbf{A}, \tag{9.207}$$

where $\mathbf{A}$ is the magnetic vector potential and q is the electric charge. In gravitation, the analogous field momentum is

$$\mathbf{p}_g = m\mathbf{v}_g. \tag{9.208}$$

Consequently, the *canonical momentum* of classical mechanics is

$$\mathbf{p}_c = \mathbf{p}_m + \mathbf{p}_g = m\mathbf{v} + m\mathbf{v}_g. \tag{9.209}$$

Only the mechanical (and not the canonical) momentum is measurable. Therefore, the Hamiltonian (or total energy) is defined using the measurable momentum only, giving us

$$\mathscr{H} = \frac{1}{2m}(\mathbf{p}_c - \mathbf{p}_g)^2 + U = \frac{1}{2}m\mathbf{v}^2 + U, \tag{9.210}$$

in the non-relativistic case with potential energy U.

If no canonical momentum is present, the Lagrangian is

$$\mathscr{L} = T - U \tag{9.211}$$

with kinetic energy T. However, to determine the Lagrangian in the presence of $\mathbf{v}_g$, instead of using Eq. (9.211), we have to use *Hamilton's equation*:

$$\mathscr{H} = \sum_j p_j \dot{q}_j - \mathscr{L}, \tag{9.212}$$

where p_j are the components of the canonical momentum and q_j are the generalized coordinates. The result is

$$\mathscr{L} = \frac{1}{2} m\mathbf{v}^2 + m\mathbf{v} \cdot \mathbf{v}_g - U \tag{9.213}$$

(see computer algebra code [168]). There is a product term of velocities, but no quadratic term in $\mathbf{v}_g$. This derivation follows the procedures of standard mechanics [104]. In the papers of ECE theory, the Hamiltonian

$$\mathscr{H}_1 = \frac{1}{2m} \mathbf{p}_c^2 + U \tag{9.214}$$

was used [105]. This leads to somewhat different results, and to complications in the Euler-Lagrange equations and gravitational Lorentz force (see below), although both methods are consistent internally. For internal consistency checks, see [168].

In the following discussion, we assume a uniform rotation of spacetime (which corresponds to a rotation of a solid disk), and represent that through the vector potential. This allows a number of special cases to be introduced in the Hamiltonian. We focus on the results here; for the full (and sometimes lengthy) calculations see [106].

We use a cylindrical coordinate system ($r, \ \theta, \ Z$), and assume that the gravitomagnetic field points in the direction of the unit vector $\mathbf{k}$, which gives us

$$\mathbf{\Omega}_g = \Omega_{gZ} \mathbf{k}. \tag{9.215}$$

Because

$$\mathbf{\Omega}_g = \boldsymbol{\nabla} \times \mathbf{v}_g, \tag{9.216}$$

$\mathbf{v}_g$ has only a θ component, i.e., it is in the rotation plane and perpendicular to the radius vector. Next, it can be shown [105] that

$$\mathbf{v}_g = \frac{1}{2} \mathbf{\Omega}_g \times \mathbf{r}, \tag{9.217}$$

and that the (constant) angular velocity of spacetime rotation is

$$\Omega = \frac{1}{2} |\mathbf{\Omega}_g| = \frac{1}{2} \Omega_{gZ}. \tag{9.218}$$

This is analogous to the electromagnetic Larmor frequency:

$$\omega_L = \frac{e}{2m} B, \tag{9.219}$$

in a magnetic field B. In the mechanical case, both constants are masses; therefore, the factor reduces to 1/2. For the modulus of $\mathbf{v}_g$, it follows that

$$v_g = \Omega r. \tag{9.220}$$

Using the definition of angular momentum,

$$\mathbf{L} = \mathbf{p} \times \mathbf{r}, \tag{9.221}$$

and the Hamiltonian of ECE theory (9.214), which can be rewritten as

$$\mathscr{H}_1 = \frac{1}{2}m\left(\mathbf{v}^2 + \mathbf{v}_g^2\right) + m\mathbf{v}\cdot\mathbf{v}_g + U, \tag{9.222}$$

the product term on the right side can be written as

$$m\mathbf{v}\cdot\mathbf{v}_g = \frac{1}{2}\mathbf{p}\cdot\mathbf{\Omega}_g \times \mathbf{r} = -\frac{1}{2}\mathbf{p}\times\mathbf{r}\cdot\mathbf{\Omega}_g = -\frac{1}{2}\mathbf{L}\cdot\mathbf{\Omega}_g. \tag{9.223}$$

By construction, $\mathbf{L}$ and $\mathbf{\Omega}_g$ are parallel. Therefore, $\mathbf{L}\cdot\mathbf{\Omega}_g = L\Omega_g$, and with (9.218) it follows that

$$m\mathbf{v}\cdot\mathbf{v}_g = -L\Omega, \tag{9.224}$$

where L is a constant of motion. Then, the Hamiltonian (9.222) is

$$\mathscr{H}_1 = \frac{1}{2}m\left(\mathbf{v}^2 + \mathbf{v}_g^2\right) - L\Omega + U. \tag{9.225}$$

A positive precession frequency Ω leads to a decrease in total energy, while a negative precession frequency (in the direction of elliptic motion, see next example) increases total energy so that it is possible to change the orbit to an open conic section, if $E > 0$ is reached.

■ **Example 9.9** The Euler-Lagrange equations of the Lagrangian (9.213) are computed in plane polar coordinates. We define the vector potential $\mathbf{v}_g$ for a uniformly rotating spacetime by

$$\mathbf{v}_g = \begin{bmatrix} 0 \\ r\omega_0 \end{bmatrix}, \tag{9.226}$$

where the two vector components are the radial and angular components. This is a purely rotational potential with a fixed frequency ω_0. The resulting Euler-Lagrange equations from the Lagrangian (9.213) are

$$\ddot{r} = r\dot{\theta}^2 + 2r\omega_0\dot{\theta} - \frac{MG}{r^2}, \tag{9.227}$$

$$\ddot{\theta} = -\frac{2\dot{r}}{r}\left(\omega_0 + \dot{\theta}\right). \tag{9.228}$$

There is an additional term of ω_0 in both equations. A numerical solution (see computer algebra code [169]) shows that a negative ω_0 leads to forward precession, while a positive ω_0 produces retrograde precession. The constant of motion is the angular momentum

$$L = mr^2(\omega_0 + \dot{\theta}). \tag{9.229}$$

The angular velocity of spacetime rotation adds to that of the undisturbed ellipse. ■

■ **Example 9.10** The vector potential of the terrestrial orbit is derived from Eq. (9.220). The orbital precession (apsidal precession) is 1.862e-7 radians/year, which gives

$$\Omega = 1.862e\text{-}7/(365.25\cdot 24\cdot 3600)\text{ rad/s} = 5.900e\text{-}15\text{ rad/s}. \tag{9.230}$$

With the average orbit radius of r=1.495e11 m, we obtain

$$v_g = \Omega r = 8.821e\text{-}4\text{ m/s}. \tag{9.231}$$

This is orders of magnitude smaller than the orbital velocity of the Earth around the Sun, which is 2.979e4 m/s. ■

The gravitational Lorentz force

Finally, we show that the *gravitational Lorentz force* can be derived from the Lagrangian (9.213). Analogously to the electromagnetic Lorentz force (7.46), the gravitational Lorentz force is defined by Eq. (7.49):

$$\mathbf{F}_L = m\mathbf{g}_L = m\mathbf{v} \times \mathbf{\Omega}_g, \tag{9.232}$$

which can be written as

$$m\mathbf{g}_L = m\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}_g). \tag{9.233}$$

To derive this force from the Lagrangian, we use a cartesian coordinate system in three dimensions with position and velocity vectors

$$\mathbf{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix}. \tag{9.234}$$

The gravitomagnetic vector potential is

$$\mathbf{v}_g = \begin{bmatrix} v_{gX} \\ v_{gY} \\ v_{gZ} \end{bmatrix} \tag{9.235}$$

and, without a potential energy, the Lagrangian is

$$\mathscr{L} = \frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + m\left(v_{gX}\dot{X} + v_{gY}\dot{Y} + v_{gZ}\dot{Z}\right). \tag{9.236}$$

In computer algebra code [170] it is shown that, for example, the X component of the acceleration is

$$\ddot{X} = \left(\frac{\partial}{\partial X}v_{gZ} - \frac{\partial}{\partial Z}v_{gX}\right)\frac{\partial}{\partial t}Z + \left(\frac{\partial}{\partial X}v_{gY} - \frac{\partial}{\partial Y}v_{gX}\right)\frac{\partial}{\partial t}Y. \tag{9.237}$$

Ultimately, computer algebra gives

$$\ddot{\mathbf{r}} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}_g) = \mathbf{v} \times \mathbf{\Omega}_g, \tag{9.238}$$

which is the acceleration in the gravitomagnetic force equation (9.232). The gravitomagnetic field, $\mathbf{\Omega}_g$, is the gravitational equivalent of magnetic induction in the gravitational version of the Ampère-Maxwell law of ECE2 electrodynamics:

$$\boldsymbol{\nabla} \times \mathbf{\Omega}_g = \frac{4\pi G}{c^2}\mathbf{J}_g, \tag{9.239}$$

where $\mathbf{J}_g$ is a local mass current density that is analogous to the current density in electrodynamics. The gravitomagnetic force gives rise to forces in addition to the centrifugal and Coriolis forces.

Field momentum in one dimension

The concept of the canonical momentum also holds for one-dimensional motion, say in the X direction. We will explore whether the gravitational force on the Earth's surface can be counteracted by a field momentum. This field momentum can be produced only by electromagnetism, and has the well-known vector form given by Eq. (9.207). Here, we use it just in one dimension:

$$p_A = qA \tag{9.240}$$

with charge q and vector potential A. The Lagrangian (9.236) also holds for one-dimensional motion, in which it is

$$\mathscr{L} = \frac{m}{2}\dot{X}^2 + qA\dot{X} - U \tag{9.241}$$

with

$$U = mgX \tag{9.242}$$

being the potential energy in the constant gravitational field, and g=9.81 m/s^2. When the Lagrangian is reduced to one dimension, it follows from solution (9.237) that

$$\ddot{X} = -g, \tag{9.243}$$

because the only remaining component v_g, corresponding to A, has no dependence on the Y or Z coordinate. However, if we assume a time dependence of the vector potential, the Lagrangian then becomes

$$\mathscr{L} = \frac{m}{2}\dot{X}^2 + qA(t)\dot{X} - mgX, \tag{9.244}$$

and the Euler-Lagrange equation becomes

$$\ddot{X} = -\frac{q}{m}\dot{A}(t) - g. \tag{9.245}$$

■ **Example 9.11** We consider examples of Eq. (9.245) with growing $X(t)$ (see computer algebra code [171]), which indicates that gravity is counteracted by the momentum of the vector potential. For

$$A(t) = -A_0 t^n, \tag{9.246}$$

the A term in the Euler-Lagrange equation becomes positive and outperforms the gravitational term $-g$ after a short time. To obtain growing curves, it has to have $n \geq 1$. Two examples, with $n = 1$ and $n = 2$, are graphed in Fig. 9.18 using a logarithmic scale. The results can also be obtained analytically and are polynomials in t. ■

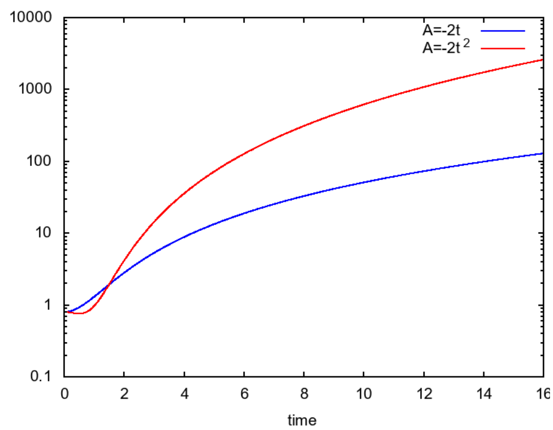


Figure 9.18: Trajectories $X(t)$ for two values of n, according to Eq. (9.246).

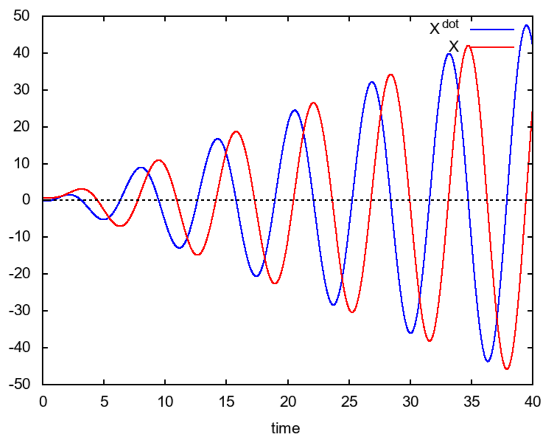


Figure 9.19: Trajectories $\dot{X}(t)$ and $X(t)$ for a driven harmonic oscillator.

■ **Example 9.12** As another example, we consider a harmonic oscillator in an external momentum field (see computer algebra code [172]). Instead of a driving force, we use an oscillating vector potential:

$$A(t) = A_0 \cos(\omega t). \tag{9.247}$$

The potential of the repulsive force is

$$U = \frac{1}{2} k X^2 \tag{9.248}$$

with spring constant k. In a driven oscillator, the resonance frequency is

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{9.249}$$

The Euler-Lagrange equation is

$$\ddot{X} + \frac{k}{m} X = A_0\, \omega \frac{q}{m} \sin(\omega t), \tag{9.250}$$

which is an equation of an undamped, forced oscillation that gives resonance at $\omega = \omega_0$. The vector potential term qA acts as a driving force, even though it is a momentum and not a force. For $\omega \to \omega_0$, the amplitude grows to infinity. The benefit of this approach is that no minimum value of the driving momentum amplitude is required, so that a resonance can be produced even by small values of A_0 and q. An example is graphed in Fig. 9.19.

