THREE DIMENSIONAL ORBITS FROM ECE THEORY

by

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ABSTRACT

The three dimensional orbit from the inverse square law of attraction is analyzed in spherical polar coordinates. The general solution is shown to be the beta ellipse, which is equivalent to precessing ellipses in the angles ϕ and θ of the spherical polar coordinates. The resulting orbits are graphed in spherical polar representation and equations given for their animation. In general, the orbit of a mass m attracted to a mass M by an inverse square law is three dimensional. The theory applies unchanged to the three dimensional classical orbit of an electron around a proton. This quantizes to the well known orbitals of quantum mechanics. Eckardt quantization occurs when the precession constant is an integer. This theory produces three dimensional fractal conical sections in mathematics.

Keywords: ECE theory, three dimensional orbits, graphics and animation.

UFT 269

1. INTRODUCTION

In recent papers of this series {1 - 10} the x theory has been developed from ECE theory and applied to several phenomena self consistently. To date the x theory has been developed in terms of planar ellipses and a great deal of new information obtained. In general however the orbit due to the inverse square law of attraction is three dimensional. There is no reason why an orbit should be planar, and planar orbits evolve from early three dimensional orbits. In order to analyze three dimensional orbits the spherical polar coordinates are used in Section 2 to develop the theory. The angular momenta are evaluated from basic first principles of geometry, the spherical polar coordinates being a special case of Cartan geometry on which ECE theory is based directly. This procedure defines the angular momentum components. A lagrangian analysis is used to find the constants of motion of the system in terms of the angular momenta and the general solution shown to be the beta ellipse where beta is defined in terms of the angles ϕ and θ of the spherical polar coordinate system. It is shown that the beta ellipse is precisely equivalent to precessing ellipses in the two polar angles. The orbits are graphed in Section 3 as spherical polar plots. Two other solutions are analyzed in terms of the Z component of the hamiltonian and the component of the hamiltonian defined by $L^2 - L_Z^2$, where L is the total angular momentum and L_Z its Z component. These are the components used in quantum mechanics as is well known. These orbits are also graphed and analyzed in Section 3. In general the precessing three dimensional ellipses evolve into three dimensional fractal conical sections with many interesting properties. This formalism is fundamental, so is applicable throughout mathematics, physics and astronomy. It characterizes three dimensional orbits in general. Equations are given for the time evolution of the three dimensional orbit in preparation for animation. The animation would give the trajectory in three dimensions of m around M as governed by the inverse square law of attraction. The same theory exactly applies to the classical three dimensional motion of an electron around a proton attracted by the Coulombic inverse square law of attraction. This classical three dimensional motion quantizes to orbitals as is well known. The Eckardt orbits are defined when the precession constants are integral, and this type of orbit is also graphed in Section 3. The two dimensional Eckardt quantization leads to an ellipse with superimposed de Broglie wave structure akin to Bohr quantization.

2. SOLUTION FOR THREE DIMENSIONAL ORBITS.

Consider the angular momentum vector in three dimensions {11, 12}:

$$\underline{L} = \underline{r} \times \underline{p} \qquad -(1)$$

where $\underline{r}$ is the position vector and $\underline{p}$ the linear momentum vector. The angular momentum vector is conserved:

$$\frac{d\underline{L}}{dt} = \underline{0}. \qquad -(2)$$

In spherical polar coordinates {11}:

$$\underline{r} = r\sin\theta\cos\phi\,\underline{i} + r\sin\theta\sin\phi\,\underline{j} + r\cos\theta\,\underline{k} \qquad -(3)$$

and:

$$\begin{aligned}\underline{p} = m\Big(&(\dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi})\,\underline{i} \\ &+ (\dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi})\,\underline{j} \\ &+ (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\,\underline{k}\Big) \qquad -(4)\end{aligned}$$

Therefore the Cartesian components of angular momentum are:

$$L_X = -mr^2\left(\dot{\theta}\sin\phi + \dot{\phi}\sin\theta\cos\theta\cos\phi\right) \qquad -(5)$$

$$L_Y = mr^2\left(\dot{\theta}\cos\phi - \dot{\phi}\sin\theta\cos\theta\sin\phi\right) \qquad -(6)$$

and

$$L_z = mr^2 \sin^2\theta \, \dot{\phi}. \quad -(7)$$

Each Cartesian component is conserved and is a constant of motion:

$$\frac{dL_x}{dt} = \frac{dL_y}{dt} = \frac{dL_z}{dt} = 0. \quad -(8)$$

The total angular momentum is:

$$L^2 = L_x^2 + L_y^2 + L_z^2 = m^2 r^4 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right) \quad -(9)$$

in which:

$$L^2 = m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta + \sin^4\theta \, \dot{\phi}^2\right)$$
$$= m^2 r^4 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right) \quad -(10)$$

and:

$$L_x^2 + L_y^2 = L^2 - L_z^2$$
$$= m^2 r^4 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \cos^2\theta\right) \quad -(11)$$

and these are also conserved.

The hamiltonian is:

$$H = \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right)\right) - \frac{k}{r}$$
$$= \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r} \quad -(12)$$

and the lagrangian is:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right)\right) + \frac{k}{r} \quad -(13)$$

The angle beta is defined by:

$$\dot{\beta}^2 = \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \quad -(14)$$

so the hamiltonian becomes:

$$H = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) - \frac{k}{r} \quad -(15)$$

The solution of Eq. (15) is the three dimensional function {1 - 10, 12}:

$$r = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(16)$$

which has the structure of an ellipse or conical section in general {12}. The half right latitide and ellipticity are also three dimensional functions defined by

$$\alpha = \frac{L^2}{mk}, \quad \epsilon^2 = 1 + \frac{2EL^2}{mk^2}. \quad -(17)$$

