Chapter 8

Resonance Solutions Of The ECE Field Equations

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Abstract

Resonance solutions of the Einstein Cartan Evans (ECE) field equations are obtained by developing them in terms of the electromagnetic potential to give linear inhomogeneous differential equations whose solutions were first discovered by the Jacobi's and Euler (1739 - 1743). There are four such resonance equations, and in a well defined approximation it is shown that resonance absorption from ECE space-time occurs. The net result is that electric power from space-time is available in copious quantities given the circuit or material design to take resonant energy from ECE space-time.

Keywords: Einstein Cartan Evans (ECE) unified field theory; resonant absorption from ECE space-time, energy from ECE space-time.

8.1 Introduction

The mathematical structure of Einstein Cartan Evans (ECE) unified field theory is that of standard differential geometry [1] – [35] within a scalar valued factor $A^{(0)}$, a vector potential magnitude. Thus, for example, the relation between the electromagnetic field form (F) and electromagnetic potential form $((A))$ is given by the first Cartan structure equation, and the field equations for F and its Hodge dual $\widetilde{F}$ are given by the first Bianchi identity. The Cartan structure equations and the Bianchi identities are standard equations of Cartan geometry. We use for clarity of mathematical structure a "barebones" or index suppressed notation [1] – [35] to give these equations as follows:

$$F = d \wedge A + \omega \wedge A, \tag{8.1}$$

$$d \wedge F = \mu_0 j, \tag{8.2}$$

$$d \wedge \widetilde{F} = \mu_0 J. \tag{8.3}$$

Here j is the homogeneous current and J the inhomogeneous current, and μ_0 is the S.I. permeability in vacuo. The symbol $\wedge$ is the wedge product, $d\wedge$ is the exterior derivative and ω is the spin connection. These quantities and notation are fully defined elsewhere [1]– [35]. The Hodge dual of Eq.8.1 is denoted:

$$\widetilde{F} = d\widetilde{\wedge}A + \omega\widetilde{\wedge}A \tag{8.4}$$

$$= d \wedge B + \omega \wedge B. \tag{8.5}$$

From Eqs.8.1 and 8.2:

$$d \wedge (d \wedge A + \omega \wedge A) = \mu_0 j \tag{8.6}$$

and from Eqs.8.3 and 8.5:

$$d \wedge (d \wedge B + \omega \wedge B) = \mu_0 J. \tag{8.7}$$

Eq.8.6 is the fundamental resonance equation of ECE field theory and Eq.8.7 is its Hodge dual. Eq.8.6 is a development of the well known linear inhomogeneous equation whose resonance solutions [36] were first given by the Bernoulli's and Euler (1739 - 1743). In general such equations give amplitude resonance, potential and kinetic energy resonance, Q factors, transient and equilibrium solutions, phase lags and other features of interest in many aspects of physics and electrical engineering, notably circuit theory [36]. In Eq.8.6:

$$j = \frac{A^{(0)}}{\mu_0} (R \wedge q - \omega \wedge T) \tag{8.8}$$

where

$$T = d \wedge q + \omega \wedge q \tag{8.9}$$

is the torsion form [1]– [35] and where

$$R = d \wedge \omega + \omega \wedge \omega. \tag{8.10}$$

R is the Riemann form of standard differential geometry. Eqs.8.9 and 8.10 are the first and second Cartan structure equations, sometimes known as the master equations of differential geometry. Therefore Eq.8.6 is in general:

$$j = \frac{1}{\mu_0} d \wedge (d \wedge A + \omega \wedge A) \tag{8.11}$$

where

$$A = A^{(0)} q. \tag{8.12}$$

Thus, the current j is a source of resonance absorption from ECE spacetime. A similar conclusion can be reached for the Hodge dual resonance equation 8.7. The potential A also obeys the ECE Lemma [1]– [35]:

$$\square A = RA \tag{8.13}$$

where

$$R = -kT \tag{8.14}$$

is a well defined scalar curvature, T is the index contracted canonical energy-momentum tensor, and k is Einstein's constant. Therefore the ECE Lemma is the subsidiary proposition of the ECE wave equation [1]– [35]:

$$(\square + kT)\, A = 0. \tag{8.15}$$

Therefore the fundamental mathematical structure of standard differential geometry gives three equations, 8.6, 8.7 and 8.15 with which to investigate resonant absorption of energy from ECE space-time.

In the standard model:

$$F = d \wedge A, \tag{8.16}$$

$$d \wedge F = 0, \tag{8.17}$$

$$d \wedge \widetilde{F} = \mu_0 J. \tag{8.18}$$

Eqs.8.16 and 8.17 give the Poincaré Lemma [37]:

$$d \wedge (d \wedge A) = 0 \tag{8.19}$$

and the current j is missing. The current J in the standard model is introduced empirically and is not recognized to originate in Cartan geometry. Therefore many key resonance features are missing from the standard model, notably the ability of ECE theory to take electric power from space-time in the shape of the currents j and J. Within the factor $A^{(0)}/\mu_0$ these currents are defined completely by the structure or geometry of space-time itself. In the standard model of classical electrodynamics (the Maxwell Heaviside field equations) space-time has no structure, it is the flat or Minkowski space-time and in consequence classical electrodynamics in the standard model cannot be unified with gravitation, in which space-time is structured. Therefore electric power cannot be taken from space-time in the standard model. This is contrary to the reproducible and repeatable experiments [38] of the Mexican Group, which has observed amplification of power levels in excess of one hundred thousand in given circuit designs, and amplification that is due to resonant absorption from ECE space-time. This paper is the first to offer a detailed explanation of this important phenomenon.