This resonance mechanism can be used to extract energy from spacetime, and to counteract gravitation directly. The important point is that the driving force has to operate without a feedback effect. The mechanism that generates A must be independent of the amplitude X of the oscillator, in order to guarantee that the use of the vector potential by the oscillator does not impact the mechanism that is creating the vector potential. The vector potential may arise from the vacuum or aether. It can be created by a magnetic field, for example. The condition of decoupling is satisfied for a permanent magnet, where the vector potential is replenished from the vacuum via the elementary magnetic dipoles, even if the amplitude of the vector potential is diminished by driving the oscillator. ■

9.3.3 Rotation of the relativistic line element

Next, we will consider the effects of a rotating spacetime by investigating frame rotations, mainly in a relativistic context. Such a context is described by a rotating line element. A principal origin of rotating orbits can be found from the line elements of special and general relativity. In Minkowski space with planar polar coordinates, this line element is

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2, \tag{9.251}$$

where τ is proper time, the time in the local frame of the system under consideration. The time t is defined in the observer frame. With the squared observer velocity

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 \tag{9.252}$$

and the relativistic γ factor

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \tag{9.253}$$

the above equation can be developed into

$$\begin{aligned} ds^2 = c^2 d\tau^2 &= c^2dt^2 - dr^2 - r^2 d\phi^2 \\ &= c^2dt^2 - \frac{dr^2}{dt^2}dt^2 - \frac{d\phi^2}{dt^2}dt^2 \\ &= c^2dt^2\left(1-\frac{v^2}{c^2}\right) \\ &= c^2\frac{dt^2}{\gamma^2}. \end{aligned} \tag{9.254}$$

For the time transformation, it follows that

$$dt = \gamma\, d\tau. \tag{9.255}$$

Thomas precession

In the standard view of special relativity, *Thomas precession* is described in the following way. A particle moves on an arbitrary curved line in Minkowski space. Because such a motion requires accelerations, it is assumed that the particle is at rest in an inertial frame at each instant of time. Thus, special relativity can be applied at each instant. This leads to a complicated transformation to an external observer frame [107]. The result is that in this frame an additional rotation is seen, an angular velocity of Thomas precession, which is

$$\omega_T = -\frac{1}{c^2}\frac{\gamma^2}{\gamma+1}\mathbf{v}\times\mathbf{a} = -(\gamma-1)\frac{\mathbf{v}\times\mathbf{a}}{v^2} \tag{9.256}$$

with $\mathbf{a} = \dot{\mathbf{v}}$ being the acceleration. For non-relativistic velocities with $v \ll c,\ \gamma \to 1$, this angular velocity is approximately

$$\omega_T \approx -\frac{1}{2}\frac{\mathbf{v}\times\mathbf{a}}{c^2}. \tag{9.257}$$

The factor 1/2 is called the Thomas half, which appears in quantum mechanics (for example, in spin-orbit coupling, where electrons move with relativistic velocities).

This effect becomes evident when a mass orbits a center of gravitation with a high velocity, which leads to a frame rotation of the orbit in the observer frame. The acceleration is the central gravitational acceleration $\mathbf{g}$. In a generally covariant theory such as ECE2, the term $\mathbf{v}\times\mathbf{g}$ in Eq. (9.257) is part of the gravitomagnetic field of the transformation law (7.43):

$$\mathbf{\Omega}'_g = -\gamma\frac{1}{c^2}\mathbf{v}\times\mathbf{g} - \frac{\gamma^2}{1+\gamma}\frac{\mathbf{v}}{c}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{\Omega}_g\right). \tag{9.258}$$

The primed field is the one in the observer frame. There is, however, no gravitomagnetic field local to the rotating frame; therefore, the equation simplifies to

$$\mathbf{\Omega}'_g = -\gamma\frac{1}{c^2}\mathbf{v}\times\mathbf{g}. \tag{9.259}$$

The gravitomagnetic field $\mathbf{\Omega}'_g$ effects a rotation. This is comparable to the electromagnetic case, where a magnetic moment of a moving charge q in an external magnetic field B leads to the *Larmor precession*:

$$\omega_L = \frac{q\,g_L}{2m}B \tag{9.260}$$

(see [108]). g_L is the Landé factor, which is unity in classical physics. The corresponding equation for the gravitomagnetic field is

$$\omega_g = \frac{1}{2}\Omega'_g, \tag{9.261}$$

which is analogous to the definition of the gravitomagnetic moment in Example 7.3. Therefore, the frequency of Thomas precession in ECE2 theory, in non-relativistic approximation, is

$$\omega_T = \frac{1}{2}\Omega'_g = \frac{1}{2}\frac{1}{c^2}\mathbf{v}\times\mathbf{g}, \tag{9.262}$$

and is identical to the modulus of Eq. (9.257), which has been obtained only in complicated form from standard theory. The result (9.262) is valid for a spacetime with curvature and torsion.

In a simplified perspective, the angular difference between the frames is caused by the time dilation between observer time dt and proper time $d\tau$. It is assumed that the rotational frequency ω of the orbiting mass is the same in both frames. Then, the angular difference seen in the observer frame is

$$\Delta\phi = \omega(dt - d\tau). \tag{9.263}$$

Using relation (9.255), the additional angle seen in the observer frame is

$$\Delta\phi = \omega\, dt\left(1 - \frac{1}{\gamma}\right) \tag{9.264}$$

and, in the local frame of the mass, it is

$$\Delta\phi = \omega\, d\tau\,(\gamma - 1). \tag{9.265}$$

When we expand the γ factor to second order, we have

$$\left(1 - \frac{1}{\gamma}\right) \approx (\gamma - 1) \approx \frac{1}{2}\frac{v^2}{c^2}, \tag{9.266}$$

i.e., in this approximation the deviation $\Delta\phi$ in both frames is identical. This value is usually related to one full rotation of 2π; therefore, we have a relative precession of

$$\frac{\Delta\phi}{2\pi} \approx \frac{1}{2}\frac{v^2}{c^2}. \tag{9.267}$$

Here, the Thomas half appears again, and we can directly compute the precession angle $\Delta\phi$ from the orbital velocity v. Since v is not constant except in circular orbits, we have to take either an average value or the maximum at periastron.

de Sitter and Lense-Thirring precessions

The common explanation for *de Sitter* or *geodetic precession* uses Einsteinian general relativity. The line element with non-constant metric factors is defined by

$$ds^2 = c^2 d\tau^2 = c^2 m(r,t)dt^2 - \frac{dr^2}{m(r,t)} - r^2 d\phi^2, \tag{9.268}$$

where $m(r,t)$ is a metric function to be determined by Einstein's obsolete field equation. For the Schwarzschild metric, it follows that

$$m(r,t) = 1 - \frac{2MG}{rc^2} =: 1 - \frac{r_0}{r} \tag{9.269}$$

with the *Schwarzschild radius* r_0. Geodetic precession is then derived in the following way. The rotation angle of plane polar coordinates in the observer frame is

$$d\phi' = d\phi + \omega dt, \tag{9.270}$$

where we have assumed that the frame of the orbiting mass rotates with full angular frequency ω, i.e., the mass is at rest in its own frame. Then, the line element is

$$\begin{aligned} ds^2 &= c^2 m(r,t)dt^2 - \frac{dr^2}{m(r,t)} - r^2 d\phi'^2 \\ &= c^2 m(r,t)dt^2 - \frac{dr^2}{m(r,t)} - r^2(d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2). \end{aligned} \tag{9.271}$$

Using the standard relation for polar coordinates,

$$d\phi = \omega dt, \tag{9.272}$$

we obtain

$$ds^2 = \left(c^2 m(r,t) - 3\omega^2 r^2\right) dt^2 - \left(\frac{dr^2}{m(r,t)} + r^2 d\phi^2\right). \tag{9.273}$$

We define

$$v_\phi = \omega r \tag{9.274}$$

and

$$v_1 = \frac{1}{m(r,t)} \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2, \tag{9.275}$$

which gives us

$$ds^2 = \left(m(r,t) - \frac{3v_\phi^2 + v_1^2}{c^2}\right) c^2 dt^2, \tag{9.276}$$

and with

$$v_2^2 = v_1^2 + 3v_\phi^2, \tag{9.277}$$

the line element can be written as

$$ds^2 = \frac{c^2}{\gamma_2^2} dt^2 \tag{9.278}$$

with

$$\gamma_2 = \frac{dt}{d\tau} = \left(m(r,t) - \frac{v_2^2}{c^2}\right)^{-1/2}. \tag{9.279}$$

The phase change by de Sitter rotation is given by

$$\Delta\phi = 2\pi(\gamma_2 - 1) \approx 2\pi \frac{v_2^2}{c^2}, \tag{9.280}$$

for one orbit.

The *Lense-Thirring precession* is an additional frame dragging of spacetime, caused by rotating masses, and is normally quite small, compared to geodetic precession. In standard theory, it is derived from a solution of Einstein's field equation in the weak field approximation. In ECE2 theory, it is explained by the gravitomagnetic field (see Example 7.1).

Precession by general frame rotation

We now consider a general frame rotation, where an angular frequency is added to the angular velocity of the orbiting mass. The rotation angle in the observer frame is defined by

$$\phi' = \phi + \omega_1 t, \tag{9.281}$$

where ω_1 is a frame rotation that is independent of the "regular" orbital angular velocity:

$$\omega = \frac{d\phi}{dt}. \tag{9.282}$$

ω_1 is allowed to have an arbitrary time dependence, and therefore

$$d\phi' = d\phi + \omega_1\, dt + t\, d\omega_1. \tag{9.283}$$

Similarly to Eq. (9.271), the line element is

$$\begin{aligned}
ds^2 &= c^2 dt^2 - dr^2 - r^2 \left(d\phi + \omega_1\ dt + t d\omega_1\right)^2 \qquad (9.284)\\
&= c^2 dt^2 - dr^2 - r^2 \left(d\phi^2 + \omega_1^2 dt^2 + t^2 d\omega_1^2 + 2(\omega_1\ d\phi\ dt + t\ d\phi\ d\omega_1 + t\omega_1 d\omega_1\ dt)\right)\\
&= \left(c^2 - \left(\frac{dr}{dt}\right)^2\right) dt^2 - r^2 \left(\omega^2 dt^2 + \omega_1^2 dt^2 + t^2 d\omega_1^2 + 2(\omega_1\ \omega\ dt^2 + (\omega + \omega_1)\ t\ d\omega_1\ dt)\right)\\
&= \left(c^2 - \left(\frac{dr}{dt}\right)^2 - r^2(\omega^2 + 2\omega_1\omega + \omega_1^2)\right) dt^2\\
&\quad - r^2 \left(t^2 \left(\frac{d\omega_1}{dt}\right)^2 dt^2 + 2(\omega + \omega_1)\ t\ \frac{d\omega_1}{dt}\ dt^2\right)\\
&= \left(c^2 - \left(\frac{dr}{dt}\right)^2 - r^2(\omega + \omega_1 + t\ \frac{d\omega_1}{dt})^2\right) dt^2\\
&= c^2 \left(1 - \frac{v_1^2}{c^2}\right) dt^2.
\end{aligned}$$

In the last line we have used the replacement:

$$v_1^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\omega + \omega_1 + t\ \frac{d\omega_1}{dt}\right)^2. \tag{9.285}$$

The corresponding relativistic γ factor is

$$\gamma_1 = \frac{dt}{d\tau} = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2}. \tag{9.286}$$

This derivation can be extended to the metric (9.268), which has a non-constant function $m(r,t)$. As before, the phase change of one rotation is approximately

$$\Delta\phi = 2\pi \frac{v_1^2}{c^2}. \tag{9.287}$$

This result holds for forward precession. For retrograde precession, the sign has to be changed. We can compare this result with x theory, from which we obtain (see Eq. (9.200)):

$$\Delta\phi = 2\pi(1 - x). \tag{9.288}$$

Equating both results (for forward precession) gives

$$x = 1 - \frac{v_1^2}{c^2} = \frac{1}{\gamma_1^2}. \tag{9.289}$$

This is a useful connection between non-relativistic x theory and relativistic gravitational theory. The velocity v_1 is not constant; therefore, it is meaningful to apply the v_1 value at the perihelion (or the periastron, respectively), which is the point where precession is most effective.

The frame rotation by an angular frequency ω_1 can also be characterized kinematically. The phase change for one rotation is

$$\Delta\phi = \omega_1 T, \tag{9.290}$$

where T is the time period for this rotation in the observer frame. This is an equation of classical mechanics, but it does not follow from Newtonian gravitation, which cannot describe frame rotation.

■ **Example 9.13** For the precession formula (9.287), we compare computed results with experimental data for three astronomical objects (see Table 9.3, and computer algebra code [173]). For the experimental values, see Tables 9.1 and 9.2 and Example 9.10.

Since the contribution of precession to the velocity at periastron is much smaller than the linear velocity v, we assume that $v_1 \approx v$. The $\Delta\phi$ value for the Earth is too small by a factor of 3, and conformance can be improved by increasing v_1 (see adjusted value in Table 9.3). One has to bear in mind that the relativistic contribution to the Earth's precession is only 1/232 of the measured value. The main contribution comes from the influences of other planets.

For the Hulse-Taylor double star system, a factor of 4.6 appears between computed and experimental precession. It is not known whether influences of other celestial bodies play a role. For the S2 star, the computed value is well within the experimentally determined range, albeit this is quite large. As mentioned earlier, the spin connections of spacetime influence the motion of celestial bodies, and therefore we cannot expect to obtain exact agreement. In particular, the Solar System is not well suited to studying these effects, because the precession of planets is very small.

	v_1 [m/s]	$\Delta\phi$ [theory]	$\Delta\phi$ [exp.]
Earth	29 766	6.1937e-8	1.8627e-7
Earth (adjusted)	52 090	1.8968e-7	1.8627e-7
Hulse-Taylor	450 000	1.4156e-5	6.5209e-5
S2	7.75296e6	0.00420	-0.017 ... +0.035

Table 9.3: Comparison of results from precession formula (9.287) with experimental values.