In planetary motion of a mass m around a mass M the constant k is defined by:

$$k = mMG \quad -(18)$$

where G is Newton's constant. In Coulombic dynamics:

$$k = \frac{e^2}{4\pi\epsilon_0} \quad -(19)$$

where e is the proton charge and ϵ_0 the vacuum permittivity. The total energy E is defined by the experimentally measurable semi major axis of the three dimensional ellipse:

$$a = \frac{\alpha}{1 - \epsilon^2} = \frac{k}{2E} \quad -(20)$$

The Binet equation of the beta ellipse is {1 - 10, 12}:

$$F = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right)+\frac{1}{r}\right) \quad -(21)$$

For an inverse square law of attraction:

$$F = m\ddot{r} - \frac{L^2}{mr^3} = -\frac{k}{r^2} \quad -(22)$$

so for an ellipse:

$$F = -\frac{L^2}{m\alpha r^2} = -\frac{k}{r^2} \quad -(23)$$

self consistently. For a precessing beta ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\beta)} \quad -(24)$$

and the force law from the Binet equation becomes:

$$F = -x^2\frac{k}{r^2} + (x^2-1)\frac{L^2}{mr^3} \quad -(25)$$

giving the potential:

$$V = -x^2\frac{k}{r^2} + (x^2-1)\frac{L^2}{2mr^2} \quad -(26)$$

The hamiltonian is changed from:

$$H = \frac{p^2}{2m} - \frac{k}{r} \quad -(27)$$

to:

$$H = \frac{p^2}{2m} - x^2\frac{k}{r} + (x^2-1)\frac{L^2}{2mr^2} \quad -(28)$$

Upon quantization the energy levels of the H atom are changed to:

$$E = \langle H \rangle = \left\langle \frac{p^2}{2m} \right\rangle - x^2 \left\langle \frac{k}{r} \right\rangle + \frac{(x^2-1)}{2m} \left\langle \frac{L^2}{r^2} \right\rangle \quad -(29)$$

where the expectation values are:

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{me^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(30)$$

$$\left\langle \frac{k}{r} \right\rangle = \frac{-me^4}{16\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(31)$$

$$\langle L^2 \rangle = l(l+1)\hbar^2 \quad -(32)$$

So the new energy levels are:

$$E = \frac{(1-2x^2)me^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} + (x^2-1)\frac{\hbar^2}{2m} l(l+1) \left\langle \frac{1}{r^2} \right\rangle. \quad -(33)$$

As described in detail in Note 269(3) the x factor can be defined as follows by comparison with the fermion equation's spin orbit coupling hamiltonian:

$$(x^2-1)\left(\frac{me^4}{16\pi^2 \epsilon_0^2 \hbar^2 n^2} + l(l+1)\frac{\hbar^2}{2m} \left\langle \frac{1}{r^2} \right\rangle \right) = \frac{e^2 \hbar^2}{16\pi \epsilon_0 m^2 c^2 r_{B0}^2} \left(\frac{j(j+1) - l(l+1) - s(s+1)}{n^3 l(l+\frac{1}{2})(l+1)} \right) \quad -(34)$$

where:

$$r_{B0} = \frac{4\pi \epsilon_0 \hbar^2}{me^2}. \quad -(35)$$

Here n is the main quantum number, j is the total angular momentum quantum number:

$$j = \ell + s, \ell + s - 1, \ldots, |\ell - s| \quad -(36)$$

and s is the spin quantum number.

The classical beta lagrangian is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\beta}^2\right) + \frac{k}{r}. \quad -(37)$$

The Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) = 0 \quad -(38)$$

gives the conserved total classical angular momentum

$$L = mr^2\dot{\beta} \quad -(39)$$

so:

$$\frac{d\beta}{dt} = \frac{L}{mr^2} = \frac{L}{m\alpha^2}\left(1 + \epsilon\cos\beta\right)^2 \quad -(40)$$

and

$$t = \frac{m\alpha^2}{L}\int \frac{d\beta}{(1+\epsilon\cos\beta)^2}. \quad -(41)$$

This equation can be animated to give the trajectory r or beta as a function of time. The integral is analytical and is given by computer algebra or by the expressions in notes 269(6) and 269(7).

The Euler Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0 \quad -(42)$$

gives

$$mr^2 \sin\theta \cos\theta \dot{\phi}^2 = \frac{d}{dt}(mr^2 \dot{\theta}) \quad -(43)$$

so:

$$L^2 = L_\theta^2 + m^2 r^4 \dot{\phi}^2 \sin^4\theta \quad -(44)$$

where

$$L_\phi = mr^2 \dot{\phi} \sin^2\theta. \quad -(45)$$

These results are the same as those given earlier in this section from basic fundamentals of geometry. So the analysis is correct and self consistent.

It follows that:

$$\frac{d\beta}{dt} = \frac{L}{mr^2}, \quad \frac{d\phi}{dt} = \frac{L_\phi}{mr^2 \sin^2\theta} \quad -(46)$$

and that:

$$\frac{d\beta}{d\phi} = \frac{L}{L_\phi} \sin^2\theta. \quad -(47)$$

So:

$$\beta = \frac{L}{L_\phi} \phi \sin^2\theta \quad -(48)$$

and the integration of Eq. (14) has been achieved. We arrive at the remarkable result that the beta ellipse is precisely equivalent to a precessing ellipses in ϕ :

$$r = \frac{\alpha}{1 + \epsilon \cos\left(\frac{L}{L_\phi} \sin^2\theta \, \phi\right)} = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(49)$$

In general therefore three dimensional orbits precess on the classical non relativistic level.