In Section 8.2 the fundamental resonance equation:

$$d \wedge (d \wedge A + \omega \wedge A) = \mu_0 j \tag{8.20}$$

is developed into four resonance equations in the vector notation used in electrical engineering and circuit theory. One of these vector equations is solved analytically using appropriate approximations. The result is resonance from a driven undamped inhomogeneous structure. This simple analytical exercise achieves our aim of showing that resonant absorption is possible from ECE space-time, as observed experimentally [38]. Driven undamped resonance produces an infinite Q factor and infinite amplitude resonance at the fundamental frequency [36]. More generally [36] the solutions of the linear inhomogeneous equation give finite Q factors and phase factors, transient and steady state effects, and various types of resonances. These are briefly reviewed in Section 8.3

for the simplest type of linear inhomogeneous second order differential equation [36]. Eq.8.20 is expected to have all these features in general, and several more, and numerical methods will reveal all of them straightforwardly given initial and boundary conditions. Most generally resonance from ECE space-time is described by solving Eqs.8.6, 8.7 and 8.15 simultaneously with given initial and boundary conditions. However the simplest type of linear inhomogeneous structure (Section 8.3) is sufficient to give the features expected, most importantly the ability of a circuit or material of given design to absorb j and J from ECE spacetime and amplify them greatly.

8.2 The Resonance Equations

The source of electric current from ECE space-time is its torsion. In barebones notation the currents are given by:

$$j = \frac{A^{(0)}}{\mu_0} d \wedge T \tag{8.21}$$

$$J = \frac{A^{(0)}}{\mu_0} d \wedge \widetilde{T}. \tag{8.22}$$

The torsion is defined by the tetrad and spin connection in the first Cartan structure equation of differential geometry and the tetrad in turn is defined by the eigenvalues of the ECE Lemma, Eq.8.13. The tetrad is the fundamental field in the Palatini variation of general relativity and is a wave of space-time. The potential field is governed by resonance equations, and within a factor $A^{(0)}$ is the tetrad. In this section the resonance equation 8.20 is developed into vector notation for use in engineering. The spin connection is always defined by the second Bianchi identity, and for the free electromagnetic field is the dual of the tetrad in the tangent space-time [1]– [35]. The scalar curvature is defined as eigenvalues of the ECE Lemma and is proportional to the index contracted energy-momentum tensor through the Einstein Ansatz 8.14. Therefore energy and momentum are transferred from R to j and J, and total energy and momentum are conserved. Total charge-current density is also conserved.

In the standard notation of differential geometry [1]– [35] the relevant equations are:

$$j^a = \frac{A^{(0)}}{\mu_0} d \wedge T^a, \tag{8.23}$$

$$J^a = \frac{A^{(0)}}{\mu_0} d \wedge \widetilde{T}^a, \tag{8.24}$$

$$\Box q^a = R q^a \tag{8.25}$$

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b, \tag{8.26}$$

$$D \wedge (D \wedge \omega^a{}_b) = 0. \tag{8.27}$$

In the standard notation the tangent space-time indices appear but the base manifold indices are the same on both sides of a given equation and are not written out [1]– [35]. If we restore these indices for the sake of illustration and completeness Eqs.8.23 to 8.27 become:

$$j^a{}_{\mu\nu\rho} = \frac{A^{(0)}}{\mu_0} \left(d \wedge T^a\right)_{\mu\nu\rho}, \tag{8.28}$$

$$J^a{}_{\mu\nu\rho} = \frac{A^{(0)}}{\mu_0} \left(d \wedge \widetilde{T}^a\right)_{\mu\nu\rho}, \tag{8.29}$$

$$\Box q^a{}_\mu = R q^a{}_\mu \tag{8.30}$$

$$T^a{}_{\mu\nu} = \left(d \wedge q^a\right)_{\mu\nu} + \omega^a{}_{\mu b} \wedge q^b{}_\nu, \tag{8.31}$$

$$D \wedge \left(D \wedge \omega^a{}_{\mu b}\right) = 0. \tag{8.32}$$

Therefore the barebones and standard notations must always be interpreted as implying the presence of the various indices that appear in Eqs.8.28 to 8.32. The advantage of the barebones notation is that it gives the basic structure with greatest clarity. These equations and notations are fully developed and explained in the literature [1]– [35] in differential form, tensor and vector notation. The vector notation is used in this section because it is the notation universally used in engineering. However all three notations are equivalent and contain the same mathematical information. The differential form notation is the most concise and elegant.

