PROOF OF THE ANTISYMMETRY OF THE CHRISTOFFEL CONNECTION FROM THE CARTAN IDENTITY

by

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ABSTRACT

By considerations of the Cartan identity of differential geometry, it is proven conclusively that the Christoffel connection is always antisymmetric for non zero torsion and curvature. This proof overturns a century of dogma which asserted arbitrarily that the Christoffel connection is symmetric. The Einsteinian general relativity is based on this dogma and is therefore incorrect and obsolete. The Einstein Cartan Evans theory uses the correct antisymmetry and non zero spacetime torsion.

Keywords : ECE theory, Cartan identity, proof of the antisymmetry of the Christoffel connection, non-zero torsion.

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1. INTRODUCTION

In recent papers of this series {1 - 10} the Einsteinian general relativity (EGR) has been refuted conclusively using simple algebra, as in UFT202 on www.aias.us. The EGR was based on a century of dogma that originated in incorrect geometry. The precise point at which the mathematics took a wrong turn was the allocation of an incorrect symmetry to the Christoffel connection {11}. In the textbooks of the twentieth century the EGR is almost always described in terms of a connection that is symmetric in its lower two indices. The connection was introduced by Christoffel in the eighteen sixties. Riemann inferred only the metric. At around the turn of the twentieth century Levi Civita, Ricci and co workers inferred the curvature tensor. The latter is almost always mis-attributed to Riemann. All the geometry of EGR was based on a symmetric connection, which came to be taken as axiomatic. For example the first and second Bianchi identities of the EGR geometry are true if and only if the connection is symmetric. Einstein based his field equation directly on the second Bianchi identity {1 - 11}, so the field equation is true if and only if the connection is symmetric. All inferences based on the field equation are true if and only if the connection is symmetric. All solutions of the equation are true if and only if the connection is symmetric.

Einstein's major contribution to physics was the idea that it can be based on a type of geometry that is more general than that of the ancient philosophers such as Euclid. It was natural at the time of development of the Einstein field equation (about 1905 to 1915) to use the then new geometry of Levi-Civita, Ricci, Bianchi and others, a geometry based on a symmetric connection. These ideas were conveyed to Einstein by the mathematician Grossman. In historical retrospect it is not clear why the connection was assumed to be symmetric. There is no really convincing argument. The field equation was published in November 1915 and immediately criticised by Schwarzschild in December 1915 {12}.

Shortly thereafter it was criticised by other leading intellectuals such as Bauer and Schroedinger. Unfortunately Eddington and colleagues claimed incorrectly in about 1919 to have verified the theory by observations of light bending, but it was rejected by many others and continued to be rejected. There has always been an uneasy feeling about EGR within intellectual circles competent enough to understand it. Critics later included Levi-Civita, Dirac and Eddington himself.

In the early nineteen twenties Cartan {1 - 11} made several important contributions to mathematics, notably the Cartan identity of differential geometry. This is an exact identity that involves curvature, and spacetime torsion. The latter is missing completely from EGR. In Section 2 the Cartan identity is summarized in different kinds of notation and written out fully in tensor notation. In Section 3 it is proven conclusively that the Cartan identity implies that the Christoffel connection must be antisymmetric in its lower two indices, otherwise the geometry collapses into Minkowski geometry in which the curvature and torsion both vanish. This means that EGR is meaningless and obsolete. The proof is not technically difficult and for scientists means the end of the Einstein era. It shows that the various other arguments for an antisymmetric connection {1 - 10} are correct.

2. THE CARTAN IDENTITY

In the shorthand notation of the ECE series of papers {1 - 10} the identity is simple:

$$D \wedge T := R \wedge q \quad (1)$$

Here D ^ is the exterior derivative with an added spin connection term {11}, T denotes torsion, R denotes curvature, q denotes tetrad and ^ denotes the wedge product. These ideas were inferred mainly by Cartan, and his colleague Maurer, about seven or eight years after

EGR had been developed without considering T. Cartan informed Einstein of the basic incorrectness of his geometry, but by then Einstein had been catalysed into fame, and had become an idol of the cave. This seems to be the only way to explain why Einstein ignored Cartan's reasoning. There ensued a catastrophic era for physics, an era in which it degenerated into the endless dogma of EGR.