The Eckardt ellipses are defined by:

$$\frac{L}{L_\phi} \sin^2\theta = n = 1, 2, \ldots \quad -(50)$$

The trajectories of the three dimensional orbits are given by:

$$t = \frac{m\alpha^2}{L} \int \frac{d\beta}{(1+\epsilon\cos\beta)^2} \quad -(51)$$

i. e. :

$$L = mr^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right)^{1/2} \quad -(52)$$

$$= \text{constant}$$

and

$$t = \frac{m\alpha^2}{L} \int \frac{d\phi}{\left(1+\epsilon\cos\left(\frac{L}{L_\phi}\sin^2\theta\,\phi\right)\right)^2} \quad -(53)$$

and the trajectories can be animated as functions of the constants:

$$L \text{ and } L_\phi = L_Z \quad -(54)$$

defined by:

$$L^2 = L_\theta^2 + L_\phi^2 = L_\theta^2 + L_Z^2 \quad -(55)$$

The above is a complete solution for any three dimensional orbit, and reduction to a two dimensional orbit occurs when:

$$\theta = \frac{\pi}{2}, \quad L_\theta = 0. \quad -(56)$$

It is also possible to graph the orbits and properties associated with the Z component of the angular momentum:

$$L_Z = mr^2 \dot{\phi} \sin^2\theta \quad -(57)$$

whose hamiltonian is:

$$H_Z = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{k}{r}. \quad -(58)$$

The solution of this hamiltonian is:

$$r = \frac{\alpha_1}{1+\epsilon_1 \cos\phi} \quad -(59)$$

where:

$$\alpha_1 = \frac{L_1^2}{mk} = \frac{L_Z^2}{mk\sin^4\theta} \quad -(60)$$

and:

$$\epsilon_1^2 = 1 + \frac{2EL_1^2}{mk^2} = 1 + \frac{2EL_Z^2}{mk^2\sin^4\theta} \quad -(61)$$

This type of ellipse, and its associated functions, are also graphed in Section 3.

A third type of ellipse can be analyzed by using the result from fundamental geometry:

$$L^2 = m^2r^4\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\cos^2\theta + \dot{\phi}^2\sin^4\theta\right) \quad -(62)$$

so:

$$L_X^2 + L_Y^2 = L^2 - L_Z^2 = m^2r^4\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\cos^2\theta\right). \quad -(63)$$

Using:

$$\dot{\phi} = \frac{L_Z}{mr^2 \sin^2\theta} \quad -(64)$$

it is found that:

$$L_X^2 + L_Y^2 = m^2 r^4 \dot{\theta}^2 + L_Z^2 \cot^2\theta. \quad -(65)$$

The complete hamiltonian:

$$H = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{k}{r} \quad -(66)$$

is expressed as:

$$H = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L_Z^2}{2mr^2} + \frac{L^2 - L_Z^2}{2mr^2} - \frac{k}{r} \quad -(67)$$

and can be analyzed in terms of L_Z^2 and $L^2 - L_Z^2$. The hamiltonian

$$H_1 = H_Z = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L_Z^2}{2mr^2} - \frac{k}{r} \quad -(68)$$

has already been analyzed. The type two hamiltonian is:

$$H_2 = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{L^2 - L_Z^2}{2mr^2} - \frac{k}{r} \quad -(69)$$

where:

$$\frac{d\theta}{dt} = \frac{\left(L^2 - L_Z^2\right)^{1/2}}{mr^2} \quad -(70)$$

and the type two ellipse is:

$$r = \frac{\alpha_2}{1 + \epsilon_2 \cos\theta} \quad -(71)$$

where:

$$\alpha_2 = \frac{1}{mk}\left(L^2 - L_Z^2\left(1+\cot^2\theta\right)\right) \quad -(72)$$

and

$$\epsilon_2^2 = 1 + \frac{2E}{mk^2}\left(L^2 - L_Z^2\left(1+\cot^2\theta\right)\right) \quad -(73)$$

These functions are also graphed and analyzed in Section 3.

In conclusion, the analysis of three dimensional orbits reveals a far richer structure than the analysis of two dimensional orbits with the same inverse square law and this opens up new subject areas in mathematics, physics and astronomy in both classical and quantum mechanics in non relativistic and relativistic theories.

3. GRAPHICAL ANALYSIS.

Section by Dr. Horst Eckardt.

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3 Graphical analysis

The analysis of three-dimensional elliptic orbits starts with the β ellipse representing the Hamiltonian (15). The angular momentum is given by L^2 and L_ϕ as constants of motion and the angular coordinates are coupled by Eq.(48). With the given θ coordinate we obtain

$$\beta = \frac{L}{L_\phi} \phi \sin^2(\theta). \tag{74}$$

This defines the elliptic surface

$$r = \frac{\alpha}{1 + \epsilon \cos\left(\frac{L}{L_\phi} \phi \sin^2(\theta)\right)}. \tag{75}$$

The used parameters for the graphs are

$$\alpha = 1, \tag{76}$$

$$\epsilon = 0.5, \tag{77}$$

$$L_\phi = 3,\ L_\phi = 0.5 \tag{78}$$

$$L = 3. \tag{79}$$

Fig. 1 shows the elliptic orbital surface for $L_\phi = 3$. The ellipsoid is opened at one side, the orbits are not closed. The same surface is graphed in Fig. 2 with $L_\phi = 0.5$. Now the ratio L/L_ϕ is larger and the orbits are much more structured to a kind of 3D spiral.

For the Eckardt quantization the factor x in

$$r = \frac{\alpha}{1 + \epsilon \cos(x\ \phi)} \tag{80}$$

has to be constant and integral, leading to the condition

$$x = \frac{L}{L_\phi} \sin^2(\theta) = n = \text{const.} \tag{81}$$

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or

$$\theta = \operatorname{asin}\left(\sqrt{\frac{L_\phi\, x}{L}}\right). \tag{82}$$

The orbital surface for $x = 3$ is graphed in Fig. 3. According to Eckardt quantization, it has a threefold symmetry which can be seen better when the surface is projected to ghe XY plane (Fig. 4). This is a three-dimensional extension of the three-fold orbit in Figs. 6 and 7 of UFT Paper 266.