In the standard model

$$j = 0, \tag{8.33}$$

$$R = 0, \tag{8.34}$$

and there can be no battery powered by space-time, even on a qualitative level. The reason for this is that classical electrodynamics in the standard model is still the Maxwell Heaviside theory, which is a nineteenth century theory of special relativity in which the field is thought of as a separate entity superimposed on a Minkowski frame in four dimensions. To Maxwell, space and time were still separate concepts, and there could be no structure to space-time. At the time when Heaviside developed Maxwell's quaternion equations into vector notation (late nineteenth century), space and time were still thought of as separate. Only when Lorentz and Poincaré developed the tensor notation of the Maxwell-Heaviside field equations did space and time become unified into space-time. This occurred at the beginning of the twentieth century. Even then however, the electromagnetic field was still through of as an entity superimposed on a SEPARATE Minkowski frame with metric diag (-1, 1, 1, 1). The concept of a curving space-time appeared only in 1916, in the Einstein Hilbert (EH) theory of general relativity, but that theory was applied only to gravitation, and not to electromagnetism. In EH theory a field was thought of for the first time as the curving frame of reference ITSELF, not as something superimposed on a separate frame of reference. ECE theory, developed from 2003 onwards [1]– [35] is a rigorously objective theory of general relativity in which the electromagnetic field is the torsion of space-time itself and in which currents can be generated by the torsion of space-time itself through Eqs.8.21 and 8.22. These currents are real, observable and physical, and can be used for engineering. In ECE theory electromagnetism is unified with gravitation using differential geometry and space-time currents are a new source of energy that conserves Noether's

Theorem. This section is designed to show how the currents can be maximized by resonance. In the standard model, again, there is no concept of spin connection, because the latter is the mathematical description of a spinning and curving frame. When a frame itself spins or curves (or both spins and curves) the spin connection must be non-zero. In electromagnetism the non-zero spin connection is observed through the Evans spin field [1] – [35] using the phenomenon of magnetization by a circularly polarized electromagnetic field. This is known as the inverse Faraday effect, and is rigorously reproducible and repeatable, occurring in all materials and at all frequencies of the applied electromagnetic field. The Evans spin field is therefore the definitive proof of general relativity in the electromagnetic field. In the standard model the inverse Faraday effect must be explained by assuming the existence of a cross product of complex conjugates of the potential [1]– [35] or equivalently of the electric field or magnetic field. Even this purely empirical description (occurring in non-linear optics [1]– [35]) did not appear until the mid fifties of the twentieth century and therefore was not present in the original Maxwell theory and was not considered by Maxwell or Heaviside. In summary one cannot describe the inverse Faraday effect self-consistently and objectively without general relativity, which asserts that ALL of the equations of physics must be generally covariant. This means that all must retain their structure under the general coordinate transformation, i.e. all of physics must be geometrical in nature. This is the very essence of general relativity, and until this is realized field unification cannot occur in an objective manner. The Maxwell Heaviside equations do not obey this fundamental requirement, because they retain their mathematical (tensorial) structure only under the Lorentz transformation, as described in many texts [39]. In order for the equations of electrodynamics to be generally covariant as required by general relativity, the spin connection must be non-zero, and the Evans spin field must be non-zero [1]– [35]. This is exactly what is shown experimentally by the inverse Faraday effect.

The resonance equations developed in vector notation in this section originate in the "master" equation 8.20, which in standard notation is:

$$d \wedge \left(d \wedge A^a + \omega^a{}_b \wedge A^b\right) = \mu_0 j^a, \tag{8.35}$$

i.e.

$$d \wedge F^a = \mu_0 j^a \tag{8.36}$$

where

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b. \tag{8.37}$$

In tensor notation Eq.8.37 is [1]– [35]:

$$F^a{}_{\mu\nu} = -F^a{}_{\nu\mu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + \omega^a{}_{\mu b} A^b{}_\nu - \omega^a{}_{\nu b} A^b{}_\mu. \tag{8.38}$$

This equation can be developed into the electric field components:

$$F^a{}_{01} = -F^a{}_{10} = \partial_0 A^a{}_1 - \partial_1 A^a{}_0 + \omega^a{}_{0b} A^b{}_1 - \omega^a{}_{1b} A^b{}_0, \tag{8.39}$$

$$F^a{}_{02} = -F^a{}_{02} = \partial_0 A^a{}_2 - \partial_2 A^a{}_0 + \omega^a{}_{0b} A^b{}_2 - \omega^a{}_{2b} A^b{}_0, \tag{8.40}$$

$$F^a{}_{03} = -F^a{}_{30} = \partial_0 A^a{}_3 - \partial_3 A^a{}_0 + \omega^a{}_{0b} A^b{}_3 - \omega^a{}_{3b} A^b{}_0, \tag{8.41}$$

and the magnetic field components:

$$F^a{}_{12} = -F^a{}_{21} = \partial_1 A^a{}_2 - \partial_2 A^a{}_1 + \omega^a{}_{1b} A^b{}_2 - \omega^a{}_{2b} A^b{}_1, \tag{8.42}$$

$$F^a{}_{13} = -F^a{}_{31} = \partial_1 A^a{}_3 - \partial_3 A^a{}_1 + \omega^a{}_{1b} A^b{}_3 - \omega^a{}_{3b} A^b{}_1, \tag{8.43}$$

$$F^a{}_{23} = -F^a{}_{32} = \partial_2 A^a{}_3 - \partial_3 A^a{}_2 + \omega^a{}_{2b} A^b{}_3 - \omega^a{}_{3b} A^b{}_2. \tag{8.44}$$

The vector description of the electric and magnetic fields follows by using the following definitions in covariant/contra-variant notation [40]:

$$A^a{}_\mu = (A^a{}_0, -\mathbf{A}^a), \quad \omega^a{}_{\mu b} = (\omega^a{}_{ob}, -\boldsymbol{\omega}^a{}_b), \tag{8.45}$$

$$A^{a\mu} = \left(A^{a0}, \mathbf{A}^a\right), \quad \omega^{a\mu}{}_b = \left(\omega^{a0}{}_b, \boldsymbol{\omega}^a{}_b\right), \tag{8.46}$$