■

General frame rotation with Lagrange theory

In Example 9.6, the relativistic equations of gravitational motion were derived using Lagrange theory. The Lagrangian in plane polar coordinates is

$$\mathscr{L} = -\frac{mc^2}{\gamma} - U \tag{9.291}$$

with gravitational potential energy

$$U = m\Phi = -\frac{mMG}{r} \tag{9.292}$$

and γ factor

$$\gamma = \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{-1/2}. \tag{9.293}$$

The constant of motion is the relativistic angular momentum

$$L = \gamma m r^2 \dot{\phi}, \tag{9.294}$$

and the Hamiltonian is

$$\mathscr{H} = \gamma m c^2 + U. \tag{9.295}$$

Application of frame rotation consists of the replacement

$$\phi \to \phi' = \phi + \omega_1 t \tag{9.296}$$

with $\omega_1(t)$ being an arbitrary rotation frequency. For relativistic motion, the γ factor (9.293) is replaced with

$$\begin{aligned}\gamma &= \left(1 - \frac{\dot{r}^2 + r^2\left(\frac{d}{dt}(\phi + \omega_1 t)\right)^2}{c^2}\right)^{-1/2} \\ &= \left(1 - \frac{\dot{r}^2 + r^2\left(\dot{\phi} + \omega_1 + \dot{\omega}_1 t\right)^2}{c^2}\right)^{-1/2}.\end{aligned} \tag{9.297}$$

Inserting this into the Lagrangian (9.291), and then evaluating the Euler-Lagrange equations for ϕ and r, gives us the equation set:

$$\begin{aligned}\ddot{\phi} &= \underbrace{-\ddot{\omega}_1 t - \frac{2\dot{\omega}_1 \dot{r} t}{r} + \frac{\dot{\omega}_1 GM \dot{r} t}{\gamma c^2 r^2} + \frac{\omega_1 GM \dot{r}}{\gamma c^2 r^2} - 2\dot{\omega}_1 - \frac{2\omega_1 \dot{r}}{r}}_{\Omega_\phi \Phi} \\ &\quad + \frac{GM\dot{\phi}\dot{r}}{\gamma c^2 r^2} - \frac{2\dot{\phi}\dot{r}}{r},\end{aligned} \tag{9.298}$$

$$\begin{aligned}\ddot{r} &= \underbrace{\dot{\omega}_1^2 r t^2 + 2\dot{\omega}_1 \dot{\phi} r t + 2\omega_1 \dot{\omega}_1 r t + {\omega_1}^2 r + 2\omega_1 \dot{\phi} r}_{\Omega_r \Phi} \\ &\quad + \frac{GM\dot{r}^2}{\gamma c^2 r^2} + \dot{\phi}^2 r - \frac{GM}{\gamma r^2}.\end{aligned} \tag{9.299}$$

These equations have to be solved simultaneously for a given function ω_1. For $\omega_1 \to 0$, this equation set becomes identical to (9.121, 9.122), as required for consistency:

$$\ddot{\phi} = \frac{GM\dot{\phi}\dot{r}}{\gamma c^2 r^2} - \frac{2\dot{\phi}\dot{r}}{r}, \tag{9.300}$$

$$\ddot{r} = \frac{GM\dot{r}^2}{\gamma c^2 r^2} + \dot{\phi}^2 r - \frac{GM}{\gamma r^2}. \tag{9.301}$$

The additional terms in Eqs. (9.298, 9.299) contain the rotation frequency ω_1, and can be interpreted as spin connection terms of a vector spin connection:

$$\mathbf{\Omega} = \begin{bmatrix}\Omega_r \\ \Omega_\phi\end{bmatrix}. \tag{9.302}$$

With this spin connection, the gravitational acceleration becomes

$$\ddot{\mathbf{r}} = -\boldsymbol{\nabla}\Phi + \boldsymbol{\Omega}\Phi. \tag{9.303}$$

The components $\Omega_\phi\Phi$ and $\Omega_r\Phi$ have been defined in Eqs. (9.298, 9.299). A rotating frame is a manifestation of a Cartan spin connection of ECE theory. Such effects have been discussed in Section 8.3.3, for example.

■ **Example 9.14** We will now study a numerical example of a general frame rotation. The equation set (9.298, 9.299) has been solved numerically for a demonstrational system with parameters near unity (see computer algebra code [174]). The relativistic angular momentum with general rotation is defined by

$$L = \gamma m r^2 \left(\dot\phi + \omega_1 + \dot\omega_1 t\right), \tag{9.304}$$

and comes out of the Euler-Lagrange equations as a constant of motion. Its non-relativistic, Newtonian counterpart is

$$L_N = m r^2 \left(\dot\phi + \omega_1 + \dot\omega_1 t\right). \tag{9.305}$$

The total energy (without rest energy) is

$$E = mc^2(\gamma - 1) - \frac{mMG}{r}, \tag{9.306}$$

and the corresponding Newtonian expression is

$$E_N = \frac{1}{2} m \left(\dot r^2 + r^2(\dot\phi + \omega_1 + \dot\omega_1 t)^2\right) - \frac{mMG}{r}. \tag{9.307}$$

To solve the frame-rotated equations, we used the rotation function

$$\omega_1 = c_1 (c_2 t^2 + c_3 \exp(-c_4 t)) \tag{9.308}$$

with positive parameters $c_1, \ldots c_4$. This is a quite exotic example, as the results show. We chose a negative frame rotation, so that the orbiting mass changes its direction of rotation (see Fig. 9.20). Even a small loop is visible. The angular velocity $\omega = \dot\phi$ is graphed in Fig. 9.21. The directional change takes place where ω reverses sign. Nonetheless, the radial velocity $\dot r$ keeps its shape over time, and the orbit does not increase or shrink, even though the additional rotation is massive. This is a consequence of the structure of spin connections in the equations of motion, which describe a pure rotation.

Despite the exotic orbit, the motion is physical in the sense that angular momentum and total energy are maintained. This can be seen from Figs. 9.22 and 9.23, where both quantities are presented in relativistic and non-relativistic form, as given in Eqs. (9.304 - 9.307). L and E are constant, while their Newtonian counterparts are not, as expected. The deviation is largest where the orbital velocity is at maximum. ■

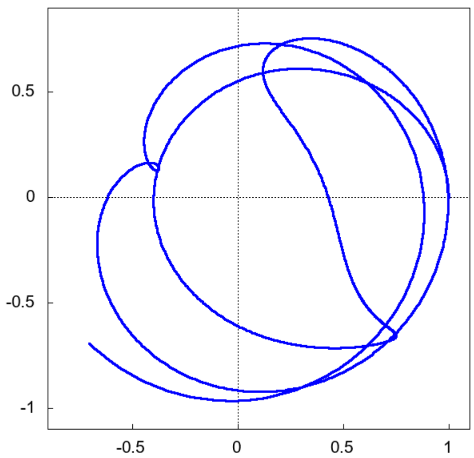


Figure 9.20: Orbit in a rotating frame.

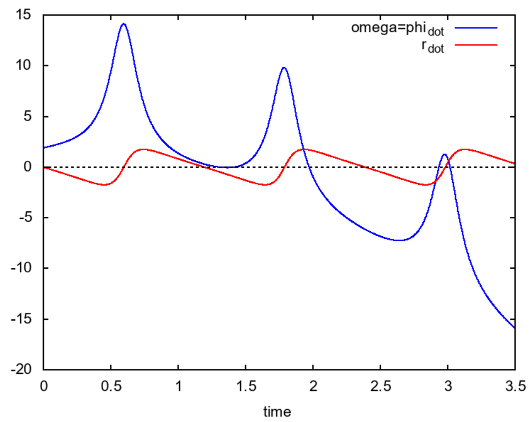


Figure 9.21: Velocity components ω and $\dot{r}$ in a rotating frame.

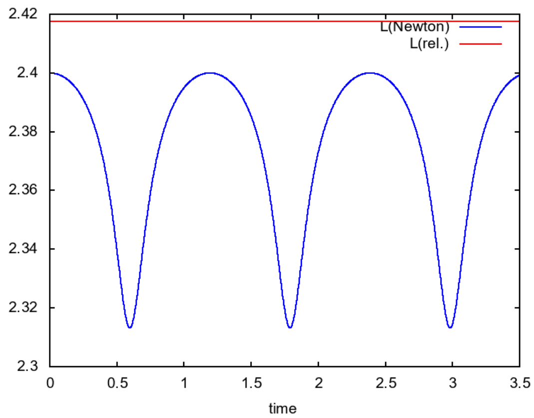


Figure 9.22: Relativistic and Newtonian angular momentum.

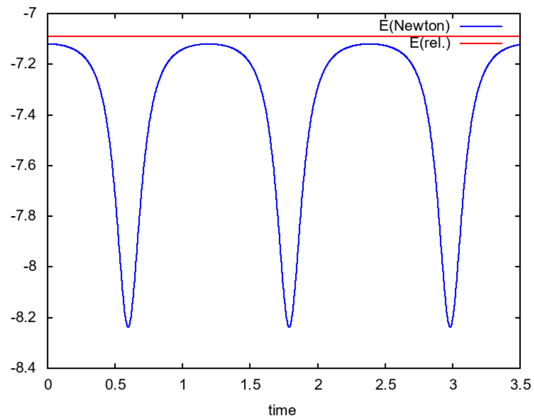
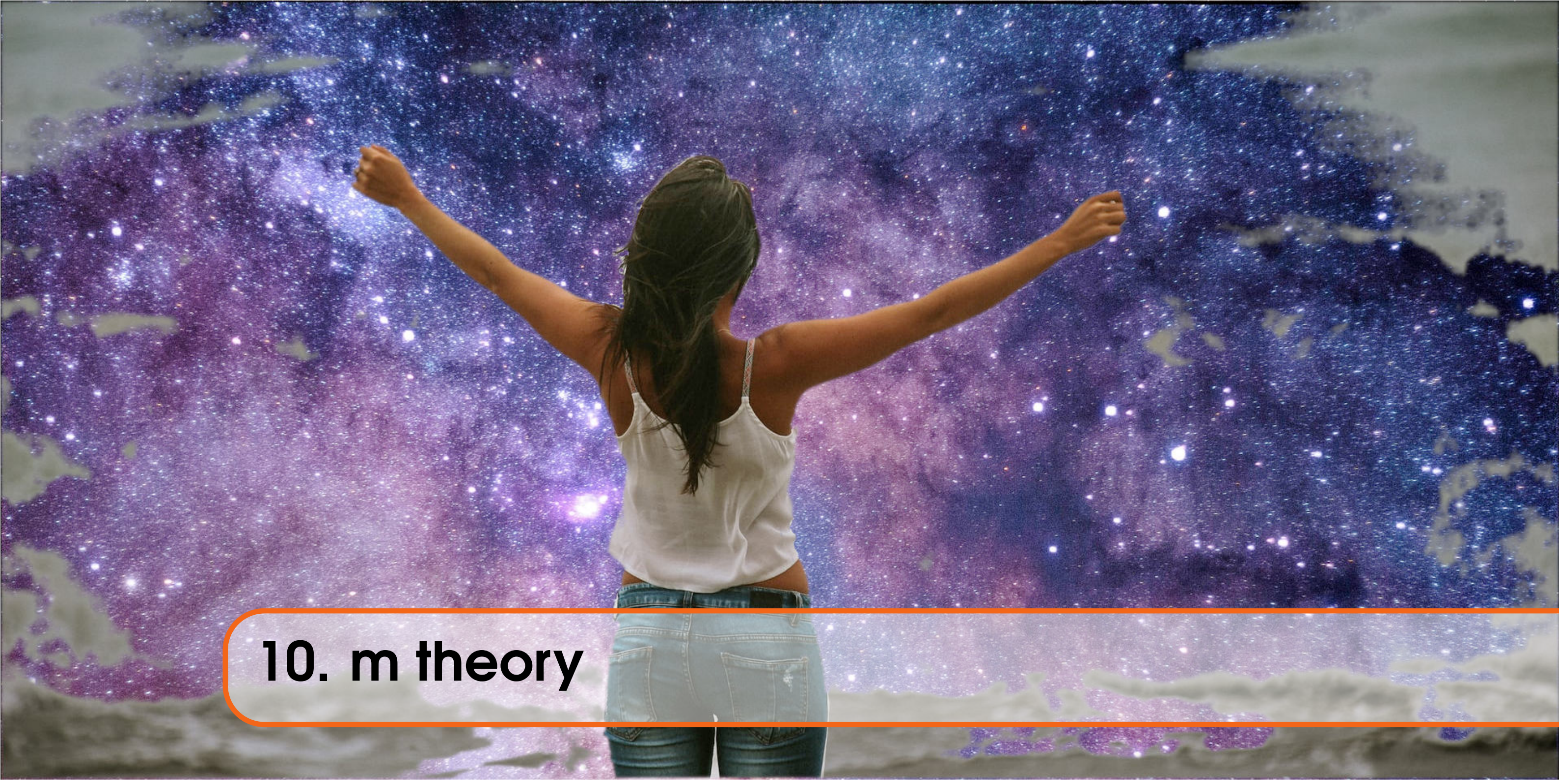


Figure 9.23: Relativistic and Newtonian total energy.



10. m theory

In Chapter 9, we have worked mainly with equations of relativistic mechanics, which are formally derived from special relativity. In the relativistic mechanics of the Standard Model, special relativity is extended in an ad-hoc manner by allowing the velocity of objects to be variable, which leads to the relativistic γ factor not being constant. The basis of this extension continues to be the Minkowski space of special relativity, and no curvature or torsion is introduced. To our knowledge, the relativistic mechanics of the Standard Model has seldom been applied to gravitational problems.

For precession, we started by using the same extended model with variable velocity in Minkowski space. We then showed that rotational motion introduces spin connections, which means that it actually takes place in a space with curvature and torsion. This extension of Minkowski space becomes visible when the additional terms in the equations of motion (due to precession) are interpreted as spin connections.

Our model goes far beyond the initial (extended) model because we use the field equations of ECE theory, which are derived from a spacetime with curvature and torsion, and are thus covariant in the sense of general relativity. Therefore, our handling of relativistic effects can be considered to be a method of general relativity, even if we use a formal extension of special relativity.

In contrast, the Standard Model considers precession to be a property of “true general relativity”, which means that it has to be investigated by using Einstein’s obsolete field equation. This significantly increases complexity. The Standard Model never did try to describe precession through an approach based on Minkowski space. However, as we have shown in previous sections, it is possible to describe precession in a much simpler way, by using a variable γ factor in Minkowski space.

Throughout this chapter, we will use the metric of a non-constant, centrally symmetric spacetime that is different from Minkowski space to extend the methods that we have used so far. With respect to spacetime structure, our method will be on the same level as the “true general relativity” of Einstein’s field equation, but without having to involve that erroneous equation.

10.1 Equations of motion

In this section, we will extend the Lagrangian method to be equivalent to “true general relativity”, and derive equations of motion that satisfy the same criteria. We will base our development on a metric that is common in Einstein’s theory, but we will develop our method within ECE2 theory, i.e., Cartan geometry.