The L_Z Hamiltonian (58) leads to angle-dependent ellipse parameters ϵ_1 and α_1 given by Eqs.(59-61). For an ellipse the eccentricity is required to be in the range $\epsilon_1 \geq 0$ and $\epsilon_1 < 1$. For the first condition we obtain from (61):

$$\frac{2\, {L_Z}^2\, E}{k^2\, m \sin^4(\theta)} + 1 \geq 0 \tag{83}$$

which can be rewritten to

$$\sin^4(\theta) \geq -\frac{2\, {L_Z}^2\, E}{k^2\, m}. \tag{84}$$

The energy E has to be negative. Condition (84) defineds a minimum angle θ which is demonstrated in Fig. 5. We see that for the given parameters

$$L = 4, \tag{85}$$

$$L_Z = 1, \tag{86}$$

$$E = -0.05, \tag{87}$$

$$k = m = 1 \tag{88}$$

θ is not defined below 0.60 and above 2.54, that means the orbit is constrained to an angular range of θ. This can directly be seen from the elliptic function

$$r_1 = \frac{\alpha_1}{1 + \epsilon_1 \cos(\theta)} \tag{89}$$

which is plotted in Fig. 5 too for $\phi = 0$. The ϵ_1 surface is a torus (Fig. 6), but is not smooth at the origin as shown in Fig. 7 where only a quarter circle of the ϕ coordinate has been shown. The α_1 function is graphed in Fig. 8. It is a double cone with a hole at the centre. The full surface $r_1(\theta, \phi)$ is an ellipsoid combined with a double cone at one side (Fig. 9).

The third angular momentum orbits are for $L^2 - L_Z^2 = L_X^2 + L_Y^2$. This gives a θ surface with variable ϵ_2 and α_2 (Eqs.(71-73) similar as before. The two conditions $\epsilon_1 \geq 0$ and $\epsilon_1 < 1$ now give the restrictions

$$\theta \geq \operatorname{acot}\left(\frac{\sqrt{L - L_Z}\,\sqrt{L + L_Z}}{L_Z}\right) \tag{90}$$

and

$$\theta < \operatorname{acot}\left(\frac{1}{\sqrt{2}\, L_Z}\sqrt{2\, L^2 - 2\, {L_Z}^2 + \frac{k^2\, m}{E}}\right). \tag{91}$$

The range is relatively small for the parameters given in (85-88) as can be seen in Fig. 10 where $\epsilon_2(\theta)$ and $r_2(\theta)$ are graphed for $\phi = 0$ (cf. Fig. 5). From Figs. 11-13 it is obvious that the small range of θ gives flat, rotationally symmetric structures for ϵ_2, α_2 and $r_2(\theta, \phi)$.

Finally we investigated the time function $t(\theta, \phi)$ as given by Eqs.(51-53). Choosing the ϕ representation (53), this integral takes the form

$$t = \frac{m\alpha^2}{L} \int \frac{d\phi}{(1 + \epsilon \cos(x\phi))^2} = \frac{m\alpha^2}{L\,x} \int \frac{d\phi'}{(1 + \epsilon \cos(\phi'))^2} \tag{92}$$

with

$$\phi' = x\,\phi = \frac{L}{L_\phi} \sin^2(\theta)\,\phi. \tag{93}$$

The integral is solvable analytically giving

$$t = \frac{m\,\alpha^2}{L\,x} \left(\frac{\epsilon \sin(\phi')}{(\epsilon^2 - 1)\,(\epsilon \cos(\phi') + 1)} - \frac{2\,\mathrm{atan}\left(\frac{(1-\epsilon)\tan\left(\frac{\phi'}{2}\right)}{\sqrt{1-\epsilon^2}} \right)}{\sqrt{1-\epsilon^2}\,(\epsilon^2 - 1)} \right) \tag{94}$$

The result is graphed in Fig. 14 for $L = 3$, $x = 1.1$. At $\phi = x\,\pi$ there is a jump in the time scale because of the principal values of trigonometric functions. The inverse curve $\phi(t)$ shows the typical behaviour of elliptic dynamics: Velocity of the orbiting mass is at minimum and maximum near to the focal points, for $\phi = 0$ and $\phi = \pi$.

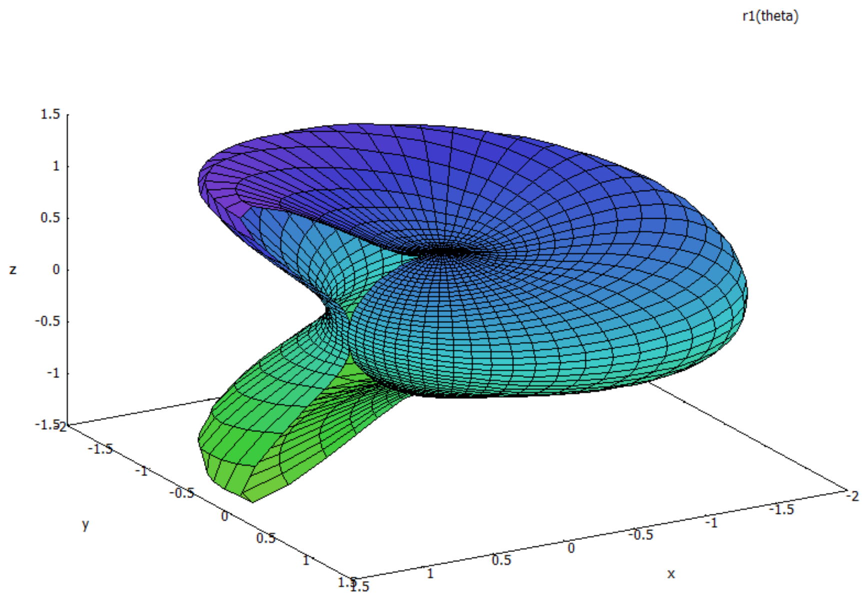

Figure 1: Orbital surface for $L = 3$, $L_\phi = 3$.

