THE DEVELOPMENT OF GENERAL RELATIVITY WITH THE MINKOWSKI METRIC.

by

M. W. Evans and H. Eckardt

Civil List, AIAS and UPITEC

(www.webarchive.org.uk, www.aias.us, www.atomicprecision.com, www.upitec.org, www.et3m.net)

ABSTRACT

The Minkowski metric in a plane is used to develop a theory of all orbits. The transition from special to general relativity is considered through use of a Lorentz transform with variable velocity v. The only constant of motion in this theory is the hamiltonian, half the rest energy of an orbiting object of mass m. Various orbital equations are developed and the Newtonian limit considered. The theory is illustrated through its use in orbits of the solar system and galaxies. The theory does not predict orbits, but rationalizes them in terms of the Minkowski metric, orbits being considered as a constraint on the metric.

Keywords: ECE theory, orbital laws and equations based on the Minkowski metric.

UFT 203

1. INTRODUCTION

It is well known that Einstein and others aimed to develop general relativity from special relativity, a theory in which one frame of reference moves at a constant velocity $\underline{v}$ with respect to another, the two frames being related by the Lorentz transform {1 - 11}. This paper considers the natural generalization achieved by allowing $\underline{v}$ to vary in the Lorentz transform. The infinitesimal line element of the theory is therefore the line element based on the Minkowski metric. In Section 2 this metric is defined in cylindrical polar coordinates in a plane, the plane of the orbit. The orbital equation is deduced using the methods of general relativity {11}, methods which are well known, and consist of defining a lagrangian from the line element without using the concept of potential energy. The lagrangian is therefore the same as the hamiltonian H, which is half the rest energy of an orbiting object of mass m. The Euler Lagrange equations are used to define the total energy E and the total angular momentum L. In the classical dynamics these quantities are constants of motion, but in general relativity they depend in general on time t and the radial coordinate r. The hamiltonian H is not the same as the total energy E in general relativity. The theory developed in Section 2 is a special case of general relativity in which the velocity of the Lorentz transform is not constant, and which the Minkowski metric is constrained by the orbit being described. The orbit introduces a relation between the infinitesimals dr and $d\theta$, so the Minkowski metric is constrained. In Section 3 this theory is compared with general relativity in the most general spherical spacetime, and is preferred by Ockham's Razor as a simpler and complete description of all orbits.

2. THE ORBITAL EQUATIONS

Consider the Minkowski metric in cylindrical polar coordinates:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(1)$$

in the plane:

$$dZ^2 = 0. \quad -(2)$$

Here the coordinates are (r, θ), τ is the proper time, t the time in the observer frame, and c the assumed constant speed of light in vacuo. The lagrangian and hamiltonian are defined as in general relativity {1 - 10}:

$$\mathcal{L} = H = \frac{1}{2} mc^2$$
$$= \frac{1}{2} mc^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{d\theta}{d\tau}\right)^2 r^2 \quad -(3).$$

The Euler Lagrange equations give the total energy:

$$E = \gamma mc^2 = \left(\frac{dt}{d\tau}\right) mc^2 \quad -(4)$$

and the total angular momentum:

$$L = \gamma mr^2 \frac{d\theta}{dt}. \quad -(5)$$

Not that the total energy E and the hamiltonian H are not the same. This is a well known feature of general relativity. By definition:

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\theta^2 \quad -(6)$$

so the velocity of the orbiting particle of mass m is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(7)$$

in the observer frame. From Eqs. (1) and (7),

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad - (8)$$

which is the Lorentz factor. If v is constant the total energy E is constant, but not otherwise.

To illustrate this point further consider the most general metric {11} of spherically symmetric spacetime in the plane (2):

$$ds^2 = c^2 d\tau^2 = m(r,t)c^2 dt^2 - n(r,t)dr^2 - r^2 d\theta^2. \quad - (9)$$

Here m(r, t) and n(r, t)are both functions of time and of the radial coordinate r. The hamiltonian and lagrangian from the metric (9) are:

$$\mathcal{L} = H = \frac{1}{2}mc^2$$
$$= \frac{1}{2}m\,m(r,t)c^2\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}m\,n(r,t)\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{2}mr^2\left(\frac{d\theta}{d\tau}\right)^2 \quad - (10)$$

and are the same as for the Minkowski metric, half the rest energy. However the total energy from (10) is:

$$E = m(r,t)mc^2\frac{dt}{d\tau} \quad - (11)$$

and the total angular momentum from (10) is:

$$L = mr^2\omega\frac{dt}{d\tau}. \quad - (12)$$

By definition:

$$\underline{dr} \cdot \underline{dr} = n(r,t)dr^2 + r^2 d\theta^2 \quad - (13)$$

so:

$$\gamma = \frac{dt}{d\tau} = \left(m(r,t) - \frac{v^2}{c^2}\right)^{-1/2} \quad -(14)$$

from the hamiltonian (10). Written out in full, the total energy is:

$$E = m(r,t)\left(m(r,t) - \frac{v^2}{c^2}\right)^{-1/2} mc^2 \quad -(15)$$

and in general is not constant. The hamiltonian, however, is constant.