10.1.1 Line element and relativistic energy

According to Section 2.1.3, Eq. (2.53), the squared line element in a space with curvature and torsion is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{10.1}$$

In a Minkowski space for a spherically symmetric spacetime, this takes the form

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin(\theta)^2 d\phi^2 \tag{10.2}$$

with a time coordinate $x_0 = ct$. Furthermore, r is the radius coordinate, θ is the polar angle and ϕ is the azimuthal angle (see Fig. 2.3). In a general spherically symmetric spacetime with torsion and curvature, the line element has to be generalized as described in Chapter 7 of [8]:

$$ds^2 = c^2 \mathrm{m}(r,t) dt^2 - \mathrm{n}(r,t) dr^2 - r^2 d\theta^2 - r^2 \sin(\theta)^2 d\phi^2. \tag{10.3}$$

$\mathrm{m}(r,t)$ and $\mathrm{n}(r,t)$ are general functions describing the distortion of spacetime by a central point mass at $r = 0$. Only the radial and time coordinates are affected. The angular parts remain unchanged because of the rotational symmetry. It was shown in [8] that the line element can be simplified further by the replacement

$$\mathrm{n}(r,t) = \frac{1}{\mathrm{m}(r,t)}, \tag{10.4}$$

and that the time dependence of $\mathrm{m}(r,t)$ can be rolled over to the time coordinate. Therefore, the simplified line element reads:

$$ds^2 = c^2 \mathrm{m}(r) dt^2 - \frac{dr^2}{\mathrm{m}(r)} - r^2 d\theta^2 - r^2 \sin(\theta)^2 d\phi^2. \tag{10.5}$$

This form is used in Einstein’s field equation, in which the Ricci tensor is zero, to derive an expression for $\mathrm{m}(r)$ in the vacuum, leading to the Schwarzschild metric. This simplified line element was already used for the metric of the general rotation in Section 9.3.3. In ECE2 theory, we use this form for simplicity, but we can freely define the function $\mathrm{m}(r)$. By comparing Eqs. (10.5) and (10.1), we can see that the metric is diagonal and that the metric coefficients are

$$g_{00} = \mathrm{m}(r), \quad g_{11} = -\frac{1}{\mathrm{m}(r)}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin(\theta)^2. \tag{10.6}$$

This metric-based theory, which we call *m theory*, can also be developed from Cartan geometry itself. In Cartan geometry, the basis element is the tetrad, and the metric follows from the tetrad (see Eq. (2.204)) by

$$g_{\mu\nu} = n\, q^a{}_\mu q^b{}_\nu \eta_{ab}, \tag{10.7}$$

where $q^a{}_\mu$ are the tetrad elements, η_{ab} is the Minkowski metric of tangent space, and $n = 4$ is the dimension of the base manifold. The metric does not generally allow the tetrad to be determined uniquely. In this case, however, the metric is diagonal. We can assume that the tetrad matrix is diagonal also, because we are not considering specific polarization effects of Cartan geometry. Therefore, Eq. (10.7) reduces to the diagonal elements in both the base manifold and tangent space, and we obtain

$$q^{(0)}{}_0 = \frac{1}{2}\sqrt{\mathrm{m}(r)}, \quad q^{(1)}{}_1 = \frac{1}{2\sqrt{\mathrm{m}(r)}}, \quad q^{(2)}{}_2 = \frac{r}{2}, \quad q^{(3)}{}_3 = \frac{r\sin(\theta)}{2}. \tag{10.8}$$

This is how m theory connects to Cartan geometry.

In order to obtain simpler equations of motion, we restrict consideration to a two-dimensional plane polar coordinate system. Then, the line element (10.5) reads:

$$ds^2 = c^2\mathrm{m}(r)dt^2 - \frac{dr^2}{\mathrm{m}(r)} - r^2d\phi^2 = c^2\mathrm{m}(r)dt^2 - d\mathbf{r}^2. \tag{10.9}$$

According to Section 2.1.2,

$$d\mathbf{r} = \frac{\partial\mathbf{r}}{\partial r}dr + \frac{\partial\mathbf{r}}{\partial\phi}d\phi, \tag{10.10}$$

where $\mathbf{r}$ is the position vector in m space. According to (10.9), the scalar product $d\mathbf{r}^2 = d\mathbf{r}\cdot d\mathbf{r}$ must be

$$d\mathbf{r}^2 = \left(\frac{\partial\mathbf{r}}{\partial r}dr + \frac{\partial\mathbf{r}}{\partial\phi}d\phi\right)^2 = \frac{dr^2}{\mathrm{m}(r)} + r^2d\phi^2. \tag{10.11}$$

A possible solution of this equation is

$$\left(\frac{\partial\mathbf{r}}{\partial r}\right)^2 dr^2 = \frac{1}{\mathrm{m}(r)}dr^2, \tag{10.12}$$

$$\left(\frac{\partial\mathbf{r}}{\partial\phi}\right)^2 d\phi^2 = r^2d\phi^2, \tag{10.13}$$

$$\frac{\partial\mathbf{r}}{\partial r}\cdot\frac{\partial\mathbf{r}}{\partial\phi} = 0. \tag{10.14}$$

This gives us

$$\frac{\partial\mathbf{r}}{\partial r} = \frac{1}{\mathrm{m}(r)^{1/2}}\mathbf{e}_r, \tag{10.15}$$

$$\frac{\partial\mathbf{r}}{\partial\phi} = r\mathbf{e}_\phi, \tag{10.16}$$

and

$$\frac{\partial\mathbf{r}}{\partial r}\cdot\frac{\partial\mathbf{r}}{\partial\phi} = 0, \tag{10.17}$$

which confirms Eq. (10.14). Therefore, the position vector in m space is

$$\mathbf{r} = \frac{r}{\mathrm{m}(r)^{1/2}}\mathbf{e}_r. \tag{10.18}$$

The velocity in m space is

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}\left(\frac{r}{\mathrm{m}(r)^{1/2}}\right)\mathbf{e}_r + \frac{r\dot{\phi}}{\mathrm{m}(r)^{1/2}}\mathbf{e}_\phi. \tag{10.19}$$

Since r describes a trajectory coordinate, it depends implicitly on time, which gives us

$$\frac{d}{dt}\left(\frac{r}{\mathrm{m}(r)^{1/2}}\right) = \frac{\dot{r}}{\mathrm{m}(r)^{1/2}} - \frac{\frac{\partial \mathrm{m}(r)}{\partial r} r\dot{r}}{2\mathrm{m}(r)^{3/2}} = \frac{\dot{r}}{\mathrm{m}(r)^{1/2}}\left(1 - \frac{\frac{\partial \mathrm{m}(r)}{\partial r} r}{2\mathrm{m}(r)}\right). \tag{10.20}$$

We can neglect the second term with $\frac{\partial \mathrm{m}(r)}{\partial r}$, assuming a slow change of $\mathrm{m}(r)$ with r. Thus, we avoid having the expression for the velocity become extraordinarily complex. We set

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{1}{\mathrm{m}(r)^{1/2}}\left(\dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi\right). \tag{10.21}$$

Sometimes we use the variable name

$$\mathbf{r}_1 = r_1\mathbf{e}_r = \frac{r}{\mathrm{m}(r)^{1/2}}\mathbf{e}_r, \tag{10.22}$$

so that the velocity becomes

$$\mathbf{v}_1 = \dot{\mathbf{r}}_1 = \dot{r}_1\mathbf{e}_r + r_1\dot{\phi}\mathbf{e}_\phi. \tag{10.23}$$

A new time variable can be defined by

$$t_1 = \mathrm{m}(r)^{1/2} t, \tag{10.24}$$

where $\mathbf{r}_1$ and t_1 are the characteristic variables of m space.

From the line element (10.9) of m space, it follows that

$$ds^2 = c^2\mathrm{m}(r)dt^2 - \left(\frac{d\mathbf{r}_1}{dt}\right)^2 dt^2 = c^2 dt_1^2 - \mathbf{v}_1^2 dt^2. \tag{10.25}$$

In plane polar coordinates related to the observer space, this becomes

$$ds^2 = c^2\left(\mathrm{m}(r) - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{\mathrm{m}(r)c^2}\right)dt^2 = \frac{c^2 dt^2}{\gamma^2}. \tag{10.26}$$

Thus, the generally relativistic γ factor of m space is defined by

$$\gamma = \left(\mathrm{m}(r) - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{\mathrm{m}(r)c^2}\right)^{-1/2}. \tag{10.27}$$

The linear momentum of m space is

$$\mathbf{p}_1 = \gamma m\mathbf{v}_1 = \gamma m\frac{\mathbf{v}}{\mathrm{m}(r)^{1/2}}. \tag{10.28}$$

The angular momentum in polar coordinates then becomes

$$\begin{aligned}\mathbf{L}_1 &= \mathbf{r}_1 \times \mathbf{p}_1 = \gamma m\mathbf{r}_1 \times \mathbf{v}_1 \\ &= \gamma m r_1\mathbf{e}_r \times (\dot{r}_1\mathbf{e}_r + r_1\dot{\phi}\,\mathbf{e}_\phi) = \gamma m r_1^2\dot{\phi}\,\mathbf{k} \\ &= \gamma m\frac{r^2}{\mathrm{m}(r)}\dot{\phi}\,\mathbf{k},\end{aligned} \tag{10.29}$$

where **k** is the unit vector perpendicular to the $(r,\ \phi)$ plane.

The relativistic total energy in m space can be derived in the following way. The line element (10.25) is

$$ds^2 = c^2 d\tau^2 = c^2 dt_1^2 - \mathbf{v}_1^2 dt^2 = \mathrm{m}(r) c^2 dt^2 - \frac{\mathbf{v}^2}{\mathrm{m}(r)} dt^2, \tag{10.30}$$

which leads to

$$c^2 \left(\frac{d\tau}{dt}\right)^2 = \mathrm{m}(r) c^2 - \frac{\mathbf{v}^2}{m(r)}. \tag{10.31}$$

Multiplying by $\gamma^2 \mathrm{m}(r) m^2 c^2$ gives

$$\mathrm{m}(r) m^2 c^4 = \gamma^2 \mathrm{m}(r)^2 m^2 c^4 - \gamma^2 v^2 m^2 c^2. \tag{10.32}$$

By inserting the relativistic momentum of m space,

$$p^2 = \gamma^2 m^2 v^2, \tag{10.33}$$

we obtain

$$\mathrm{m}(r) m^2 c^4 = \gamma^2 \mathrm{m}(r)^2 m^2 c^4 - c^2 p^2, \tag{10.34}$$

which is Einstein's energy equation:

$$E_0^2 = E^2 - c^2 p^2 \tag{10.35}$$

with rest energy

$$E_0 = \sqrt{\mathrm{m}(r)}\, m c^2 \tag{10.36}$$

and total energy

$$E = \gamma \mathrm{m}(r) m c^2. \tag{10.37}$$

Therefore, Einstein's energy equation can be written in standard form as

$$E^2 = E_0^2 + c^2 p^2 \tag{10.38}$$

or

$$\gamma^2 \mathrm{m}(r)^2 m^2 c^4 = \mathrm{m}(r) m^2 c^4 + c^2 p^2. \tag{10.39}$$

This is an equation of general relativity. The form shown in (10.39) holds for a free particle, because there is no potential energy. In special relativity, we have $\mathrm{m}(r) \to 1$, and consequently

$$\gamma^2 m^2 c^4 = m^2 c^4 + c^2 p^2. \tag{10.40}$$

The term $\mathrm{m}(r) c^2 p^2$ is not the kinetic energy, but a "momentum energy". The kinetic energy is not a constant of motion, in contrast to the conservation of total energy and momentum. If the particle is at rest, the total energy is equal to the rest energy. The relativistic kinetic energy in m space can be computed analogously to Einstein's theory [109]. When a particle moves from point 1 to 2, the work done is

$$W_{12} = \int_1^2 \mathbf{F}\, d\mathbf{r} = T_2 - T_1. \tag{10.41}$$

This is the force integral along a path from 1 to 2, and it gives the difference of kinetic energies $T_2 - T_1$. The force is the time derivative of the momentum in m space, transformed into observer space:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(\gamma m \mathbf{v}). \tag{10.42}$$

Starting from rest, we obtain

$$W = T = \int \frac{d}{dt}(\gamma m \mathbf{v}) \cdot \mathbf{v} dt = m \int_0^v v d(\gamma v). \tag{10.43}$$

This equation can be integrated by parts, which gives us

$$\begin{aligned} T &= \gamma m v^2 - m \int_0^v \frac{v\, dv}{\sqrt{\mathrm{m}(r) - \frac{v^2}{\mathrm{m}(r)c^2}}} \\ &= \gamma m v^2 + mc^2 \sqrt{\mathrm{m}(r) - \frac{v^2}{\mathrm{m}(r)c^2}}\Bigg|_0^v \\ &= \gamma m v^2 + mc^2 \sqrt{\mathrm{m}(r) - \frac{v^2}{\mathrm{m}(r)c^2}} - mc^2. \end{aligned} \tag{10.44}$$

After some algebraic manipulation, this equation becomes

$$T = mc^2 \mathrm{m}(r) \left(\mathrm{m}(r)\gamma - \sqrt{\mathrm{m}(r)} \right) \tag{10.45}$$

(see computer algebra code [175]). For $\mathrm{m}(r) \to 1$ we obtain

$$T \to mc^2(\gamma - 1), \tag{10.46}$$

which is the well-known result from special relativity. The result of classical mechanics is obtained by using an appropriate approximation of the γ factor:

$$T \to mc^2(1 + \frac{1}{2}\frac{v^2}{c^2} - 1) = \frac{1}{2} m v^2. \tag{10.47}$$

The Hamiltonian is the total energy, including the potential energy. The latter is not contained in Einstein's energy equation, as already mentioned. The potential energy of a central gravitational field in m theory is

$$U = -\frac{mMG}{r_1} = -\sqrt{\mathrm{m}(r)}\frac{mMG}{r}. \tag{10.48}$$

Therefore, the Hamiltonian of m theory is

$$\mathscr{H} = E + U = \gamma \mathrm{m}(r) mc^2 - \sqrt{\mathrm{m}(r)}\frac{mMG}{r}. \tag{10.49}$$

This quantity is conserved, and therefore

$$\frac{d\mathscr{H}}{d\tau} = \frac{d\mathscr{H}}{dt} = 0. \tag{10.50}$$

Along with the total energy, the angular momentum (10.29),

$$L = \gamma m \frac{r^2}{\mathrm{m}(r)} \dot{\phi}, \tag{10.51}$$

is a constant of motion:

$$\frac{dL}{d\tau} = \frac{dL}{dt} = 0. \tag{10.52}$$

We call these conservation equations the *Evans-Eckardt equations*:

$$\boxed{\begin{aligned} \frac{d\mathscr{H}}{dt} &= 0, \qquad (10.53)\\ \frac{dL}{dt} &= 0. \qquad (10.54) \end{aligned}}$$

They can be used to determine the equations of motion for the coordinates r and ϕ of m space. We will first develop the Lagrange theory of m space as an alternative, and then compute the equations of motion from the Evans-Eckardt equations, as a cross-check.