In the notation of differential geometry, also inferred mainly by Cartan, the identity is:

$$D \wedge T^{a} := R^{a}{}_{b} \wedge q^{b} \quad -(2)$$

When the wedge products are written out fully, eq. (2) becomes:

$$\partial_\mu T^{a}_{\nu\rho} + \partial_\rho T^{a}_{\mu\nu} + \partial_\nu T^{a}_{\rho\mu} + \omega^{a}_{\mu b} T^{b}_{\nu\rho} + \omega^{a}_{\rho b} T^{b}_{\mu\nu} + \omega^{a}_{\nu b} T^{b}_{\rho\mu} := R^{a}_{\mu\nu\rho} + R^{a}_{\rho\mu\nu} + R^{a}_{\nu\rho\mu} \quad -(3)$$

in which the indices appear in full {1 - 11}. In eq. (3), $\omega^{a}_{\mu b}$ is Cartan's spin connection, another major contribution to mathematics. The torsion tensor is defined by:

$$T^{a}_{\mu\rho} = \left(\Gamma^{\lambda}_{\mu\rho} - \Gamma^{\lambda}_{\rho\mu}\right) q^{a}_{\lambda} \quad -(4)$$

where $\Gamma^{\lambda}_{\mu\rho}$ is the Christoffel connection. In EGR the latter is symmetric in its lower two indices:

$$\Gamma^{\lambda}_{\mu\rho} =? \Gamma^{\lambda}_{\rho\mu} \quad -(5)$$

and in consequence the torsion vanishes and the Cartan identity becomes the obsolete and incorrect first Bianchi "identity":

$$R^{a}_{\mu\nu\rho} + R^{a}_{\rho\mu\nu} + R^{a}_{\nu\rho\mu} =? 0 \quad -(6)$$

The Leibniz Theorem {11} implies:

$$\partial_\mu\left(\left(\Gamma^{\lambda}_{\nu\rho}-\Gamma^{\lambda}_{\rho\nu}\right)q\nabla^{a}_{\lambda}\right)=q\nabla^{a}_{\lambda}\partial_\mu\left(\Gamma^{\lambda}_{\nu\rho}-\Gamma^{\lambda}_{\rho\nu}\right)+\left(\Gamma^{\lambda}_{\nu\rho}-\Gamma^{\lambda}_{\rho\nu}\right)\partial_\mu q\nabla^{a}_{\lambda} \qquad -(7)$$

and so:

$$\partial_\mu T^{a}_{\nu\rho}+\omega^{a}_{\mu b}T^{b}_{\nu\rho}=\left(\partial_\mu\Gamma^{\lambda}_{\nu\rho}-\partial_\mu\Gamma^{\lambda}_{\rho\nu}\right)q\nabla^{a}_{\lambda}+\left(\partial_\mu q\nabla^{a}_{\lambda}+\omega^{a}_{\mu b}q\nabla^{b}_{\lambda}\right)\left(\Gamma^{\lambda}_{\nu\rho}-\Gamma^{\lambda}_{\rho\nu}\right) \qquad -(8)$$

Now relabel the summation indices in the second term on the right hand side of Eq. (8):

$$\lambda\rightarrow\sigma \qquad -(9)$$

to obtain:

$$\partial_\mu T^{a}_{\nu\rho}+\omega^{a}_{\mu b}T^{b}_{\nu\rho}=\left(\partial_\mu\Gamma^{\lambda}_{\nu\rho}-\partial_\mu\Gamma^{\lambda}_{\rho\nu}\right)q\nabla^{a}_{\lambda}+\left(\partial_\mu q\nabla^{a}_{\sigma}+\omega^{a}_{\mu b}q\nabla^{b}_{\sigma}\right)\left(\Gamma^{\sigma}_{\nu\rho}-\Gamma^{\sigma}_{\rho\nu}\right) \qquad -(10)$$