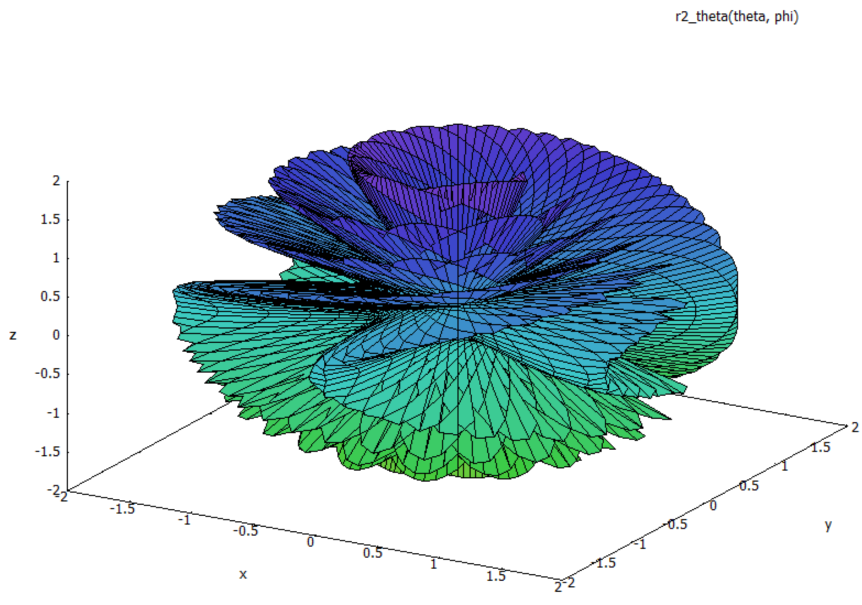

Figure 2: Orbital surface for $L = 3$, $L_\phi = 0.5$.

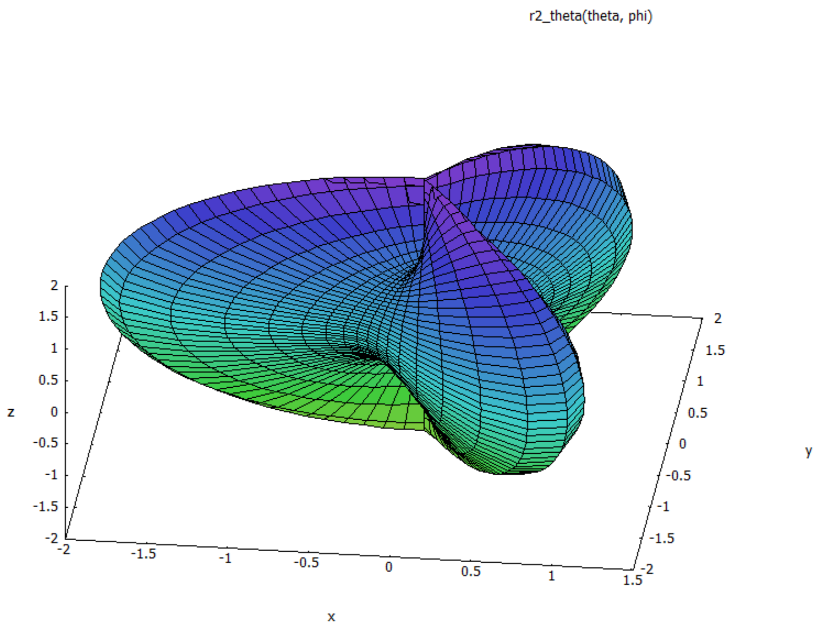

Figure 3: Orbital surface for Eckardt quantization, $x = 3$.

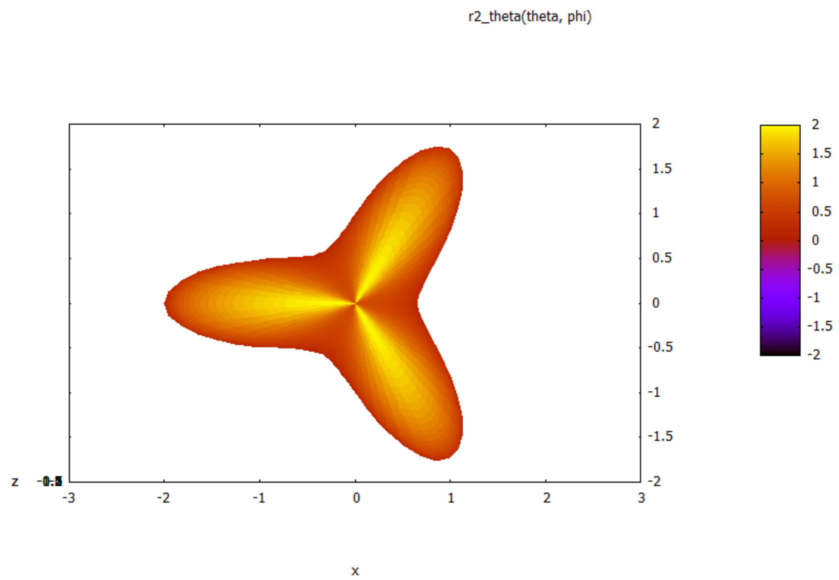

Figure 4: XY Projection of the orbital surface for Eckardt quantization, $x = 3$.

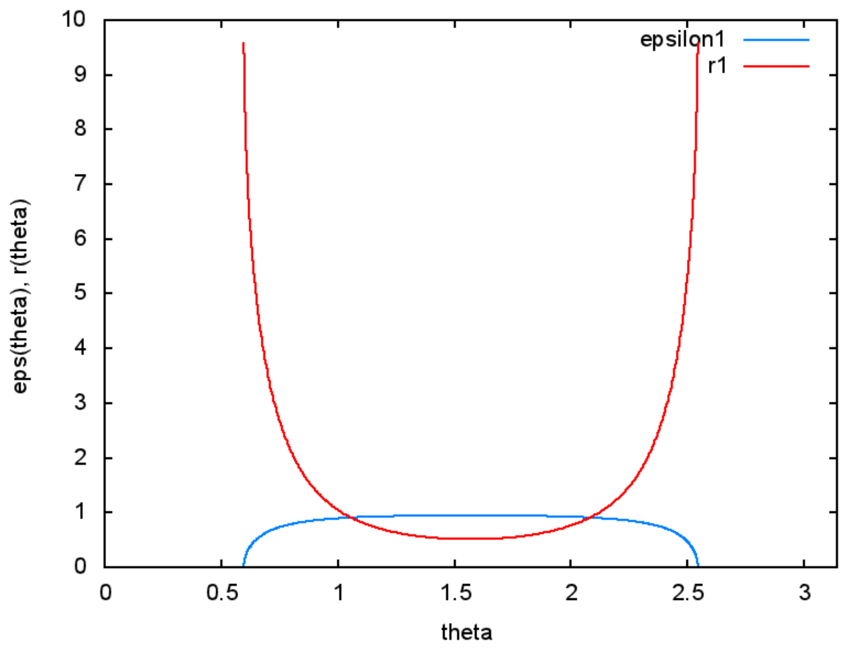

Figure 5: θ angle restriction for L_Z: $\epsilon_1(\theta)$ and $r_1(\theta)$ for $\phi = 0$.