10.1.2 Lagrange equations

The relativistic Lagrangian used so far has been

$$\mathscr{L} = -\frac{mc^2}{\gamma} - U. \tag{10.55}$$

The form of the Lagrangian is not defined a priori. It must be chosen in a way that allows the correct equations of motion to be produced. The Hamiltonian is connected with the Lagrangian via the following relation [110]:

$$\mathscr{H} = \sum_i \dot{q}_i p_i - \mathscr{L}, \tag{10.56}$$

where q_i are the generalized coordinates and p_i are the generalized momenta. In our case, we have

$$\dot{q}_i = v_1, \quad p_i = p_1 = \gamma m v_1. \tag{10.57}$$

By inserting them and Eq. (10.49) into Eq. (10.56), and using

$$v^2 = c^2 \mathrm{m}(r) \left(\mathrm{m}(r) - \frac{1}{\gamma^2} \right), \tag{10.58}$$

we obtain the Lagrangian of m theory:

$$\mathscr{L} = -\frac{mc^2}{\gamma} + \frac{mMG\sqrt{\mathrm{m}(r)}}{r}, \tag{10.59}$$

whose kinetic part is formally identical to that of special relativity, except that the γ factor is that of generally relativistic m theory. We can now develop the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\phi}} - \frac{\partial \mathscr{L}}{\partial \phi} = 0, \tag{10.60}$$

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{r}} - \frac{\partial \mathscr{L}}{\partial r} = 0, \tag{10.61}$$

which are defined for coordinates in observer space. The γ factor complicates the calculation significantly. In particular, for the time derivative of $\mathrm{m}(r)$ we have to use

$$\frac{d\mathrm{m}(r)}{dt} = \frac{d\mathrm{m}(r)}{dr}\dot{r}. \tag{10.62}$$

Computer algebra then gives the following results:

$$\ddot{\phi} = \dot{\phi}\,\dot{r}\left(\frac{1}{\mathrm{m}(r)}\frac{d\,\mathrm{m}(r)}{dr}\left(2-\frac{GM}{2\gamma c^2 r\sqrt{\mathrm{m}(r)}}\right)\right.$$
$$\left.+\frac{GM}{\gamma c^2 r^2\sqrt{\mathrm{m}(r)}}-\frac{2}{r}\right), \tag{10.63}$$

$$\ddot{r} = \frac{d\,\mathrm{m}(r)}{dr}\left(-\frac{2\dot{\phi}^2 r^2}{\mathrm{m}(r)}+c^2\left(\mathrm{m}(r)-\frac{3}{2\gamma^2}\right)+\frac{GM}{2\gamma^3 r\sqrt{\mathrm{m}(r)}}\right.$$
$$\left.+\frac{GM\dot{\phi}^2 r}{2\gamma c^2\mathrm{m}(r)^{3/2}}\right)$$
$$-\frac{GM\dot{\phi}^2}{\gamma c^2\sqrt{\mathrm{m}(r)}}-\frac{GM\sqrt{\mathrm{m}(r)}}{\gamma^3 r^2}+\dot{\phi}^2 r. \tag{10.64}$$

The angular momentum comes out as a constant of motion, which proves Eq. (10.51). As an additional check, we have computed the equations of motion from the Evans-Eckardt equations (10.53, 10.54) (see computer algebra code [176]). This cross-check gives the same equations, and confirms the correctness of the calculations[1].

The equations of motion contain terms with $d\mathrm{m}(r)/dr$. These are critical for the computation. In particular, one term weights this derivative by a factor of c^2 in the numerator. This gives a behavior like a limit of $0\cdot\infty$, which can converge to any value or diverge to infinity. For a constant $\mathrm{m}(r)$, this critical term disappears; however, other terms containing $\mathrm{m}(r)$ remain. In the limit $\mathrm{m}(r)\to 1$, only the terms of Eqs. (9.121, 9.122) remain, which are the standard equations of relativistic motion in polar coordinates, with exception of a factor of $1/\gamma^3$ in the gravitational force. This is the factor appearing in the relativistic Newton equation (9.89).

The relativistic Newton equation of m space can be derived as shown in Section 9.2.3, which gives us

$$\mathbf{F} = \frac{d\mathbf{p}_1}{dt} = \frac{d}{dt}\left(\gamma m\frac{\mathbf{v}}{\sqrt{\mathrm{m}(r)}}\right) \overset{!}{=} -\frac{mMG}{r_1^3}\mathbf{r}_1 = -\mathrm{m}(r)\frac{mMG}{r^2}\mathbf{e}_r. \tag{10.65}$$

We do not include the time dependence of the trajectory $r(t)$ in $\mathrm{m}(r)$. Therefore, we only have to evaluate the time derivative of $\gamma\mathbf{v}$ (see computer algebra code [175]). The result is

$$\mathbf{F} = m\sqrt{\mathrm{m}(r)}\gamma^3\frac{d\mathbf{v}}{dt} = -\mathrm{m}(r)\frac{mMG}{r^2}\mathbf{e}_r, \tag{10.66}$$

and the relativistic Newtonian acceleration in m space is

$$\ddot{\mathbf{r}} = -\sqrt{\mathrm{m}(r)}\frac{MG}{\gamma^3 r^3}\mathbf{r}. \tag{10.67}$$

The only difference from Eq. (9.89) is the factor $\sqrt{\mathrm{m}(r)}$.

■ **Example 10.1** We define models for m functions and show results of some representative calculations. m theory depends on the form of the metric function $\mathrm{m}(r)$, which is not predefined. In early research papers [111], the metric of m theory was used with two model functions for $\mathrm{m}(r)$.

[1] During the development of ECE theory, the results were initially different. This was a problem of the computer algebra system, which is challenged by these types of calculations.

In cosmological cases (spiral galaxies), an approach based on geometrical foundations of general relativity was used:

$$\mathrm{m}(r) = a - \exp\left(b\exp(-\frac{r}{R})\right) \tag{10.68}$$

with constants a, b and R. In the limit $r \to \infty$, we have to obtain m(r)=1, which leads to the constraint

$$1 = a - 1 \tag{10.69}$$

or

$$a = 2. \tag{10.70}$$

The parameter b determines the behavior close to $r = 0$. Our requirement that m(r) must be positive leads to the limit:

$$\mathrm{m}(0) = 0. \tag{10.71}$$

From (10.68), we then obtain

$$a - \exp(b) = 0 \tag{10.72}$$

or

$$b = \log(a) = \log(2). \tag{10.73}$$

The exponent-based m function is graphed in Fig. 10.1 for three values of b, with $a = 2$ and $R = 1$. Using $b < \log(2)$ leads to curves of m(r) that are quite flat.

In [111], we also introduced a metric function derived from the obsolete Schwarzschild metric (extended by an empirical term $-\alpha/r^2$, which was necessary to obtain the shrinking of orbits):

$$\mathrm{m}(r) = 1 - \frac{r_S}{r} - \frac{\alpha}{r^2} \tag{10.74}$$

with the Schwarzschild radius

$$r_S = \frac{2MG}{c^2}. \tag{10.75}$$

This function, as well as the original Schwarzschild metric with $\alpha = 0$, goes to $-\infty$ as $r \to 0$. It is used in Einsteinian theory only for $r > r_S$. In m theory, there are no such restrictions, because the problem of negative values of m(r) can be avoided a priori by a suitable choice of m(r).

The equations of motion (10.63, 10.64) have been solved numerically. The exponential m function (10.68) was used with $a = 2$, $b = \log(2)$, $R = 0.1$. The parameters in the equations were chosen in the range of unity so that a nearly ultra-relativistic case would be obtained. This was necessary because these relativistic effects would otherwise be too tiny to be recognizable in the graphics, even though they now sometimes appear drastic. When the initial velocity in the calculation is high enough, we obtain a highly elliptic orbit with forward precession (see Fig. 10.2). With smaller initial velocities, even backward precession has been produced in some cases, showing that this behavior depends strongly on the energetic state of the system.

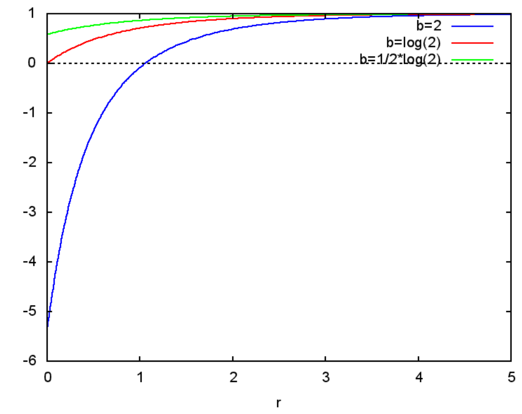


Figure 10.1: Exponential function m(r) for three values of b.

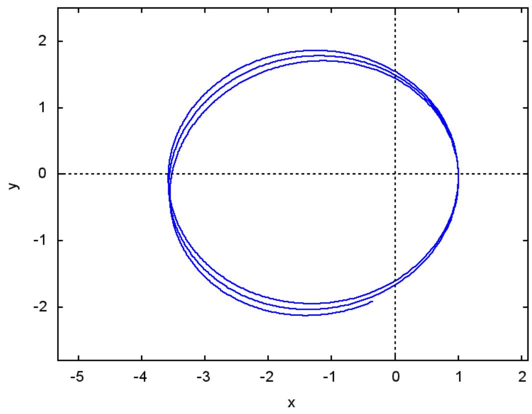


Figure 10.2: Orbit of relativistic motion with an exponential m function.

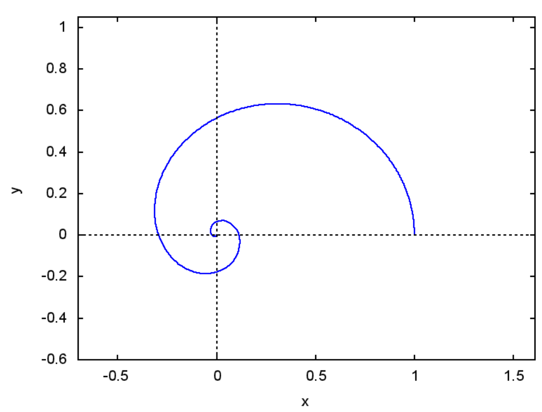


Figure 10.3: Collapsing orbit of an exponential m function.

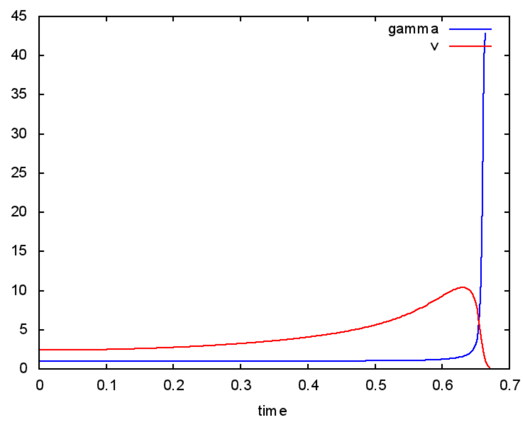


Figure 10.4: γ factor and velocity v of the collapsing orbit.

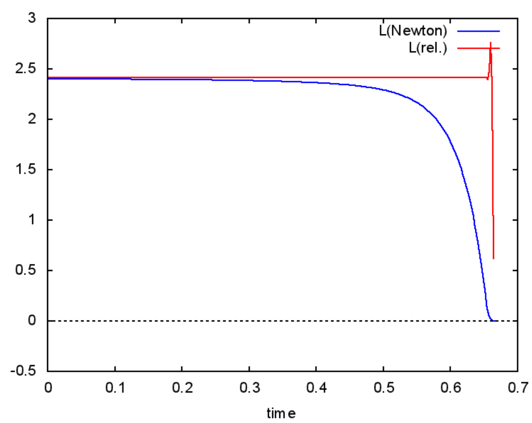


Figure 10.5: Angular momenta of the collapsing orbit.

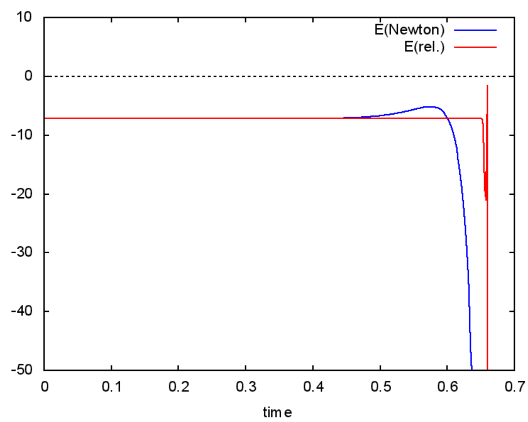


Figure 10.6: Total energies of the collapsing orbit.

In Fig. 10.3, the same calculation has been performed, but with the initial velocity reduced by about a factor of 0.6. Now, the m function comes into full effect, causing an inward spiraling of the orbiting mass until it falls into the center. This behavior is only possible in m space. The "normal" relativistic calculations always showed a stable orbit, albeit with strong precession. When the mass falls into the center, the γ factor increases dramatically, as shown in Fig. 10.4. In special relativity, one would expect the orbital velocity to go to the limit c, but in m theory this behavior is quite different. We get the surprising result that, after initially rising, the velocity drops to zero, i.e., the mass approaches the center softly.

From Fig. 10.5 we see that the relativistic angular momentum remains constant up to the last moment when the orbiting mass comes to rest and the calculation diverges. The Newtonian angular momentum does not contain the γ factor, and therefore $\dot{\phi} = 0$ at the resting point, which takes the Newtonian angular momentum to zero. A comparable behavior is seen for the total energy (see Fig. 10.6). The Newtonian values become meaningless close to the end point, but we see that such a singular orbit is correctly described by m theory. ■

■ **Example 10.2** The orbit of the S2 star is analyzed with m theory. This orbit was already investigated in Example 9.3, where relativistic effects were discussed. In Example 9.4, the relativistic force laws were compared using S2 data from older measurements. In Table 10.1, we give an update with newer experimental data from 2018 [112], when the S2 star was at its closest approach to the center of the galaxy. The detailed investigation using m theory is described in [113]. As can be seen from Table 10.1, using the experimental data directly for the initial conditions of the calculation gives larger disagreement with experiment, with respect to the period time and maximum orbital radius.

The results depend on the mass of the galactic center. In [113], it was shown that the mass value, $8.572e36$ kg, was derived by astronomers from Newtonian theory. Since results depend on the accuracy of this value, we have varied it using $m(r) = 1$ in the equations of motion of m theory. Best agreement with observation is obtained for a value of $8.3627e36$ kg (see line 3 of Table 10.1). As the calculations have shown, the γ factor is 1.0003 at maximum, i.e., it plays only a small role.