With the definitions (4) and (5), Eq. (3) becomes:

$$mc^2 = \frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{mr^2} \quad -(16)$$

and so:

$$m\left(\frac{dr}{d\tau}\right)^2 = m\left(\frac{dr}{d\theta}\right)^2\left(\frac{d\theta}{d\tau}\right)^2 = \frac{mL^2}{m^2r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{E^2}{mc^2} - mc^2 - \frac{L^2}{mr^2} \quad -(17)$$

The orbital equation is therefore:

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left(\frac{E^2 - m^2c^4}{c^2L^2} - \frac{1}{r^2}\right) \quad -(18)$$

and can be expressed as:

$$\frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right) = \chi \quad -(19)$$

where

$$\chi = \frac{E^2 - m^2c^4}{c^2L^2}. \quad -(20)$$

The orbit is considered to be observable by astronomy, and in general is:

$$f'(\theta) = \frac{dr}{d\theta} \quad -(21)$$

so Eq. (19) is the general orbital equation for all orbits. Newton's theory of orbits {12} gives the result:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{L^2}\left(2m\left(E - V - \frac{L^2}{2mr^2}\right)\right) \quad -(22)$$

where E is the classical total energy defined by:

$$E = T + V \quad -(23)$$

where T is the classical kinetic energy and V is the classical potential energy. In Eq. (22) L is the classical total angular momentum. In classical dynamics {12}, E and L are constants of motion, and there is no concept of rest mass as is well known. In classical dynamics:

$$H = E \quad -(24)$$

the hamiltonian and total energy are the same. In Newtonian dynamics, the potential energy of attraction between m and M is:

$$V = -\frac{mMG}{r} \quad -(25)$$

where G is Newton's constant. The quantity:

$$E_r = \frac{L^2}{2mr^2} \quad -(26)$$

comes from the rotational kinetic energy but is known incorrectly as the centrifugal "force" of repulsion. This is a well known fallacy of Newtonian dynamics. Accepting this for the sake of argument, the Newton theory gives an elliptical orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos\theta} \quad -(27)$$

where the half right latitude is:

$$\alpha = \frac{L^2}{m^2 M G} \quad -(28)$$

and where the eccentricity is:

$$\epsilon = \left(1 + \frac{2L^2 E}{m^3 M^2 G^2}\right)^{1/2} \quad -(29)$$

From Eq. (27):

$$\left(\frac{dr}{d\theta}\right)^2 = r^4 \left(\frac{\epsilon}{\alpha}\right)^2 \left(r^2 - \frac{1}{\epsilon^2}(\alpha - r)^2\right) \quad -(30)$$

Using the Newtonian result (22) in the orbital equation (19) gives:

$$\frac{2mT}{L^2} \longleftarrow \frac{E^2 - m^2c^4}{c^2L^2} \quad -(31)$$

i.e.:

$$\frac{1}{2m}\left(\frac{E^2}{c^2} - m^2c^2\right) \longrightarrow T \quad -(32)$$

in which the total energy is defined by Eq. (4). In the limit:

$$v \ll c \quad -(33)$$

the kinetic energy T becomes:

$$T = \frac{mc^2}{2}\left(\gamma^2 - 1\right)$$

$$= \frac{mc^2}{2}\left(\left(1 - \frac{v^2}{c^2}\right)^{-1} - 1\right) \sim \frac{mc^2}{2}\left(1 + \frac{v^2}{c^2} - 1\right) = \frac{1}{2}mv^2 \quad -(34)$$

which is the Newtonian result, self consistently. In general the Newtonian kinetic energy is not a constant unless v^2 is a constant. Eq. (32) may be interpreted as the definition of kinetic energy in a theory in which v is allowed to vary within the Lorentz transform. This is a satisfactory theory of relativity for a general v, and so is a theory of general relativity.