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \tag{8.47}$$

The contravariant electromagnetic tensor is:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{bmatrix} \tag{8.48}$$

and the contravariant four-derivative [1]– [35] is:

$$\partial^\mu = g^{\mu\nu}\partial_\nu. \tag{8.49}$$

Therefore there are electric field components such as:

$$F^{01a} = -\frac{1}{c}E^{1a} = \partial^0 A^{1a} - \partial^1 A^{0a} + \omega^{0a}{}_b A^{1b} - \omega^{1a}{}_b A^{0b}, \tag{8.50}$$

i.e.

$$-\frac{1}{c}E^a{}_x = \frac{1}{c}\frac{\partial}{\partial t}A^a{}_x + \frac{\partial}{\partial x}A^{0a} + \omega^{0a}{}_b A^b{}_x - \omega^a{}_{xb} A^{0b}, \tag{8.51}$$

and it follows that the complete electric field vector is:

$$\mathbf{E}^a = -\frac{\partial \mathbf{A}^a}{\partial t} - c\boldsymbol{\nabla} A^{0a} - c\omega^{0a}{}_b \mathbf{A}^b + c\boldsymbol{\omega}^a{}_b A^{0b}. \tag{8.52}$$

Similarly there are magnetic field components such as:

$$F^{12a} = \partial^1 A^{2a} - \partial^2 A^{1a} + \omega^{1a}{}_b A^{2b} - \omega^{2a}{}_b A^{1b} = -B^{3a}, \tag{8.53}$$

i.e.

$$-B^a{}_z = -\frac{\partial A^a{}_y}{\partial x} + \frac{\partial A^a{}_x}{\partial y} + \omega^a{}_{xb} A^b{}_y - \omega^a{}_{yb} A^b{}_x \tag{8.54}$$

and the complete magnetic field vector is:

$$\mathbf{B}^a = \boldsymbol{\nabla} \times \mathbf{A}^a - \boldsymbol{\omega}^a{}_b \times \mathbf{A}_b. \tag{8.55}$$

The classical electromagnetic field equations of ECE theory [1]– [35] in vector notation are:

$$\boldsymbol{\nabla} \cdot \mathbf{B}^a = \mu_0 \widetilde{j}^{a0}, \tag{8.56}$$

$$\boldsymbol{\nabla} \times \mathbf{E}^a + \frac{\partial \mathbf{B}^a}{\partial t} = \mu_0 \widetilde{\mathbf{j}}^a, \tag{8.57}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E}^a = c\mu_0 \widetilde{J}^{0a}, \tag{8.58}$$

$$\nabla \times \mathbf{B}^a - \frac{1}{c^2}\frac{\partial \mathbf{E}^a}{\partial t} = \frac{\mu_0}{c}\widetilde{\mathbf{J}}^a, \tag{8.59}$$

in which the currents are defined by:

$$\widetilde{j}^{a\nu} = \left(\frac{1}{c}\widetilde{j}^{a0}, \widetilde{\mathbf{j}}^a\right), \tag{8.60}$$

$$\widetilde{J}^{a\nu} = \left(\frac{1}{c}\widetilde{J}^{a0}, \widetilde{\mathbf{J}}^a\right), \tag{8.61}$$

Therefore the resonance equations are obtained by substituting Eqs.8.52 and 8.55 into each of Eqs.8.56 to 8.59.

The simplest equation is found by substituting Eq.8.55 into Eq.8.56 and using the vector identity [41]:

$$\nabla \cdot \nabla \times \mathbf{A}^a = 0 \tag{8.62}$$

to give

$$\nabla \cdot \left(\boldsymbol{\omega}^a{}_b \times \mathbf{A}^b\right) = -\mu_0 \widetilde{j}^{a0}. \tag{8.63}$$

In this equation summation is implied over repeated b indices as follows:

$$\nabla \cdot \left(\boldsymbol{\omega}^a{}_0 \times \mathbf{A}^0 + \cdots + \boldsymbol{\omega}^a{}_3 \times \mathbf{A}^3\right) = -\mu_0 \widetilde{j}^{a0} \tag{8.64}$$

Therefore the charge density available from space-time is:

$$\widetilde{j}^{a0} = -\frac{A^{(0)}}{\mu_0}\nabla \cdot \left(\boldsymbol{\omega}^a{}_b \times \mathbf{q}^b\right) \tag{8.65}$$

where $\mathbf{q}^b$ is the vector part of the tetrad.