As an alternative, we have performed calculations with the exponential m function (10.68). The results depend very sensitively on this function, and it was difficult to define the m function in a way that the terms with $d\mathrm{m}(r)/dr$ had an effect that was small enough to avoid excessive precession. Therefore, we used a constant value of

$$\mathrm{m}(r) = 0.9877. \tag{10.76}$$

Using this constant value, and the experimental central mass value of $8.572e36$ kg, gives the results shown in the fourth line of Table 10.1. Obviously, a constant $\mathrm{m}(r)$ has practically the same effect as changing the central mass. The orbit of the S2 star was graphed earlier in Fig. 9.6, which shows the high ellipticity of the orbit. We conclude that this ellipse corresponds with a generally relativistic theory, and is therefore non-Newtonian. ■

	T [y]	ε	$r_{\max}$ [10^{14} m]	$\Delta\phi$ [rad]
Experiment	16.05	0.88466	2.73464	?
Exp. initial conditions	9.65	0.83724	2.02688	6.0636e-4
Best fit for M	16.07	0.88323	2.89596	5.7702e-4
Exp. M with m=0.9877	16.07	0.88332	2.89596	5.9144e-4

Table 10.1: Parameters of the S2 star orbit (experimental data from 2018, and calculations).

10.2 Consequences of m theory

In the following subsections, we will consider selected conclusions that can be deduced from m theory.

10.2.1 Vacuum force

In the preceding section, we derived the equations of motion of m theory. We used the observer space coordinates (r, ϕ), because the metric function $\mathrm{m}(r)$ appearing in the line element is defined in these coordinates. Alternatively, we can use the coordinates of m space (r_1, ϕ), which we have already used for deriving the relativistic Newton equation, as shown in Eqs. (10.65 - 10.67). The result was an expression in observer coordinates, so this did not lead to any problems.

In this section, we start again with the force equation:

$$\mathbf{F}_1 = \frac{d\mathbf{p}_1}{dt}, \tag{10.77}$$

and stay in m space coordinates, i.e., we use the variables $\mathbf{r}_1$ and $\dot{\mathbf{r}}_1$. The time t is that of the observer space, as before. Thus, we have

$$\mathbf{p}_1 = \gamma_1 m \dot{\mathbf{r}}_1 = \frac{\gamma_1 m \dot{\mathbf{r}}}{\mathrm{m}(r)^{1/2}}. \tag{10.78}$$

With the velocity in plane polar coordinates (10.23), the γ factor of m space is

$$\gamma_1 = \left(\mathrm{m}(r) - \frac{\dot{\mathbf{r}}_1^2}{c^2}\right)^{-1/2} = \left(\mathrm{m}(r) - \frac{\dot{r}_1^2 + \dot{\phi}^2 r_1^2}{c^2}\right)^{-1/2}. \tag{10.79}$$

We compute the force in the radial direction by using the Lagrange expression

$$\mathbf{F}_1 = \frac{\partial \mathscr{L}}{\partial r_1}\mathbf{e}_r. \tag{10.80}$$

The Lagrangian in m space is

$$\mathscr{L} = -\frac{mc^2}{\gamma_1} + \frac{mMG}{r_1}. \tag{10.81}$$

When we use

$$\frac{\partial}{\partial r_1}\left(\frac{1}{\gamma_1}\right) = \frac{1}{2}\gamma_1 \frac{d\mathrm{m}(r)}{dr_1} - \frac{\gamma_1}{c^2} r_1 \dot{\phi}^2, \tag{10.82}$$

it follows that

$$F_1 = \frac{\partial \mathscr{L}}{\partial r_1} = -\frac{1}{2}\gamma_1 mc^2 \frac{d\mathrm{m}(r)}{dr_1} + \gamma_1 m r_1 \dot{\phi}^2 - \frac{mMG}{{r_1}^2} \tag{10.83}$$

(see computer algebra code [177]). If we compare this equation with the non-relativistic radial equation (9.15) of ECE theory, we see that an additional term appears that contains the derivative of $\mathrm{m}(r)$. The general form of the force is

$$F_1 = -m\boldsymbol{\nabla}\Phi + m\boldsymbol{\Omega}\Phi \tag{10.84}$$

with the vector spin connection $\boldsymbol{\Omega}$ and the gravitational potential

$$\Phi = -\frac{MG}{r_1}. \tag{10.85}$$

The term with the derivative $d\mathrm{m}(r)/dr_1$ in Eq. (10.83) is the spin connection term and is interpreted as a vacuum force:

$$F_{vac} = -\frac{1}{2}\gamma_1 mc^2 \frac{d\mathrm{m}(r)}{dr_1}. \tag{10.86}$$

The complete force is then

$$F_1 = F_{vac} + \gamma_1 m\dot{\phi}^2 - \frac{mMG}{r_1{}^2}. \tag{10.87}$$

The problem is that the vacuum force and the factor γ_1 contain both variables r_1 and r. The m function is defined in the observer space, by definition. Therefore, this problem can be solved by rewriting the derivative of the m function as

$$\frac{d\mathrm{m}(r)}{dr_1} = \frac{d\mathrm{m}(r)}{dr}\frac{dr}{dr_1} \tag{10.88}$$

with

$$r_1 = \frac{r}{\sqrt{\mathrm{m}(r)}}. \tag{10.89}$$

Computer algebra [177] gives us

$$\frac{dr_1}{dr} = -\frac{r\,\frac{d\,\mathrm{m}(r)}{dr} - 2\,\mathrm{m}(r)}{2\mathrm{m}(r)^{3/2}}, \tag{10.90}$$

and by inversion:

$$\frac{dr}{dr_1} = -\frac{2\mathrm{m}(r)^{3/2}}{r\,\frac{d\,\mathrm{m}(r)}{dr} - 2\,\mathrm{m}(r)}. \tag{10.91}$$

Inserting this into (10.86) gives for the vacuum force (in vector form):

$$\mathbf{F}_{vac} = \frac{\gamma_1 mc^2\,\mathrm{m}(r)^{3/2}\,\frac{d\,\mathrm{m}(r)}{dr}}{r\,\frac{d\,\mathrm{m}(r)}{dr} - 2\,\mathrm{m}(r)}\mathbf{e}_r, \tag{10.92}$$

which is also equal to the spin connection term

$$\mathbf{F}_{vac} = m\mathbf{\Omega}\mathbf{\Phi}. \tag{10.93}$$

This representation of the vacuum force depends on Cartan geometry and appears only in m space, and only when $\mathrm{m}(r)$ is a variable function of r. The m function may be interpreted as a density change of the vacuum or aether. This change leads to a new type of force, which is maximized when the denominator of Eq. (10.92) is approaching zero, i.e., satisfies the condition

$$r\frac{d\,\mathrm{m}(r)}{dr} = 2\,\mathrm{m}(r), \tag{10.94}$$

which is where the maximal amount of energy from spacetime is transferred, in this resonance effect. The work done by this force between space points 1 and 2 is

$$W_{12} = \int_1^2 \mathbf{F}_{vac} \cdot d\mathbf{r} = T_2 - T_1 = U_1 - U_2, \tag{10.95}$$

where $T_2 - T_1$ is the change in kinetic energy and $U_1 - U_2$ is the change in potential energy. The energy is conserved due to the Hamiltonian $\mathscr{H} = T_1 + U_1 = T_2 + U_2$, which gives the right-side equality in Eq. (10.95).

■ **Example 10.3** We discuss two types of vacuum force resonances. The vacuum force can be approximated computationally (without complex dynamics solutions), if we assume a constant factor γ_1 in Eq. (10.92). To compute the vacuum force in this way, we start with the Schwarzschild-like m function (10.74),

$$\mathrm{m}(r) = 1 - \frac{r_0}{r} - \frac{\alpha}{r^2}. \tag{10.96}$$

Inserting m(r) into the equation for the resonance condition (10.94) gives the solutions

$$r_{1,2} = \frac{3\,r_0}{4} \pm \frac{1}{4}\sqrt{9{r_0}^2 + 32\alpha}. \tag{10.97}$$

For $\alpha = 0$, the original Schwarzschild-like m function is obtained with the divergence point

$$r_1 = \frac{3r_0}{2}. \tag{10.98}$$

This vacuum force has been graphed in Fig. 10.7 for $r_S = 1$ and two values of α. There is a pole at $r = 1.5$, indicating infinite energy from spacetime at this point. Increasing α shifts the pole to the right.

The same force was graphed in Fig. 10.8 using the exponential m(r) function

$$\mathrm{m}(r) = 2 - \exp\left(\log(2)\exp(-\frac{r}{R})\right) \tag{10.99}$$

for two values of the parameter R. There is an F_{vac} minimum that moves toward $r = 0$ as $R \to 0$. This explains why, for a small R (which was used in the Lagrange solutions), the vacuum force seems to go to infinity as $r \to 0$ (like a hyperbola). This m function is much more well behaved than the Schwarzschild-like function, because it is positive and does not contain zero crossings as $r \to 0$, which would have indicated event horizons (see Section 10.2.3). The divergence point r_1 of Eq. (10.94) cannot be computed analytically for the Schwarzschild-like function, because this gives a transcendental equation. Instead, we can make a Taylor series expansion of function (10.99) (see computer algebra code [177]). The result is

$$r_1 = \frac{\sqrt{6}R}{\sqrt{\log(2)^2 + 3\log(2) + 1}}, \tag{10.100}$$

which is proportional to the R parameter.

There is still another aspect of the resonance equation (10.94) that can be investigated. This equation can be considered as a differential equation for m(r), which has the general solution

$$\mathrm{m}(r) = c_1 r^2 \tag{10.101}$$

with a constant c_1. This means that, for such a quadratic m(r), the vacuum force is infinite everywhere. However, the m function has to have the limit m(r)=1 for large r. Therefore, we compose a function that is quadratic for $r \to 0$ and constant for $r \to \infty$:

$$\mathrm{m}(r) = \begin{cases} \frac{r^2}{2a^2} & \text{for } r < a, \\ 1 - \frac{a}{4\left(r - \frac{a}{2}\right)} & \text{for } r \geq a. \end{cases} \tag{10.102}$$

It is easy to verify that m(r) is continuous and continuously differentiable at $r = a$. Both cases in the above equation give

$$\mathrm{m}(a) = \frac{1}{2}, \tag{10.103}$$

$$\frac{d\mathrm{m}(r)}{dr}(a) = \frac{1}{a}. \tag{10.104}$$

This function is graphed in Fig. 10.9 for $a = 1/2$. The corresponding vacuum force and its denominator are shown in Fig. 10.10. It can be seen that the vacuum force drops massively when r approaches 1/2. Designing the m function so that $\mathbf{F}_{vac}$ is maximized would enable the unlimited extraction of energy from spacetime. ■

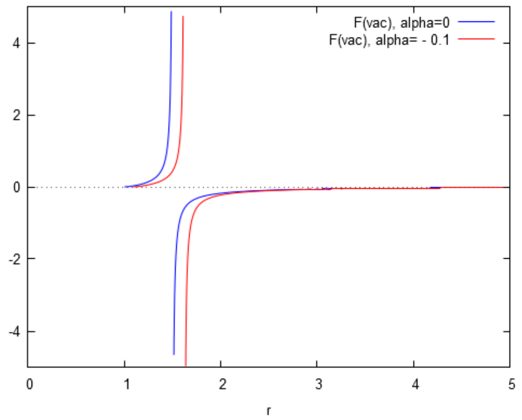


Figure 10.7: Vacuum force of Schwarzschild-like functions m(r).

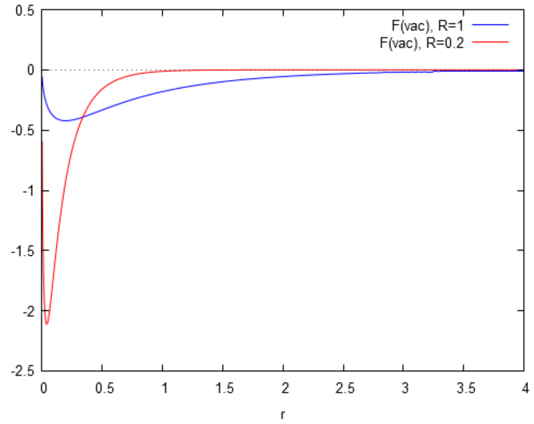


Figure 10.8: Vacuum force of exponential functions m(r).

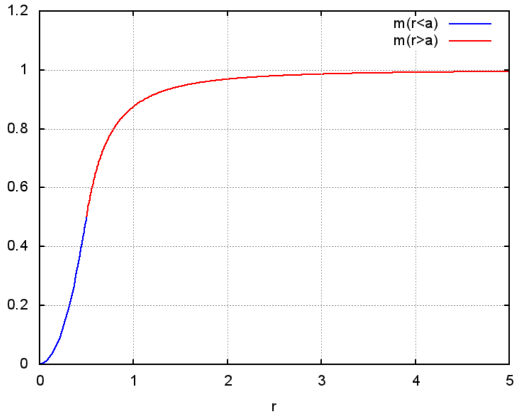


Figure 10.9: m function composed of terms r^2 and $1/r^2$.

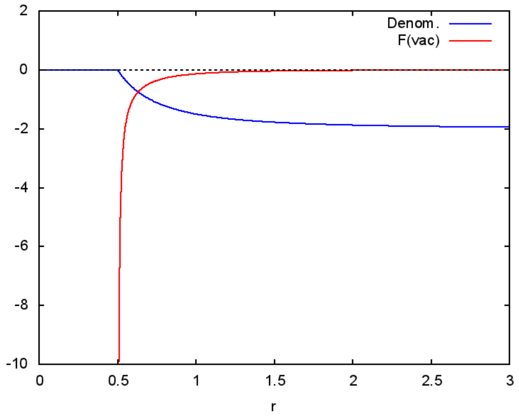


Figure 10.10: Denominator of vacuum force, and vacuum force, from Fig. 10.9.

■ **Example 10.4** A Sagnac interferometer can be used to measure the radial dependence of the function m(r). Such an interferometer was described in Example 5.4. Now, we extend the usage of this device to measure an effect of m theory and, thereby, of general relativity.