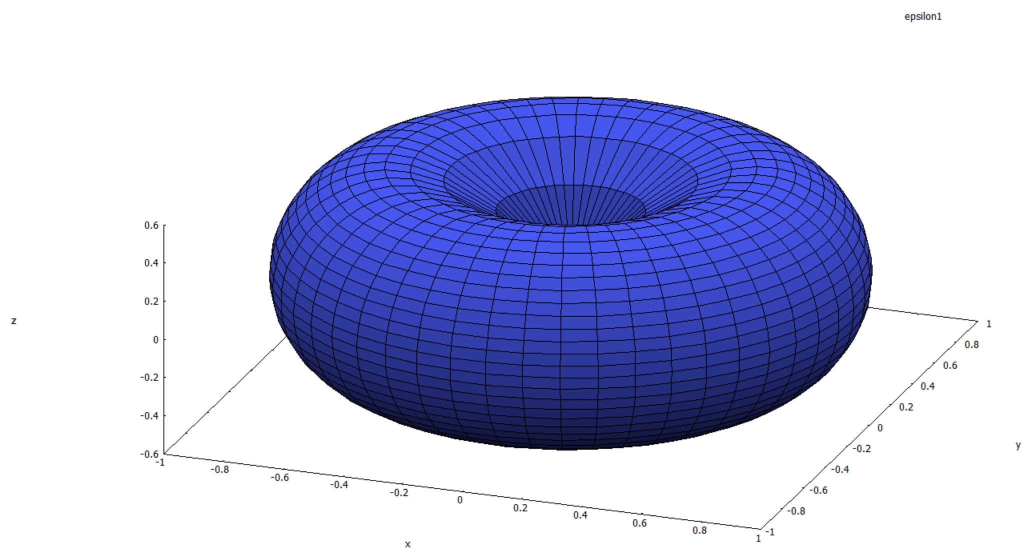

Figure 6: ϵ_1 torus for L_Z.

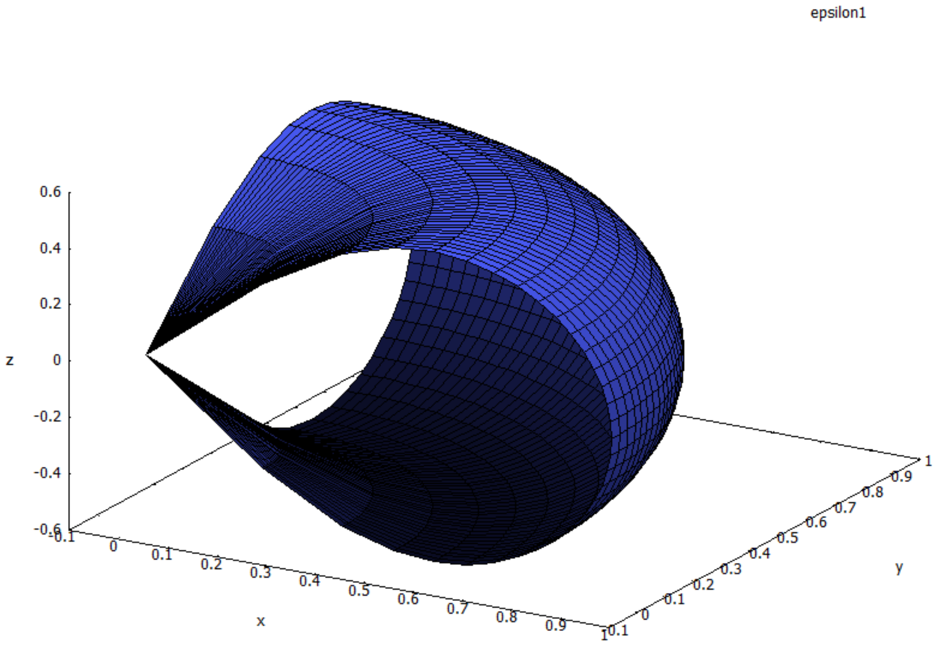

Figure 7: ϵ_1 torus for L_Z, quarter view of ϕ.

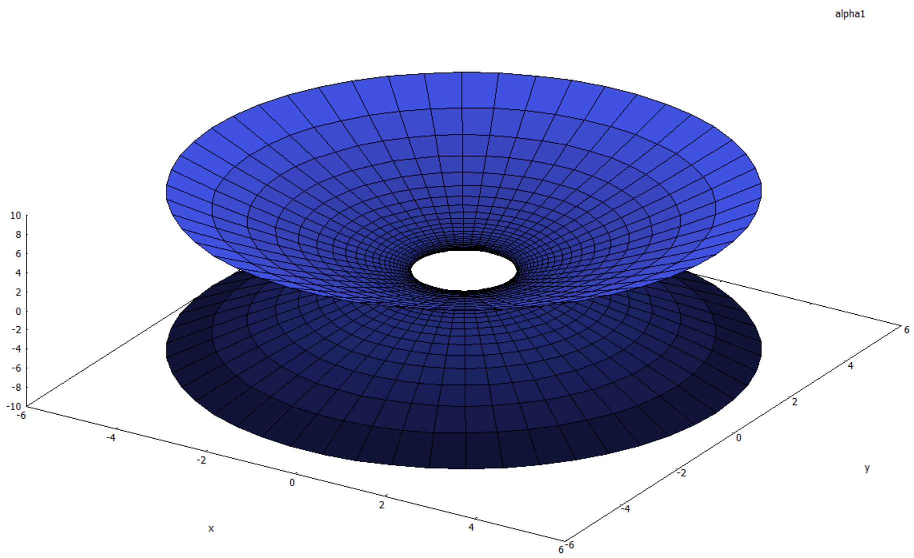

Figure 8: α_1 for L_Z.