A linear inhomogeneous [36] second order differential equation is found by substituting Eq.8.52 into Eq.8.58 to give:

$$\begin{aligned}\nabla \cdot \nabla A^{a0} + \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}^a\right) + \nabla \cdot \left(\omega^{0a}{}_b \mathbf{A}^b\right) - \nabla \cdot \left(\boldsymbol{\omega}^a{}_b A^{0b}\right) \\ = -\mu_0 \widetilde{J}^{0a}.\end{aligned} \tag{8.66}$$

As discussed further in Section 8.3, the linear inhomogeneous structure gives resonance solutions and resonances in the current $\widetilde{J}^{0a}$. This is the key to amplification of currents from ECE space-time. These concepts and equations are also used [36] in circuit theory for example, atomic absorption theory, or laser theory. Before proceeding to derive the other two resonance equations of this section the self-consistency of the mathematics being used is checked for Eqs.8.63 and 8.66 when the spin connection is dual to the tetrad [1]– [35]:

$$\omega^a{}_{\mu b} = -\kappa \epsilon^a{}_{bc} q^c{}_\mu. \tag{8.67}$$

Here κ has the units of wave-number (inverse metres) and the Levi-Civita symbol is:

$$\epsilon^a{}_{bc} = g^{ad}\epsilon_{dbc} \tag{8.68}$$

where g^{ad} is the metric of the tangent space-time (a Minkowski metric). Therefore there are components [1]– [35]:

$$\omega^1{}_{\mu 2} = -\omega^2{}_{\mu 1} = \kappa q^3{}_\mu, \tag{8.69}$$

$$\omega^2{}_{\mu 3} = -\omega^3{}_{\mu 2} = \kappa q^1{}_\mu, \tag{8.70}$$

and so on. For $a = 0$ Eq.8.63 is:

$$\widetilde{j}^{00} = -\frac{A^{(0)}}{\mu_0} \boldsymbol{\nabla} \cdot \left(\boldsymbol{\omega}^a{}_b \times \mathbf{q}^b \right) \tag{8.71}$$

where the relevant component of the spin connection is:

$$\omega^0{}_{\mu b} = -\kappa \epsilon^0{}_{bc} q^c{}_\mu \tag{8.72}$$

whose vector part is:

$$\boldsymbol{\omega}^0{}_b = -\kappa \epsilon^0{}_{bc} \mathbf{q}^c. \tag{8.73}$$

From the cyclically symmetric properties of $\epsilon^a{}_{bc}$, summation over b in Eq.8.73 would be over space-like indices, $1, 2$ and 3. From Eq.8.73:

$$\begin{aligned} \boldsymbol{\omega}^0{}_1 &= -\kappa \epsilon^0{}_{1c} \mathbf{q}^c, \\ &= -\kappa \left(\epsilon^0{}_{12} \mathbf{q}^2 + \epsilon^0{}_{12} \mathbf{q}^3 \right), \end{aligned} \tag{8.74}$$

$$\boldsymbol{\omega}^0{}_2 = -\kappa \left(\epsilon^0{}_{21} \mathbf{q}^1 + \epsilon^0{}_{23} \mathbf{q}^3 \right), \tag{8.75}$$

$$\boldsymbol{\omega}^0{}_3 = -\kappa \left(\epsilon^0{}_{31} \mathbf{q}^1 + \epsilon^0{}_{32} \mathbf{q}^2 \right). \tag{8.76}$$

Therefore in this approximation:

$$\begin{aligned} \boldsymbol{\omega}^0{}_b \times \mathbf{q}^b &= \boldsymbol{\omega}^0{}_1 \times \mathbf{q}^1 + \boldsymbol{\omega}^0{}_2 \times \mathbf{q}^2 + \boldsymbol{\omega}^0{}_3 \times \mathbf{q}^3 \\ &= -\kappa (\epsilon^0{}_{12} \mathbf{q}^2 \times \mathbf{q}^1 + \epsilon^0{}_{13} \mathbf{q}^3 \times \mathbf{q}^1 \\ &+ \epsilon^0{}_{21} \mathbf{q}^1 \times \mathbf{q}^2 + \epsilon^0{}_{23} \mathbf{q}^3 \times \mathbf{q}^3 \\ &+ \epsilon^0{}_{31} \mathbf{q}^1 \times \mathbf{q}^3 + \epsilon^0{}_{32} \mathbf{q}^2 \times \mathbf{q}^3). \end{aligned} \tag{8.77}$$

Now use the properties:

$$\epsilon^0{}_{12} = -\epsilon^0{}_{21} = 1, \tag{8.78}$$

$$\epsilon^0{}_{23} = -\epsilon^0{}_{32} = 1, \tag{8.79}$$

$$\epsilon^0{}_{31} = -\epsilon^0{}_{13} = 1, \tag{8.80}$$

and use the complex circular basis $((1), (2), (3))$ [1]– [35] to obtain:

$$\widetilde{j}^{00} = 2\kappa \frac{A^{(0)}}{\mu_0} \boldsymbol{\nabla} \cdot \left(\mathbf{q}^{(2)} \times \mathbf{q}^{(1)} + \mathbf{q}^{(1)} \times \mathbf{q}^{(3)} + \mathbf{q}^{(3)} \times \mathbf{q}^{(2)} \right). \tag{8.81}$$

For plane waves:

$$\mathbf{q}^{(1)} = \mathbf{q}^{(2)*} = \frac{1}{\sqrt{2}} \left(\mathbf{i} - i\mathbf{j} \right) e^{i\phi}, \tag{8.82}$$

and

$$\boldsymbol{\nabla} \cdot \mathbf{q}^{(2)} \times \mathbf{q}^{(1)} = 0. \tag{8.83}$$

Also [1]– [35]:

$$\mathbf{q}^{(1)} \times \mathbf{q}^{(3)} = -i\mathbf{q}^{(2)*} = -i\mathbf{q}^{(1)} \tag{8.84}$$

and

$$\mathbf{q}^{(3)} \times \mathbf{q}^{(2)} = -i\mathbf{q}^{(2)}, \tag{8.85}$$

so

$$\widetilde{j}^{00} = -2i\kappa \frac{A^{(0)}}{\mu_0} \boldsymbol{\nabla} \cdot \left(\mathbf{q}^{(1)} + \mathbf{q}^{(2)} \right) = 0. \tag{8.86}$$

Therefore it is found that:

$$\widetilde{j}^{00} = \widetilde{j}^{01} = \widetilde{j}^{02} = \widetilde{j}^{03} = 0 \tag{8.87}$$

which is self-consistent with the fact that:

$$\widetilde{j} = 0 \tag{8.88}$$

when Eq.8.67 applies, Q.E.D. Therefore the equation 8.63 is mathematically self-consistent.