In two dimensions, the infinitesimal line element of m theory (10.5) reads:

$$ds^2 = c^2 \mathrm{m}(r) dt^2 - \frac{dr^2}{\mathrm{m}(r)} - r^2 d\phi^2. \tag{10.105}$$

The interferometer is used in one fixed position, where it is spun in both directions to observe the difference between the travelling times of light in the fiber. There are two radius coordinates: r describes the distance from the gravitational center, and r_i is the radius of the interferometer windings. Both coordinates are fixed; therefore, we have $dr = 0$ in the line element and the angular coordinate is the angle of the local interferometer frame, which leads to

$$ds^2 = c^2 \mathrm{m}(r) dt^2 - r_i^2 d\phi^2. \tag{10.106}$$

Because light has a zero geodetic, the line element vanishes; consequently, with $ds^2 = 0$,

$$c^2 \mathrm{m}(r) dt^2 = r_i^2 d\phi^2. \tag{10.107}$$

The interferometer is spun at a mechanical frequency $\pm\Omega$. Therefore, the differential angle $d\phi$ has to be replaced with

$$d\phi \to d\phi \pm \Omega dt, \tag{10.108}$$

as was done in Section 9.3.3, for rotating frames. From (10.107), we obtain

$$dt = \frac{r_i}{\sqrt{\mathrm{m}(r)} c \pm \Omega r_i} d\phi. \tag{10.109}$$

Then, the time for one right and one left rotation, respectively, is

$$t_1 = \frac{2\pi r_i}{\sqrt{\mathrm{m}(r)} c - \Omega r_i} \tag{10.110}$$

and

$$t_2 = \frac{2\pi r_i}{\sqrt{\mathrm{m}(r)} c + \Omega r_i}. \tag{10.111}$$

The difference between the two rotation times is

$$\Delta t = t_1 - t_2 = \frac{2\pi r_i}{\sqrt{\mathrm{m}(r)} c - \Omega r_i} - \frac{2\pi r_i}{\sqrt{\mathrm{m}(r)} c + \Omega r_i} = \frac{4\pi r_i^2 \Omega}{\mathrm{m}(r) c^2 - \Omega^2 r_i^2}. \tag{10.112}$$

The term πr_i^2 is the area A of the interferometer, and therefore:

$$\Delta t = \frac{4A\Omega}{\mathrm{m}(r) c^2 - \Omega^2 r_i^2}. \tag{10.113}$$

The product Ωr_i is the linear rotation speed of the interferometer at its rim. Since this is much smaller than c, we can approximate:

$$\Delta t \approx \frac{4A\Omega}{\mathrm{m}(r) c^2}. \tag{10.114}$$

This result is the same as in Example 5.4, Eq. (5.126), with the additional factor of $\mathrm{m}(r)$ in the denominator. Therefore, the result Δt depends on the height of the interferometer above the Earth's surface (more precisely: the distance to the gravitational center).

The required precision of the time measurement can be assessed by setting $r_i = 1\text{m}$ and $\Omega = 2\pi \cdot 50/\text{s}$, which gives

$$\Delta t \approx 4.4e\text{-}14 \text{ s} \tag{10.115}$$

(see computer algebra code [178]). This is a very short time. Normally, the interferometer consists of a high number of turns of the optical fiber. The area factor then increases by this number, and the required minimum for time difference measurement increases by the same factor. ■

10.2.2 Superluminal motion

From the generalized γ factor,

$$\gamma_1 = \left(\mathrm{m}(r) - \frac{v^2}{\mathrm{m}(r)\,c^2}\right)^{-1/2}, \tag{10.116}$$

we see that the m function alters the effective velocity of light by

$$c^2 \to \mathrm{m}(r)\,c^2. \tag{10.117}$$

The dependence of the generalized γ factor on the m function has been graphed in Fig. 10.11. The ratio v/c has been taken as a parameter. As can be seen, the γ factor goes to infinity as $m(r) \to 0$, according to the dynamics calculations in the preceding sections. For $v/c > 1$, this limit is reached above m(r)=1. For cases where $\mathrm{m}(r) > 1$, the γ factor can even take values smaller than unity. This behavior is unknown in Einsteinian relativity. If we allow for $\mathrm{m}(r) > 1$, superluminal motion is possible.

The generalized γ factor is not restricted to positive m values. Only the total argument of the square root in Eq. (10.116) must be positive; the summands within the square root can have any sign. This allows for negative m values in a certain range. As can be seen from Fig. 10.11, there are asymptotic limits of γ for various values of v/c. Again, superluminal motion is possible, and γ may take values smaller than unity.

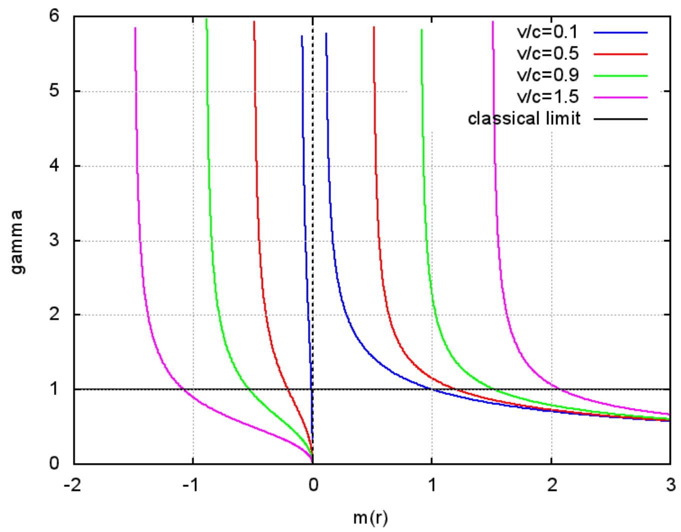


Figure 10.11: Generalized γ factor, dependent on $\mathrm{m}(r)$, for selected values of v/c.

10.2.3 Event horizons and cosmology

In Einsteinian theory of "black holes", event horizons exist. We did not find "black holes" in ECE theory, but event horizons could still exist. They would be surfaces on which the m function is zero. In m theory, we can construct such surfaces in the following way. We introduce a zero point of m(r) at $r_0 > 0$ as shown in Fig. 10.12, where we chose $r_0 = 0.3$. The m function then reads:

$$\mathrm{m}(r) = 2 - \exp\left(\log(2)\exp(\pm\frac{r-r_0}{R})\right), \tag{10.118}$$

where the plus sign holds for $r < r_0$ and the minus sign holds for $r > r_0$. We first consider the outer region: $r > r_0$. If the mass orbits a sufficient distance away from the event horizon at r_0, we obtain, in the periodic case, precessing ellipses or curves oscillating between two radii, as discussed below for the case where $r < r_0$. If the initial velocity of the calculation falls below a certain value, the mass stops at the event horizon and stays there (see Fig. 10.13). The r_1 coordinate in m space itself diverges for $\mathrm{m}(r) \to 0$, because it is inversely proportional to the square root of $\mathrm{m}(r)$:

$$r_1 = \frac{r}{\mathrm{m}(r)^{1/2}}, \tag{10.119}$$

and the denominator approaches zero. However, the relevant coordinate in observer space is r.

The periodic motion of a mass within the event horizon (the case where $r < r_0$) is graphed in Fig. 10.14. This is a precessing ellipse or a motion between two radii, as long as the mass does not come too close to the event horizon. We see that the orbits r and r_1 are different in the outer region, where they are closer to the event horizon. This is an indication of the strength of the effects of the m function. When the initial velocity exceeds a certain limit, the mass is caught by the event horizon, leading to an end of motion. This case is graphed in Fig. 10.15. The r_1 coordinate diverges again, as explained above. It has already been shown in Fig. 10.4 that the velocity goes to zero when the mass approaches an event horizon.

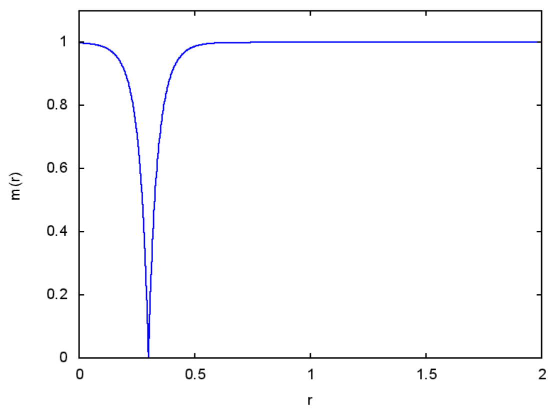


Figure 10.12: m function with the event horizon at $r = 0.3$.

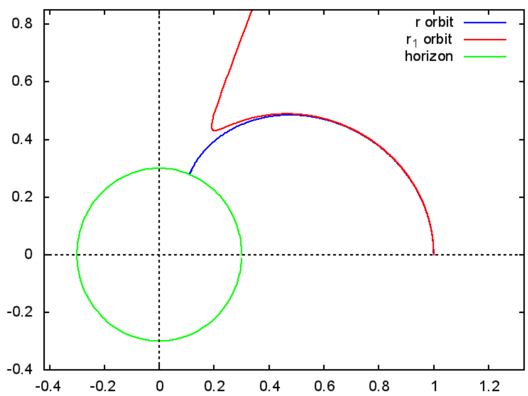


Figure 10.13: Collapsing orbits outside of the event horizon.

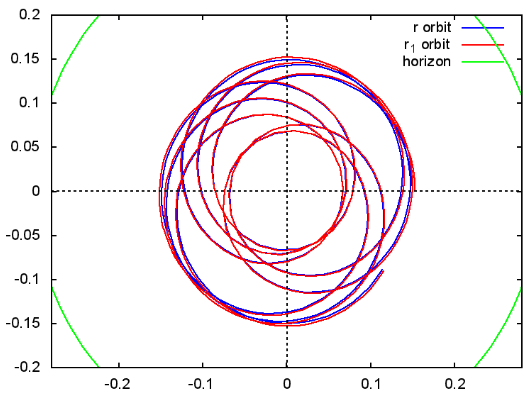


Figure 10.14: Periodic orbits inside the event horizon.

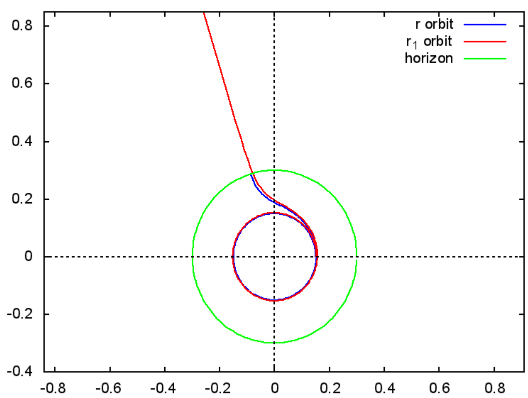


Figure 10.15: Collapsing orbits inside the event horizon.

Obviously, an event horizon is an insurmountable limit. This is different from obsolete black hole theory, wherein a mass can fall freely through the event horizon, which shows that it is only a mathematical artifact. However, for radiation, an event horizon is not a limit. To see this, consider the wave equation in the coordinates of m space:

$$\frac{1}{v_0^2}\frac{\partial^2}{\partial t_1^2}\mathbf{F} - \nabla_1^2\mathbf{F} = \mathbf{0}, \tag{10.120}$$

where $\mathbf{F}$ is a gravitational or electromagnetic field and v_0 is its propagation velocity. In one dimension, this equation reads:

$$\frac{1}{v_0^2}\frac{\partial^2}{\partial t_1^2}F(x_1,t_1) - \frac{\partial F(x_1,t_1)}{\partial x_1} = 0. \tag{10.121}$$

To concretize this, we assume a harmonic wave

$$F(x_1,t_1) = F_0 \exp(\omega_1 t_1 - k_1 x_1). \tag{10.122}$$

The parameters and coordinates are

$$\begin{aligned} x_1 &= \frac{x}{\mathrm{m}(x)^{1/2}}, \\ t_1 &= t\ \mathrm{m}(x)^{1/2}, \\ \omega_1 &= \frac{2\pi}{T_1} = \frac{2\pi}{T\ \mathrm{m}(x)^{1/2}}, \\ k_1 &= \frac{2\pi}{\lambda_1} = \frac{2\pi\ \mathrm{m}(x)^{1/2}}{\lambda}, \end{aligned} \tag{10.123}$$

with ω being the time frequency and k being the wave number, as usual. It follows that

$$\left(\frac{1}{v_0^2}\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2}\right) F_0 \exp(\omega_1 t_1 - k_1 x_1) = 0 \tag{10.124}$$

and, by insertion,

$$\left(\frac{1}{v_0^2 \mathrm{m}(x)}\frac{\partial^2}{\partial t^2} - \mathrm{m}(x)\frac{\partial^2}{\partial x^2}\right) F_0 \exp(\omega t - kx) = 0. \tag{10.125}$$

As we can see, the phase factors are independent of the coordinate system. Evaluating the derivatives gives

$$\left(-\frac{\omega^2}{v_0^2 \mathrm{m}(x)} + k^2 \mathrm{m}(x)\right) F_0 \exp(\omega t - kx) = 0. \tag{10.126}$$

Since the wave does not vanish, the factor must be zero, which gives

$$v_0^2 = \frac{\omega^2}{k^2\,\mathrm{m}(x)^2} \tag{10.127}$$

or

$$v_0 = \frac{\omega}{k\,\mathrm{m}(x)}. \tag{10.128}$$

The propagation velocity of the wave becomes infinitely high for $\mathrm{m}(x) \to 0$, and the event horizon is passed with superluminal motion. This is completely different from a mass, which is caught at the horizon. All of this could be the basis of a completely new cosmology. For example, we could hypothesize that the size of the universe is determined by the aether density diminishing to zero at the boundary of the universe, and since the m function describes the aether density, this would mean that $\mathrm{m}(r)$ would approach zero at the boundary. Consequently, spacetime would cease to exist there, and matter would "dissolve" because it is nothing more than a "condensation" of the aether. While material objects could not leave the universe, radiation could do so with superluminal speed, and even move almost instantaneously to regions where other universes could exist. If we stay with the aether model, some kind of minimal "rest aether" would have to exist there. This possibility would not require quantum effects or higher dimensions.

10.2.4 Light deflection

Einstein predicted that light coming from a distant star and passing near the Sun would be deflected by the Sun's gravitational field, and Eddington claimed to have proved this by measurements during a solar eclipse, but his data later turned out to be scientifically untenable. Nevertheless, this continues to be used as a validation of Einstein's general theory of relativity.

Einstein's derivation is based on the (quite complicated) geodetic equation, which is used with rough approximations and by inserting a trial function. Fortunately, the right result comes out, but it is not valid for massive deflection angles, which is similar to how Einstein's calculation of planetary precession is only valid for small precession angles in the Solar System, but is not valid in general. In contrast, we provide a general solution without approximations.

We start with an alternative classical approach, in which the deflection angle of light can be computed by equating the photon energy $\hbar\omega$ with a relativistic mass energy mc^2, which gives a relativistic mass according to special relativity. However, the angle of deflection then comes out to be only half of the observed value. Einstein obtained the correct result

$$\Delta\psi = \frac{4MG}{R_0 c^2}, \tag{10.129}$$

where $\Delta\psi$ is the angle of deflection, M is the mass of the Sun, and R_0 is the radius at the point where the light grazes the Sun.

During the development of ECE theory, many papers were written on different aspects of a mathematically correct explanation of light deflection by gravitation, and the photon rest mass was being estimated as early as 2010 [114]. According to the Proca equation, all particles (including photons) must have a rest mass, because particle structures are connected with the curvature and torsion of spacetime. In special relativity, however, particles moving with exactly the velocity of light in vacuo cannot have a rest mass. In the following discussion, we present the two most advanced ECE explanations of this discrepancy, and the second explanation involves m theory.