A second orbital equation may be obtained directly from Eq. (7), and is:

$$v^2 = \left(\frac{dr}{dt}\right)^2\left(1 + r^2\left(\frac{d\theta}{dr}\right)^2\right). \quad -(35)$$

The relativistic linear momentum of this theory is:

$$\underline{p} = \gamma m \underline{v} \quad -(36)$$

where $\underline{v}$ varies. In special relativity, Eq. (36) is also true, but $\underline{v}$ is a constant. From Eq. (36) it follows {1 - 10, 12} that:

$$E^2 = p^2c^2 + m^2c^4 \quad -(37)$$

so Eq. (19) may be expressed as:

$$\frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right) = \left(\frac{p}{L}\right)^2 \quad -(38)$$

in which:

$$\left(\frac{p}{L}\right)^2 = \frac{E^2 - m^2c^4}{c^2L^2} = \frac{\left(\gamma^2 - 1\right)m^2c^4}{c^2L^2} \quad -(39)$$

so the square of p is:

$$p^2 = (\gamma^2 - 1) m^2 c^2 \quad -(40)$$

Therefore the orbital equation (35) is:

$$v^2 = \left(\frac{dr}{dt}\right)^2 \left(1 + \left(\left(\frac{rp}{L}\right)^2 - 1\right)^{-1}\right) \quad -(41)$$

In the solar system the observed orbit is a precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(42)$$

where x is the precession constant. From Eq. (42):

$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{x\epsilon}{\alpha}\right)^2 r^4 \sin(x\theta) \quad -(43)$$

and the orbital equation (38) becomes:

$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^2}\left(1 + r^2 \left(\frac{x\epsilon}{\alpha}\right)^2 \sin^2(x\theta)\right). \quad -(44)$$

The orbit is described in terms of the ratio of p to L.

3. CONSTRAINED MINKOWSKI METRIC AND APPLICATIONS

An equivalent method may be developed by noting that:

$$d\theta^2 = \left(\frac{\alpha}{x\epsilon r^2 \sin(x\theta)}\right)^2 dr^2 \quad -(45)$$

where:

$$\sin^2(x\theta) = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(46)$$

Using Eq. (45) in Eq. (1) produces the constrained Minkowski line element:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - A\,dr^2 \quad -(47)$$

where:

$$A = 1 + \left(\frac{\alpha}{x\epsilon r \sin(x\theta)}\right)^2 \quad -(48)$$

By definition:

$$\underline{dr} \cdot \underline{dr} = A\,dr^2 = v^2 dt^2 \quad -(49)$$

so the velocity is:

$$v = A^{1/2}\frac{dr}{dt} \quad -(50)$$

and is not constant self consistently.

The constant of motion from Eq. (47) is defined by:

$$\mathcal{L} = H = \frac{1}{2}mc^2 = \frac{1}{2}mc^2\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2}mA\left(\frac{dr}{d\tau}\right)^2 \quad -(51)$$

and Eq. (51) is:

$$mA\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{mc^2} - mc^2 \quad -(52)$$

Defining the relativistic momentum by:

$$p = A^{1/2}m\frac{dr}{d\tau} \quad -(53)$$

Eq. (52) becomes:

$$E^2 = p^2c^2 + m^2c^4 \quad -(54)$$

which has the same format as the Einstein energy equation. It may be expressed as:

$$p^\mu p_\mu = m^2c^2 \quad -(55)$$

but is considered to be an equation of general relativity because v is not constant. If the general orbit is defined by the function:

$$f'(\theta) = \frac{dr}{d\theta} \quad -(56)$$

then the general A is:

$$A = 1 + \left(\frac{r}{f'(\theta)}\right)^2 \quad -(57)$$

In this representation the general orbital equation is:

$$\frac{dr}{d\theta} = r\left(A-1\right)^{-1/2} \quad -(58)$$

From Eq. (52):

$$\left(f'(\theta)\right)^2 = \frac{r^4}{A}\left(\frac{E^2 - m^2c^4}{c^2L^2}\right) -(59)$$

so:

$$\frac{E^2 - m^2c^4}{c^2L^2} = \left(\frac{A}{A-1}\right)\frac{1}{r^2} \quad -(60)$$

Therefore for all orbits:

$$\left(\frac{rp}{L}\right)^2 = \frac{A}{A-1} \quad -(61)$$

These results can be illustrated for spiral orbits as in Table 1.

Table 1: Results for Spiral Orbits

Spiral	Orbit	dr / dθ	A
Logarithmic	$r = r_0 e^{\alpha\theta}$	αr	$1 + \frac{1}{\alpha^2}$
Hyperbolic	$r = \frac{r_0}{\theta}$	r^2 / r_0	$1 + (r_0 / r)^2$
Archimedes	$r = a + b\theta$	b	$1 + (r / b)^2$
Fermat	$r = r_0 \theta^{1/2}$	$\frac{1}{2}(r_0^2 / r)$	$1 + (2r^2 / r_0^2)^2$
Lituus	$r = \frac{r_0}{\theta^{1/2}}$	$\frac{r^3}{2 r_0^2}$	$1 + \left(2\frac{r_0^2}{r^2}\right)^2$