In order to check the consistency of Eq.8.66 recall that in the standard model there is no spin connection, so Eq.8.66 reduces to:

$$\frac{1}{c} \cdot \frac{\partial \mathbf{A}^a}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} A^a{}_0 = 0. \tag{8.89}$$

For each polarization index a, $A^a{}_0$ is the electric scalar potential ϕ. Using the Lorentz condition:

$$\frac{1}{c} \frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{8.90}$$

it is found that Eq.8.89 reduces to:

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0 \tag{8.91}$$

i.e.

$$\Box \phi = 0 \tag{8.92}$$

which is the relativistic wave equation of the standard model for a scalar potential ϕ. In order to obtain space-time resonance however, the complete Eq.8.66 is needed.

The third resonance equation is obtained by substituting Eq.8.52 and 8.55 into Eq.8.57. Using the vector properties [41]:

$$\frac{\partial}{\partial t} \boldsymbol{\nabla} \times \mathbf{A}^a = \boldsymbol{\nabla} \times \frac{\partial \mathbf{A}^a}{\partial t} \tag{8.93}$$

and

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}^{a0} = \mathbf{0} \tag{8.94}$$

it is found that:

$$\frac{\partial}{\partial t} \left(\boldsymbol{\omega}^a{}_b \times \mathbf{A}^b \right) + c \boldsymbol{\nabla} \times \left(A^{0b} \boldsymbol{\omega}^a{}_b - \omega^a{}_b{}^0 \mathbf{A}^b \right) = \mu_0 \widetilde{\mathbf{j}}^a. \tag{8.95}$$

This is a first order differential equation in the potential. The current $\widetilde{\mathbf{j}}^a$ is non-zero if and only if the spin connection is non-zero. So the current $\widetilde{\mathbf{j}}^a$ is unique to ECE theory and general relativity and does not occur in the standard model. The self consistency of Eq.8.95 can be checked again by using Eq.8.67, in which case we obtain:

$$\begin{aligned} \boldsymbol{\omega}^a{}_b \times \mathbf{A}^b &= \boldsymbol{\omega}^1{}_2 \times \mathbf{A}^2 + \boldsymbol{\omega}^1{}_3 \times \mathbf{A}^3 \\ &= \frac{\kappa}{A^{(0)}} \left(\mathbf{A}^3 \times \mathbf{A}^2 + \mathbf{A}^2 \times \mathbf{A}^3 \right) = \mathbf{0} \end{aligned} \tag{8.96}$$

and two more equations:

$$cA^{0b}\nabla \times \boldsymbol{\omega}^1{}_b = \frac{c}{A^{(0)}}\left(A^{02}\nabla \times \mathbf{A}^3 + A^{03}\nabla \times \mathbf{A}^2\right) \tag{8.97}$$

and

$$-c\omega^{10}{}_c\nabla \times \mathbf{A}^b = \frac{c}{A^{(0)}}\left(A^{03}\nabla \times \mathbf{A}^2 + A^{02}\nabla \times \mathbf{A}^3\right) \tag{8.98}$$

which self-consistently sum to zero, Q.E.D.

The final resonance equation is obtained by substituting Eqs.8.52 and 8.55 into Eq.8.59 and is:

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}^a}{\partial t^2} + \frac{1}{c}\left(\nabla A^{0a} - A^{b0}\boldsymbol{\omega}^a{}_b + \omega^a{}_b{}^0\mathbf{A}^b\right) + \nabla \times \left(\nabla \times \mathbf{A}^a - \boldsymbol{\omega}^a{}_b \times \mathbf{A}^b\right) = \frac{\mu_0}{c}\widetilde{\mathbf{J}}^a. \tag{8.99}$$

This is a generalization of the linear inhomogeneous structure discussed further in Section 8.3 in which an analytical solution is given of Eq.8.99 in a well defined approximation. The simple type of linear inhomogeneous structure [36] is:

$$\ddot{x} + 2\beta\dot{x} + \omega^2{}_0 x = A\cos\omega t \tag{8.100}$$

which is a driven damped oscillator equation of classical dynamics. It is seen that Eq.8.99 is a generalization of Eq.8.100. Solutions of Eq.8.100 were first discovered by the Bernoulli's and Euler (1739-1743) and show resonance in the amplitude A of Eq.8.100, resonance in the kinetic and potential energies, Q factors, phase lags, transient and steady state effects. Therefore Eq.8.99 has similar solutions and is also more richly structured.