In the first approach, we introduce a hypothetical "Newtonian velocity" v_N and assume that this velocity appears in the γ factor of special relativity:

$$\gamma_N = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2}. \tag{10.130}$$

For higher velocities, the physical velocity is the relativistic velocity v, which we define by

$$v = \gamma_N v_N. \tag{10.131}$$

This is, of course, a new definition of the γ factor. If v does not come close to c, the result will not deviate significantly from the standard definition, which uses v. It follows that

$$v^2 = \frac{v_N^2}{1 - \frac{v_N^2}{c^2}} \tag{10.132}$$

and, after rewriting,

$$v_N^2 = \frac{v^2}{1 + \frac{v^2}{c^2}}. \tag{10.133}$$

When v approaches c, the Newtonian velocity has the limit

$$v_N^2 \underset{v \to c}{\longrightarrow} \frac{c^2}{2}. \tag{10.134}$$

Consequently, the γ factor does not diverge but has the limit

$$\gamma_N \underset{v \to c}{\longrightarrow} \sqrt{2} \tag{10.135}$$

(see Fig. 10.16).

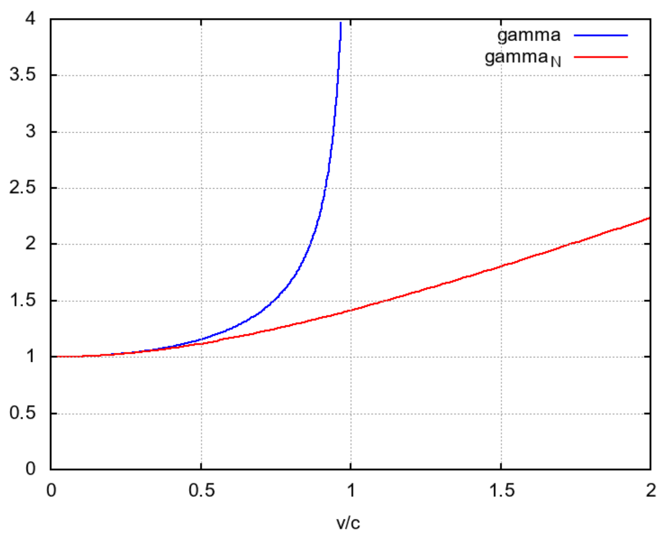


Figure 10.16: γ factors of standard velocity v and Newtonian velocity v_N.

According to Eq. (9.21), in the Newtonian theory involving conic sections, the orbital velocity of light grazing the Sun is

$$v_N^2 = MG\left(\frac{2}{R_0} - \frac{1}{a}\right). \tag{10.136}$$

Therein, a is the semi major axis

$$a = \frac{\alpha}{1 - \varepsilon^2}, \tag{10.137}$$

α is the semi latus rectum, and R_0 is the perihelion of distance of closest approach

$$R_0 = \frac{\alpha}{1 + \varepsilon}. \tag{10.138}$$

For light grazing the Sun, the ellipticity is $\varepsilon \gg 1$, thus

$$\varepsilon \approx \frac{R_0 v_N^2}{MG}. \tag{10.139}$$

The angle of deflection at closest approach is

$$\Delta\psi = \frac{2}{\varepsilon} = \frac{2MG}{R_0 v_N^2} \tag{10.140}$$

(for more details, see [115]). For a photon, the relativistic velocity approaches c, so from Eq. (10.133) we have:

$$v_N^2 \underset{v\to c}{\longrightarrow} \frac{c^2}{2} \tag{10.141}$$

and

$$\Delta\psi = \frac{4MG}{R_0 c^2}, \tag{10.142}$$

which is exactly the experimental result, and it depends only on the definition of Newtonian and relativistic velocity.

In the second approach, m theory is used to clarify the deviation of the factor γ_N, Eq. (10.130), from the γ factors of special relativity and generally relativistic ECE theory. In m theory, the generally relativistic γ factor is

$$\gamma = \left(\mathrm{m}(r) - \frac{v^2}{c^2\,\mathrm{m}(r)}\right)^{-1/2}. \tag{10.143}$$

To allow for the case $v = c$, we must have $\mathrm{m}(r) > 1$, otherwise the γ factor would diverge for $v \to c$. For this same case, because of Eq. (10.141), we have an additional requirement involving the γ factor of m theory:

$$\gamma^2(v \to c)\; v^2(v \to c) = \gamma_{\mathrm{N}}^2(v \to c)\; v_{\mathrm{N}}^2(v \to c), \tag{10.144}$$

which leads to

$$\frac{c^2}{\mathrm{m} - \frac{1}{\mathrm{m}}} = \frac{c^2}{2(1-\frac{1}{2})} = c^2 \tag{10.145}$$

or

$$\mathrm{m} - \frac{1}{\mathrm{m}} = 1, \tag{10.146}$$

where we have assumed that $\mathrm{m}(r)$ is constant. This means that $\gamma(v = c) = 1$. Eq. (10.146) is a quadratic equation for m, with solutions

$$\mathrm{m}_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{5}\right). \tag{10.147}$$

Expressing this numerically gives the surprising result:

$$\mathrm{m}_1 = -0.61803, \quad \mathrm{m}_2 = 1.61803, \tag{10.148}$$

which is known as the golden ratio Φ:

$$\mathrm{m}_1 = -(\Phi - 1), \quad \mathrm{m}_2 = \Phi. \tag{10.149}$$

m_2 can be inserted into the Lagrange equation set (10.63, 10.64), as discussed below.

As we have seen, m theory gives us $\gamma = 1$ for photons. The energy of a photon with rest mass m_0 then becomes

$$E = \hbar\omega = \mathrm{m}(r)\gamma m_0 c^2 = \Phi m_0 c^2. \tag{10.150}$$

The "standard" rest energy m_0c^2 is increased by a factor of Φ, and the rest mass depends on the wave energy $\hbar\omega$. The photon momentum is

$$p = \frac{E}{c} = \Phi m_0 c. \tag{10.151}$$

The photon with mass fits into the framework of generally relativistic theory, for example, via the Proca equation (7.52). In contrast to special relativity, there is no problem with the γ factor for $v \to c$. For $\mathrm{m}(r)$, we have obtained a self-consistent solution. Therefore, this photon theory is parameter-free.

It is possible to compute the trajectories of photons by solving the equations of motion (10.63, 10.64), which is classical theory. This trajectory computation is not achievable with special relativity, because the γ factor diverges for $v \to c$. Within m theory, however, $v = c$ is possible for $\mathrm{m}(r) > 1$. Since $\mathrm{m}(r)$ is assumed to be constant, the equations of motion reduce to

$$\ddot{\phi} = \dot{\phi}\,\dot{r}\left(\frac{GM}{\gamma c^2 r^2 \sqrt{\mathrm{m}(r)}} - \frac{2}{r}\right), \tag{10.152}$$

$$\ddot{r} = -\frac{GM\,\dot{\phi}^2}{\gamma c^2 \sqrt{\mathrm{m}(r)}} - \frac{GM\sqrt{\mathrm{m}(r)}}{\gamma^3 r^2} + \dot{\phi}^2 r. \tag{10.153}$$

Only the relativistic corrections $\sim 1/c^2$ remain. These equations do not depend on the orbiting mass m, which would be the rest mass of the photon. The scales used in such calculations require special adaptations, because the central mass and the velocities have quite high numerical values. For computing the orbit of the S2 star, for example, we had used adapted units [100], and we applied the same units here. The length, for example, is measured in 10^9m. By using the golden ratio value m_2 from Eq. (10.148), we obtain the orbit graphed in Fig. 10.17. Please note that the photon moves from right to left, and that the Sun is at $x = 0$. Since the angle of deflection is so small, the y axis had to be magnified. The deflection angle is approximately determined by

$$\Delta\psi = -\frac{\Delta y}{\Delta x}, \tag{10.154}$$

where Δx is measured from $x = 0$. The result is precise when the negative x value is far enough away from the center. The computed dependence of the deflection angle on x is shown in Fig. 10.18. The asymptotic value is $8.65 \cdot 10^{-6}$ radians, compared to the experimental value of $8.48 \cdot 10^{-6}$. This is coincidence within 2% and shows very good conformance. The orbit calculation for a photon grazing the Sun convincingly confirms the m value that has been computed analytically for an effective, constant m function. To see how the deflection angle depends on the m value, see Table 10.2, where we have varied the m value in a wider range. Obviously, there is a significant variation of $\Delta\psi$ with m, which shows that our result is non-trivial.

The rest mass of the photon can be determined from Eq. (10.150). Assuming an average frequency of light of $\omega = 10^{-15}$/s, it follows that

$$m_0 = \frac{\hbar\omega}{\Phi\, c^2} = 7.25 \cdot 10^{-37}\ \mathrm{kg}. \tag{10.155}$$

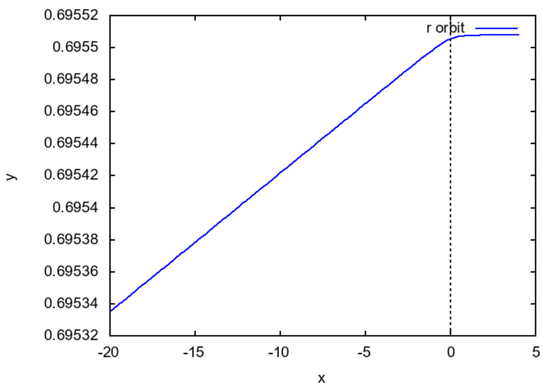


Figure 10.17: Orbit of light grazing the Sun, moving from right to left (please note the y scale).

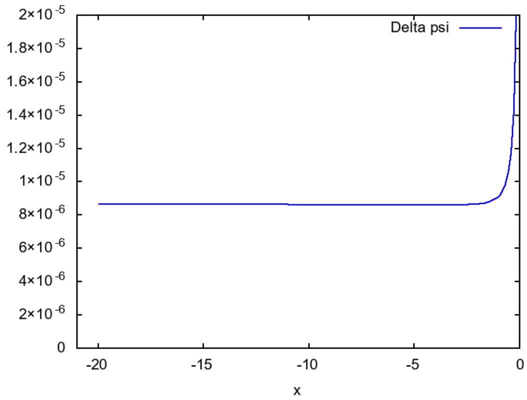


Figure 10.18: Deflection angle $\Delta\psi = -\Delta y/\Delta x$ of light grazing the Sun, moving from right to left.

m	$\Delta\psi$
1.28	$4.29 \cdot 10^{-6}$
1.43	$6.14 \cdot 10^{-6}$
1.57	$7.99 \cdot 10^{-6}$
1.61803	$8.65 \cdot 10^{-6}$
1.78	$11.0 \cdot 10^{-6}$
Exp.	$8.48 \cdot 10^{-6}$

Table 10.2: Deflection angle of light $\Delta\psi$ for several values of the m function.

From earlier results of m theory (for example, for the S2 star), we know that we have to expect m≈1 for ordinary matter. The value of m for photons is far above this range. Since our numerical calculation has convincingly confirmed the analytical result, we are led to the conclusion that light (or electromagnetic radiation in general) shows a different interaction with the spacetime or background field, compared to dense matter. This may be explained by the fact that, for a photon in any form, the mass is expanded space. In electrodynamics, light is often modeled by plane waves that are of infinite extent, in theory. A spatial restriction is possible, leading to wave packets, as in quantum mechanics. Space appears "denser" for photons than it does for ordinary matter. All of these results have been obtained from a classical theory, without involving quantum effects.

10.3 Final remarks

In the preceding sections, it was shown that the Hamiltonian, Lagrangian and equations of motion give gravitational light deflection, forward and retrograde precession of orbits, shrinking orbits and the possibility of expanding orbits, counter gravitation, superluminal signaling, and infinite potential energy from m(r). These are all major advances in classical dynamics. In additional papers that were not reviewed in this book, the compatibility of relativistic Lagrange theory with Hamilton/Hamilton-Jacobi theory has been proven [116]. The equations of motion of m theory suggest new cosmologies, while the equations of special-relativistic motion alone do not – they give only precession.

We conclude by mentioning that the set of equations of motion derived in this book can be used to investigate very intricate possibilities in cosmology. These equations have been cross-checked in various ways. For computation, the Lagrangian version was used as a basis because Lagrange theory is strongly formalized and therefore best suited for being programmed on a computer. The laws of conservation of angular momentum and total energy provide a critical cross-check for the correctness of the results. This cross-check has been applied to nearly all numerical orbit calculations of ECE theory.

The second volume of this book deals with quantum mechanics. We will see that vacuum forces give rise to microscopic effects like the Lamb shift and vacuum fluctuations. m theory will also be applied to quantum mechanics, leading to a unification of this subject with general relativity. New derived methodologies like a quantum force and quantum-Hamilton equations will be introduced. Quantum statistics will be shown to be deterministic, because the Heisenberg uncertainty principle will be proven to not be valid for all combinations of conjugated operators. Consequently, quantum mechanics will become a simpler and better understood subject than it is today.

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Computer algebra code (Maxima)

(This code is available in http://aias.us/documents/uft/ECE-Code.zip)

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[118] Ex2.5.wxm

[119] Ex2.10.wxm

[120] Ex2.10a.wxm

[121] Ex2.10b.wxm

[122] Ex2.11.wxm

[123] Ex2.12.wxm

[124] Ex2.13.wxm

[125] Ex2.14.wxm

[126] Ex2.15.wxm

[127] Ex3.1.wxm

[128] Ex3.2a.wxm

[129] Ex3.2b.wxm

[130] Ex4.1.wxm

[131] Ex4.2.wxm

[132] Ex5.2.wxm

[133] Ex5.3.wxm

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[135] Ex5.5.wxm

[136] Ex5.6.wxm

[137] Ex5.7.wxm

[138] Ex6.2.wxm

[139] Ex6.2a.wxm

[140] Ex6.6.wxm

[141] Ex6.7.wxm

[142] Ex7.1a.wxm

[143] Ex7.1b.wxm

[144] Ex7.2.wxm

[145] Ex7.4.wxm

[146] Ex7.5a.wxm

[147] Ex7.5b.wxm

[148] Ex8.1a.wxm

[149] Ex8.1b.wxm

[150] Ex8.4a.wxm

[151] Ex8.10.wxm

[152] Ex8.14.wxm

[153] Ex8.15a.wxm

[154] Ex8.15b.wxm

[155] Ex8.16.wxm

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[158] Ex9.2b.wxm

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[175] Ex10.1a.wxm

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