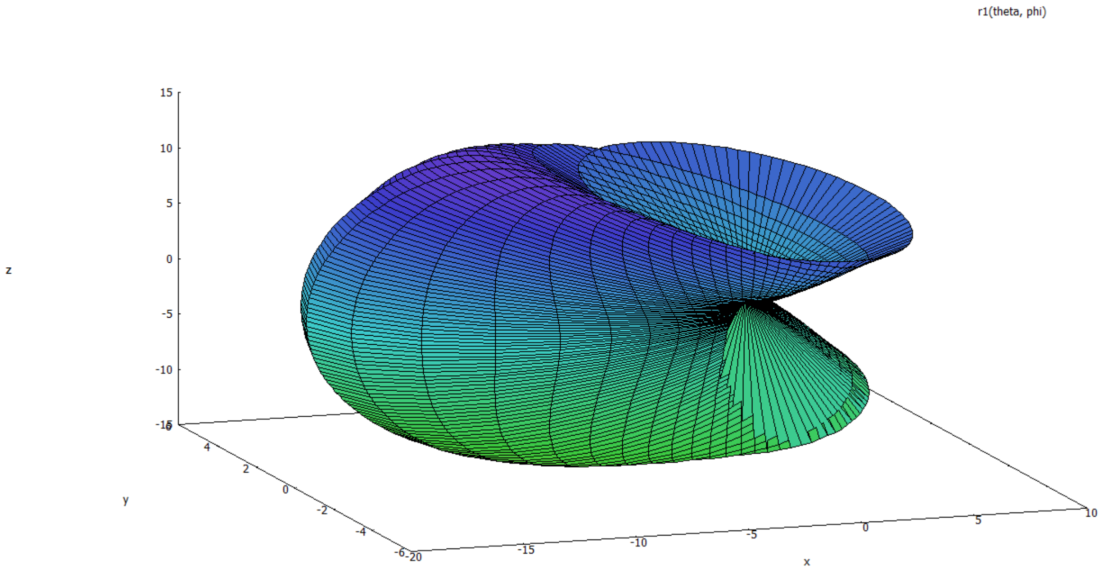

Figure 9: ϕ orbit for L_Z.

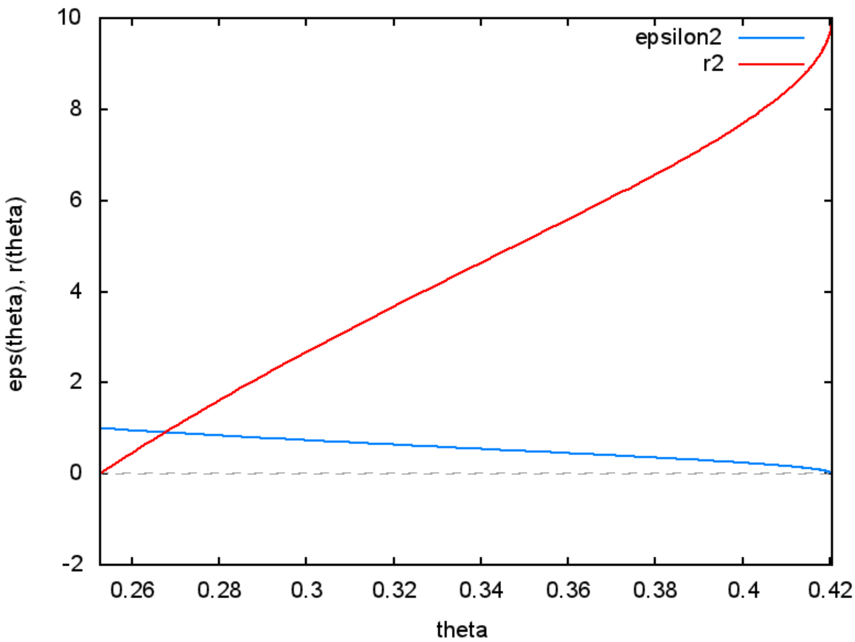

Figure 10: θ angle restriction for $L^2 - L_Z^2$: $\epsilon_2(\theta)$ and $r_2(\theta)$ for $\phi = 0$.

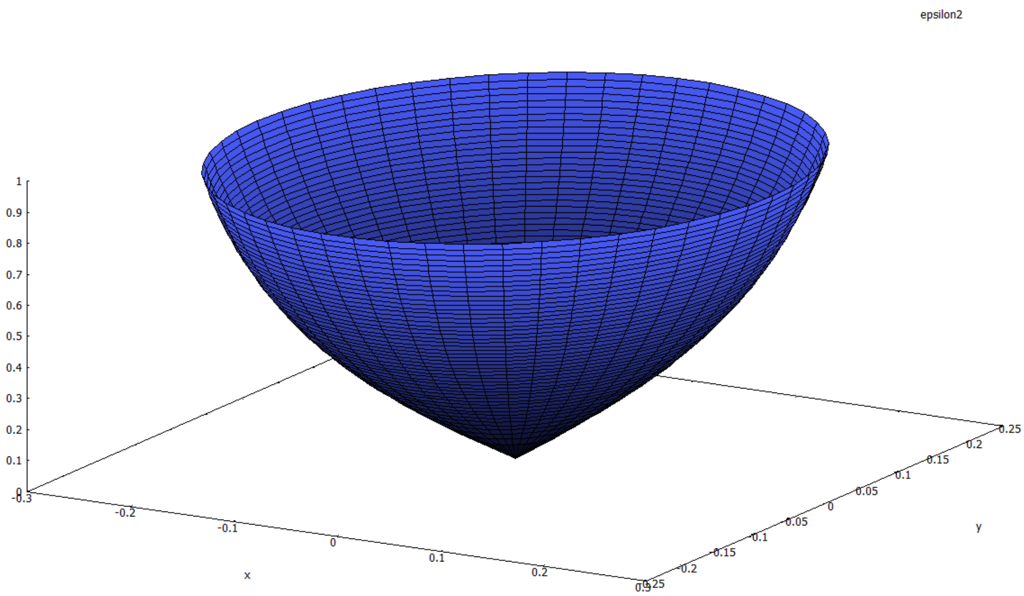

Figure 11: ϵ_2 surface for $L^2 - L_Z^2$.