8.3 Analytical Solution

Eq.8.100 is a development of the linear inhomogeneous [36] class of equations:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x). \tag{8.101}$$

In the special case:

$$f(x) = 0 \tag{8.102}$$

Eq.8.101 reduces to the linear homogeneous class

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{8.103}$$

whose general solution is:

$$y = c_1e^{r_1x} + C_2e^{r_2x}, \quad r_1 \neq r_2, \tag{8.104}$$

with the auxiliary equation

$$r^2 + ar + b = 0. \tag{8.105}$$

Eq.8.104 holds when the roots of Eq.8.103 are real and unequal, i.e. $r_1 \neq r_2$. If the roots of Eq.8.103 are imaginary $(\alpha \pm i\beta)$, then:

$$\begin{aligned} y &= e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right) \\ &= \mu e^{\alpha x}\sin\left(\beta x + \delta\right). \end{aligned} \tag{8.106}$$

Now let:

$$y = u \tag{8.107}$$

be the general solution of

$$y'' + ay' + by = 0 \tag{8.108}$$

and let

$$y = v \tag{8.109}$$

be any solution of

$$y'' + ay' + by = f(x) \tag{8.110}$$

then

$$y = u + v \tag{8.111}$$

is a solution of Eq.8.101. The function u is the complementary function and v is the particular integral. One must find by inspection a function v that satisfies:

$$v'' + av' + bv = f(x). \tag{8.112}$$

Eq.8.100 of Section 8.2 is a special case of the linear inhomogeneous class 8.100 and Eq. 8.100 can be rewritten as

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega t. \tag{8.113}$$

This is the equation of driven oscillation [37]. In Eq.8.113 the external driving force varies harmonically with time, and is applied to the oscillator. The total force on the particle is:

$$F = -kx - b\dot{x} + F_0 \cos \omega t \tag{8.114}$$

and consists of a linear restoring force, $-kx$, (Hooke's Law), and a viscous damping force $-b\dot{x}$. Therefore the master equation 8.35 of ECE theory has all these features and is also more richly structured. In this Section an analytical solution of Eq.8.99 is found in a well defined approximation using the properties of the linear inhomogeneous class of equations 8.101.

Resonance solutions of Eq.8.113 are found from the complementary function $x_c(t)$ and the particular integral $x_p(t)$. The former is:

$$x_c(t) = e^{-\beta t}\left(A_1 exp\left(\left(\beta^2 - {\omega^2}_0\right)^{1/2} t\right) + A_2 exp\left(-\left(\beta^2 - {\omega^2}_0\right)^{1/2} t\right)\right) \tag{8.115}$$

and the latter is [37]:

$$x_p(t) = D \cos\left(\omega t - \delta\right). \tag{8.116}$$

It follows that

$$x_p\left(t\right) = A\left(\left({\omega^2}_0 - \omega^2\right)^2 + 4\omega^2\beta^2\right)^{-1/2} \cos\left(\omega t - \delta\right) \tag{8.117}$$

where

$$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right) \tag{8.118}$$

The general solution is:

$$x(t) = x_c(t) + x_p(t). \tag{8.119}$$

The term $x_c(t)$ represents transient effects that depend on the initial conditions. These damp out with time because of the factor $e^{-\beta t}$. The term $x_p(t)$ represents steady state effects which dominate for $t >> 1/p$. The quantity δ is the phase difference between the driving force and the resultant motion, i.e. a delay between the application of force and the response of the system. For a fixed ω_0, as ω increases from 0, the phase increases from $\delta = 0$ at $\omega = 0$ to δ at $\pi/2$ and to π as $\omega \to \infty$.

The amplitude resonance frequency ω_R is that at which the amplitude D is a maximum. It is defined by:

$$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_R} = 0 \tag{8.120}$$

i.e.

$$\omega_R = \left({\omega^2}_0 - 2\beta^2\right)^{1/2}. \tag{8.121}$$

We see that for an equation such as 8.92 in which ω_0 and β are both zero, there is no resonance. In an equation in which ω_0 is zero but β is non-zero the resonance frequency ω_R is pure imaginary and unphysical. Therefore the requirement for resonance is that ω_0 and D be non-zero. If the amplitude D is initially zero it cannot be maximized from Eq.8.120. These conditions are very important for the resonant acquisition of energy and for resonant counter-gravitation.

The degree of damping in an oscillatory system is described by the quality factor:

$$Q = \frac{\omega_R}{2\beta}. \tag{8.122}$$

In loudspeakers for example [36] the values of Q may be a few hundred, in quartz crystal oscillators or tuning forks up to 10,000. Highly tuned electric circuits (of interest to extracting resonance energy from ECE space-time) may have Q up to 100,000 [36]. This is the order of magnitude of the amplification observed by the Mexican Group. The oscillation of electrons in atoms leads to optical radiation. The sharpness of the spectral lines is limited [36] by the damping due to loss of energy by radiation (radiation damping). The minimum width of a line is, classically, about:

$$\delta\omega = 2 \times 10^{-8}\omega. \tag{8.123}$$

The Q of such an oscillation is therefore of the order 10^7. The largest known Q occurs from radiation from a gas laser, about 10^{14}. Therefore resonant energy from ECE space-time and resonant counter-gravitation are also governed by such features. A current j (barebones notation) is set up by Eq.8.21 and can set electrons in a circuit or within a material into resonant motion, producing a resonance current from space-time as observed experimentally [1]– [35]. Eq.8.21 shows that the current is generated by the geometry of space-time itself.