Each orbit is described by the equations:

$$a = \frac{L}{mc}, \quad b = \frac{cL}{E} \quad -(62)$$

so that:

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{E^2 - m^2c^4}{c^2L^2}. \quad -(63)$$

In these equations the linear relativistic momentum is:

$$p = A^{1/2}\gamma m \frac{dr}{dt} \quad -(64)$$

and the angular relativistic momentum is:

$$L = \gamma m r^2 \omega \frac{dr}{dt}, \quad \omega = \frac{d\theta}{dt}, \quad -(65)$$

so that:

$$\left(\frac{pr}{L}\right)^2 = \frac{A}{r^2}\left(\frac{dr}{d\theta}\right)^2 = \frac{A}{A-1} \quad -(66).$$

giving the orbital equation in the format:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2}{A-1}. \quad -(67)$$

In terms of the ratio of p to L, various orbits can be described as follows. The precessing elliptical orbit is described by:

$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^2} + \left(\frac{x\epsilon}{\alpha}\sin(x\theta)\right)^2. \quad -(68)$$

The logarithmic spiral orbit is described by:

$$\left(\frac{p}{L}\right)^2 = \frac{1+\alpha^2}{r^2} \quad -(69)$$

The hyperbolic spiral orbit is described by:

$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^2} + \frac{1}{r_0^2} \quad -(70)$$

and the Newtonian ellipse by:

$$\left(\frac{p}{L}\right)^2 = \frac{2mT}{L^2}. \quad -(71)$$

Using the orbital equation in the form (35), the logarithmic spiral orbit is described by:

$$v^2 = \left(\frac{dr}{dt}\right)^2 \left(1 + \frac{1}{\alpha^2}\right) \quad -(72)$$

and the hyperbolic spiral by:

$$v^2 = \left(\frac{dr}{dt}\right)^2 \left(1 + \left(\frac{r_0}{r}\right)^2\right). \quad -(73)$$

In the limit of infinite r this equation becomes:

$$v^2 \longrightarrow \frac{dr}{dt} \quad -(74)$$

i.e. becomes an equation of special relativity in which v is constant. This result can be seen from the fact that A for the hyperbolic spiral becomes unity for infinite r. This is exactly what is observed experimentally in the well known velocity curve of a whirlpool galaxy, in which the stars are arranged in a hyperbolic spiral.

Finally, the constrained metric (47) may be used to calculate the tetrads, torsion and curvature of Cartan geometry, showing that it is indeed a metric of general relativity. It is preferred to the more complicated line element (9) by Ockham's Razor.

ACKNOWLEDGMENTS

The British Government is thanked for a Civil List Pension and rank of Armiger for MWE, and the staff of AIAS and others for many interesting discussions. Dave Burleigh is thanked for voluntary posting, Alex Hill, Robert Cheshire and Simon Clifford for translation and broadcasting. The AIAS is established under the aegis of the Newlands Family Trust.

REFERENCES

{1} M. W. Evans, Ed., J. Found. Phys. Chem., (June 2011 onwards in six issues a year, www.cisp-publishing.com).

{2} M. W. Evans, S. Crothers, H. Eckardt and K. Pendergast, "Criticisms of the Einstein Field Equation" (Cambridge International Science Publishing, CISP, www.cisp-publishing.com, 2011).

{3} The ECE websites, www.webarchive.org.uk, www.aias.us, www.atomicprecision.com, www.et3m.net, www.upitec.org).

{4} L. Felker, "The Evans Equations of Unified Field Theory" (Abramis Academic, 2007).

{5} M. W. Evans, H. Eckardt and D. W. Lindstrom, "Generally Covariant Unified Field Theory" (Abramis, 2005 - 2011) in seven volumes.

{6} K. Pendergast, "The Life of Myron Evans" (CISP, 2011).

{7} M. W. Evans and S. Kielich (Eds.), "Modern Nonlinear Optics" (Wiley, New York, 1992, 1993, 1997, 2001), in two editions and six volumes.

{8} M. W. Evans and L. B. Crowell, "Classical dn Quantum Electrodynamics and the B(3) Field" (World Scientific, 2001).

{9} M .W. Evans and J.-P Vigier, "The Enigmatic Photon (Kluwer 1994 to 2002, softback and hardback) in ten volumes.

{10} M . W. Evans and A. A. Hasanein, "The Photomagneton in Quantum Field Theory"

(World Scientific 1994).

{11} S. P. Carroll, "Spacetime and Geometry: an Introduction to General Relativity" (Addison Wesley, New York, 2004).

{12} J. B. Marion and S. T. Thornton, "Classical Dynamics of Particles and Systems" (HBC College Publishing, New York, third edition, 1988).