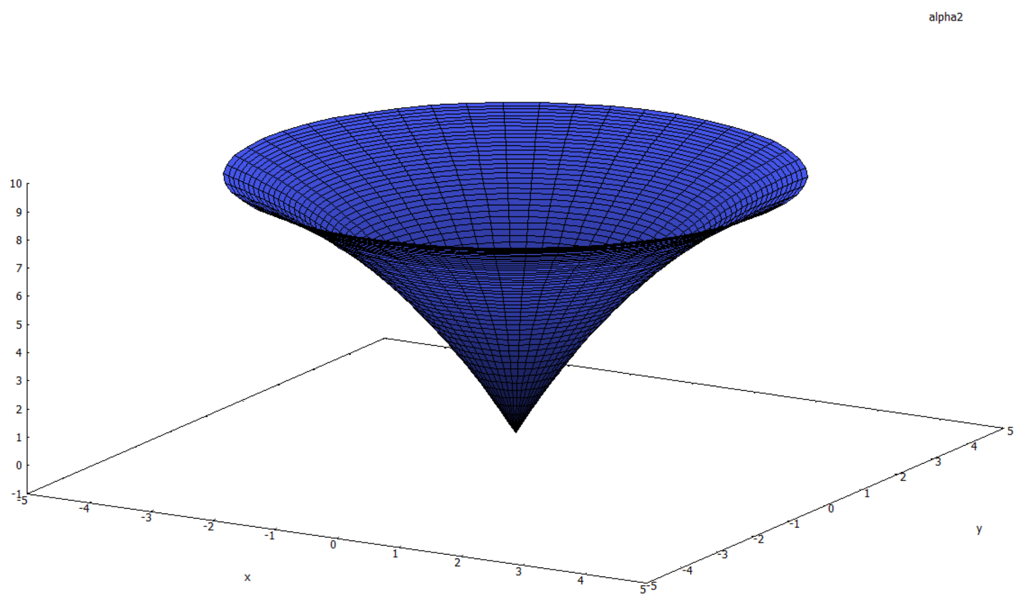

Figure 12: α_2 for $L^2 - L_Z^2$.

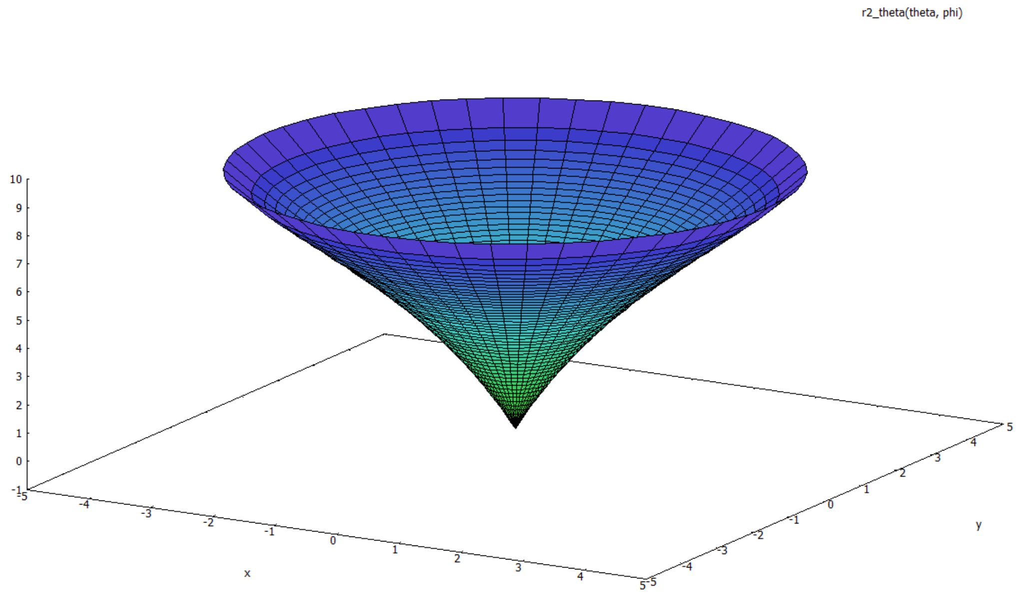

Figure 13: θ orbit for $L^2 - L_Z^2$.

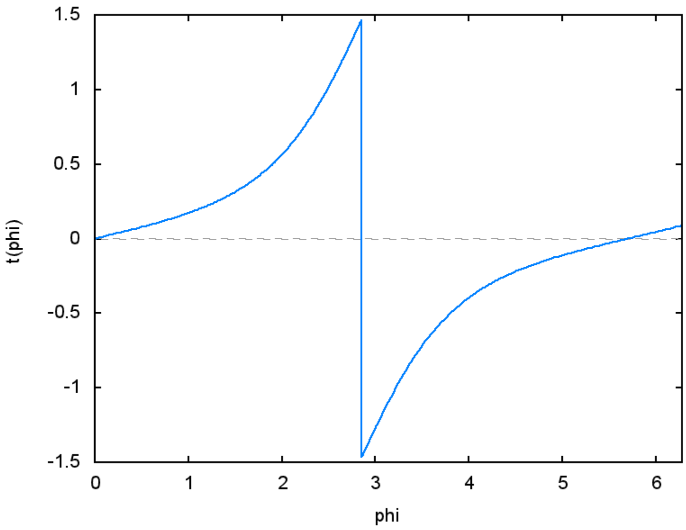

Figure 14: Time evolution of ϕ orbit.

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