Resonance in kinetic energy (T) is defined by the value of ω for which T is a maximum, where [37]:

$$T = \frac{1}{2}m\dot{x}^2 \tag{8.124}$$

It is found from:

$$\left.\frac{d\langle T\rangle}{d\omega}\right|_{\omega=\omega_E} = 0 \tag{8.125}$$

and is

$$\omega_E = \omega_0 \tag{8.126}$$

where

$$\langle T \rangle = \frac{mA^2}{4}\omega^2 \left(\left({\omega^2}_0 - \omega^2 \right)^2 + 4\omega^2\beta^2 \right)^{-1/2}. \tag{8.127}$$

The potential energy is proportional to the square of the amplitude, and occurs at the same frequency as amplitude resonance. The kinetic and potential energies resonate at different frequencies because the damped oscillator is not a conservative system [36] of dynamics. Energy is continuously exchanged with the driving system. In energy from ECE space-time energy is therefore continuously exchanged between space-time and the circuit or material, total energy being conserved by Noether's Theorem.

Atomic systems within a material taking resonant energy from ECE space-time can be represented classically as linear oscillators. When light falls on matter it causes the atoms and molecules to vibrate. Similarly ECE space-time causes the atoms and molecules to vibrate, light being ECE space-time within the factor $A^{(0)}$ of Eq.8.12. A resonant frequency occurs at one of the spectral frequencies of the system. When light (i.e. ECE space-time) having one of the resonant frequencies of the atomic or molecular system falls on the material, electromagnetic energy (i.e. energy from ECE space-time) is absorbed, causing the atom or molecule to oscillate with large amplitude. This is what happens in a circuit or material such as that of the Mexican Group [1]– [35]. A large amount of energy is resonantly absorbed from ECE space-time. This can be released as electric current or power, the governing equation is equation 8.35. Large electromagnetic fields (ECE space-time dynamics) are produced by the oscillating electric charges. Electric circuits are non-mechanical oscillations. Therefore resonance theory and electric circuit theory can be used to explain energy from space-time. The mechanism is clear from Eq.8.35, i.e.:

$$j = \frac{A^{(0)}}{\mu_0} \left(d \wedge (d \wedge q) + d \wedge (\omega \wedge q) \right). \tag{8.128}$$

The current j is picked up from ECE space-time and is represented by q and ω of Eq.8.128, a driven damped oscillator equation. Amplitude, kinetic energy and potential energy resonances occur. The electrons in a well designed circuit or material oscillate in constructive interference, producing a surge of current and electric power. This is observed experimentally in the reproducible and repeatable work of the Mexican group of AIAS [1]– [35].

These qualitative remarks are underlined as follows with an analytical solution of Eq.8.99 with well defined approximations. First use

$${\omega^a}_{\mu b} = -\kappa {\epsilon^a}_{bc} {q^c}_\mu \tag{8.129}$$

so

$$A^{b0} {\boldsymbol{\omega}^a}_b = {\omega^{a0}}_b \mathbf{A}^b, \tag{8.130}$$

$${\boldsymbol{\omega}^a}_b \times \mathbf{A}^b = \mathbf{0}. \tag{8.131}$$

Then use

$$\nabla^2 \mathbf{A}^a = -\frac{\omega_0^2}{c^2} \mathbf{A}^a, \tag{8.132}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A}^a = 0, \tag{8.133}$$

$$\partial A^{0a}/\partial t = 0, \tag{8.134}$$

with:

$$\nabla \times (\boldsymbol{\nabla} \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}). \tag{8.135}$$

Eq.8.99 then simplifies to

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}^a}{\partial t^2} + \frac{\omega_0^2}{c^2}\mathbf{A}^a = \frac{\mu_0}{c}\widetilde{\mathbf{J}}^a. \tag{8.136}$$

This is an undamped driven oscillator, it has the structure of Eq.8.100 with

$$\beta = 0. \tag{8.137}$$

From Eqs.8.132 and 8.133

$$\mathbf{A}^a = \frac{A^{(0)}}{\sqrt{2}}(\mathbf{i} - i\mathbf{j})\, e^{-i\omega_0 Z/c} \tag{8.138}$$

is a possible solution. From the analytical solution of Eq.8.100 already discussed in this Section:

$$A^{(0)} = A_c^{(0)} + A_p^{(0)} \tag{8.139}$$

where

$$A_c^{(0)} = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t} \tag{8.140}$$

$$A_p^{(0)} = D, \tag{8.141}$$

assuming:

$$\frac{\mu_0}{c}\widetilde{\mathbf{J}}^a = A^a(\mathbf{i} - i\mathbf{j})\cos\omega t \tag{8.142}$$

Resonance occurs at

$$\omega_R = \omega_0 \tag{8.143}$$

with:

$$\begin{gathered} \delta = 0, \quad Q \to \infty, \\ D \to \infty. \end{gathered} \tag{8.144}$$

In this case there is a surge of current of infinite amplitude:

$$\widetilde{\mathbf{J}}^a \to \infty \tag{8.145}$$

because there is no damping. This simple illustration, using well defined approximations, shows how resonant energy from space-time occurs mathematically within ECE theory. More realistic results with finite damping can be produced numerically from Eq.8.99, and under certian conditions will reproduce the factor of 100,000 amplification observed by the Mexican Group [1]– [35] and found independently to be reproducible and repeatable.

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