Use the tetrad postulate of Cartan {1 - 11}

$$\partial_\mu q\nabla^{a}_{\sigma}+\omega^{a}_{\mu b}q\nabla^{b}_{\sigma}=\Gamma^{\lambda}_{\mu\sigma}q\nabla^{a}_{\lambda} \qquad -(11)$$

to find that:

$$\partial_\mu T^{a}_{\nu\rho}+\omega^{a}_{\mu b}T^{b}_{\nu\rho}$$

$$=\left(\partial_\mu\Gamma^{\lambda}_{\nu\rho}-\partial_\mu\Gamma^{\lambda}_{\rho\nu}+\Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho}-\Gamma^{\sigma}_{\rho\nu}\right)\right)q\nabla^{a}_{\lambda} \qquad -(12)$$

Similar results are obtained for the other terms of the cyclical sum in Eq. (3). Using the definition:

$$R^{q}_{\rho\mu\nu} = R^{\lambda}_{\rho\mu\nu} v^{q}_{\lambda} \quad -(13)$$

the Cartan identity becomes:

$$\begin{aligned} R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu} :=\ & \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu}\right) + \\ & \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\rho\sigma}\left(\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu}\right) + \\ & \partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma}\left(\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\sigma}_{\mu\rho}\right). \end{aligned} \quad -(14)$$

It is seen that the second Cartan Maurer structure equation:

$$R^{\lambda}_{\mu\nu\rho} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \quad -(15)$$

is a solution of Eq. (14). The structure equation appears three times in the Cartan identity, which is the cyclic sum of three second structure equations. This is a supremely elegant result because it is an exact identity given the second structure equation. The latter is equivalent to the definition of the curvature tensor.

3. PROOF OF THE ANTISYMMETRIC CONNECTION

Consider the identity in the format:

$$\begin{aligned} R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu} :=\ & \partial_\mu T^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho} \\ & + \partial_\rho T^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\rho\sigma} T^{\sigma}_{\mu\nu} \\ & + \partial_\nu T^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\nu\sigma} T^{\sigma}_{\rho\mu}. \end{aligned} \quad -(16)$$

It it seen that the following equation is a solution

$$R^{\lambda}_{\mu\nu\rho} = \partial_\mu T^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho} \quad -(17)$$

and so are cyclic permutations of this equation, i.e.

$$R^{\lambda}_{\rho\mu\nu} = \partial_{\rho} T^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\rho\sigma} T^{\sigma}_{\mu\nu} \quad -(18)$$

and:

$$R^{\lambda}_{\nu\rho\mu} = \partial_{\nu} T^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\nu\sigma} T^{\sigma}_{\rho\mu} \quad -(19)$$

These solutions show that the connection is antisymmetric, otherwise both the torsion and curvature vanish and the space collapses to a flat space. Now raise indices on both sides to obtain an equation such as:

$$\partial_{\mu} T^{\lambda\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma\nu\rho} = R^{\lambda\ \ \nu\rho}_{\mu} \quad -(20)$$

Finally consider the special case:

$$\partial_{\mu} T^{\lambda\mu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma\mu\rho} = R^{\lambda\ \ \mu\rho}_{\mu} \quad -(21)$$

This is the Evans identity, Q.E.D. {1 - 10}. Eq. (21) is true for each index μ, so the repeated indices μ can be summed. The Evans identity has been proven independently in previous work {1 - 10} using Hodge duals.

It is well known {1 - 11} that the commutator of covariant derivatives acts on a vector V^{ρ} to produce:

$$[D_{\mu}, D_{\nu}] V^{\rho} = R^{\rho}_{\ \sigma\mu\nu} V^{\sigma} - T^{\lambda}_{\mu\nu} D_{\lambda} V^{\rho} \quad -(22)$$

If

$$\mu = \nu \quad -(23)$$

the commutator becomes the null operator and the connection, torsion and curvature all

vanish. The connection must be antisymmetric for non-zero torsion and curvature, Q.E.D. The connection cannot have a symmetric component. Under the general coordinate transformation an inhomogeneous term appears in the connection. This inhomogeneous term is however symmetric in its lower two indices and vanishes. The connection is therefore a tensor. In EGR it was not considered to be a tensor because of the inhomogeneous term.

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