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ECE Theory of the Earth's Gravitomagnetic Precession

by

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Abstract

The ECE equation of static gravitomagnetism is used to calculate the angle of precession due to the earth of a gyroscope carried in an orbiting satellite. The origin of the earth's gravitomagnetic angular frequency is thereby identified as space-time torsion. The ECE result is fortuitously in exact agreement with the first term of a dipole approximation in a recent result of H. Pfister from a revised theory of the standard physics' Lense Thirring effect. However, the latter is obtained from the incorrect Einstein field equation using the incorrect Kerr metric. Metrics used to calculate the so called Lense-Thirring effect from the Einstein field equation are shown to violate the fundamental dual identity of Cartan geometry and ECE theory.

Keywords: ECE theory, gravitomagnetic precession of the earth.

24.1 Introduction

It has been shown recently [1–12] that the Einstein field equation is incorrect due to its neglect of torsion. This is a fundamental error which shows up in the Hodge dual of the well known Bianchi identity as given by Cartan [13, 14]. Exact solutions of the Einstein field equation are given in terms of line elements [15], and in paper 93 of this series several such line elements were shown to violate the dual identity:

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} \tag{24.1}$$

of Cartan geometry. Here $T^{\kappa\mu\nu}$ is the torsion tensor and $R^{\kappa}{}_{\mu}{}^{\mu\nu}$ is the curvature tensor. The dual identity states that the covariant derivative of the torsion tensor is the curvature tensor. In paper 93 it was shown that the type of curvature tensor that appears in Eq. (24.1) is non-zero in general from the Einstein field equation, while the torsion tensor in that equation is by definition zero because of its use of the symmetric connection. Therefore the result is obtained that the Einstein field equation is geometrically incorrect, a severely negative result for modern physics of the standard type. In hindsight such a result was bound to occur, because the neglect of torsion by Einstein and his contemporaries was arbitrary. If the torsion is eliminated by choice of connection, the subject of general relativity becomes incorrectly constrained to curvature only. It has been known for ninety years that the Einstein equation had severe flaws in it but this criticism has gone unanswered. The result is a fiasco for the subject of natural philosophy, because the well known predictions attributed to this deeply flawed equation, over no less than ninety years, are entirely meaningless. Among these is the Lense Thirring effect [16] which is the attempted standard explanation for the earth's gravitomagnetic precession.

By use of the correct geometry due to Cartan [1–13] the Einstein Cartan Evans (ECE) theory has re-instated spacetime torsion in its rightful place in physics and has developed equations of dynamics based on the correct consideration of both torsion and curvature. One of these is the static gravitomagnetic equation, whose analogue in classical electrodynamics is the ECE Ampère law [1–12]. In Section 24.2 the simplest type of solution of this equation in the weak field approximation is found in order to give a first approximation to the angular frequency of the earth's gravitomagnetic precession. The latter gives an angle of precession when observed over a year's time - the aim of the well known Gravity Probe B experiment. The ECE theory gives the expected precession in a much simpler and much more direct way than the standard physics. In an article such as that by Pfister [16] it is seen that the history of the so-called Lense Thirring effect is convoluted, and even within the context of the standard model there are several errors in its development. It is not clear that these errors have been corrected and it is not even clear that Gravity Probe B has produced anything new experimentally. In Section 24.3, the Kerr metric and similar metrics used in the description of the Lense Thirring effect are shown to violate the dual identity (24.1) of Cartan geometry, and complete details of the computation are given.

24.2 Calculation of the Gravitomagnetic Angular Frequency

The ECE theory of gravitomagnetic precession is part of a generally covariant unified field theory [1–12] in which the equations of classical electrodynamics and dynamics have precisely the same structure based on the Bianchi identity of geometry. The ECE dynamical equation that gives gravitomagnetic precession is the precise analogue of the ECE Ampère law of classical electrodynamics, one of the law of magnetostatics. The ECE Ampère law is:

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \tag{24.2}$$

where $\mathbf{B}$ is magnetic flux density (defined by elements of spacetime torsion) and where $\mathbf{J}$ is part of the charge current four-density:

$$J^\mu = (c\rho, \mathbf{J}) \tag{24.3}$$

where ρ is charge density in coulombs per metre cubed and where c is the speed of light in vacuo. Here μ_0 is the permeability in vacuo in S.I. units. The other ECE law of magnetostatics is, for all practical purposes:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{24.4}$$

In the limit of:

$$v << c \tag{24.5}$$

the inverse Lorentz transform gives the well known Biot Savart law [17] in the form:

$$\mathbf{B} = -\frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{24.6}$$

where $\mathbf{E}$ is electric field strength in volts per metre.

In precise analogy, the mass four density is defined as:

$$j^\mu = (c\rho_m,\ \mathbf{j}) \tag{24.7}$$

where ρ_m is the mass density in kilograms per cubic metre and where $\mathbf{j}$ is the mass current density. The analogue of $\mathbf{E}$ in the ECE dynamical equations is the usual acceleration due to gravity $\mathbf{g}$ in metres per second squared. The dynamical analogue of $\mathbf{B}$ is defined as the quantity:

$$\boldsymbol{\Omega} = \frac{\mathbf{h}}{c} \tag{24.8}$$

which has the units of radians per second and which comes from the precise dynamical analogue of Eq. (24.6), i.e. from:

$$\boldsymbol{\Omega} = -\frac{1}{c^2}\mathbf{v} \times \mathbf{g}. \tag{24.9}$$

The dynamical analogue of the ECE Ampère law is therefore:

$$\boldsymbol{\nabla} \times \boldsymbol{\Omega} = \left(\frac{4\pi\, G}{c^2}\right)\mathbf{j} \tag{24.10}$$

and the precise dynamical analogue of Eq. (24.4) is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Omega} = 0. \tag{24.11}$$

The ECE Coulomb law is:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{24.12}$$

where ϵ_0 is the permittivity in vacuo, and its dynamical analogue is:

$$\boldsymbol{\nabla} \cdot \mathbf{g} = 4\pi\, G\, \rho_m \tag{24.13}$$

which in the weak field approximation gives the Newton inverse square law where G is Newton's gravitational constant. Gravity Probe B [16] is a satellite that orbits at 650 kilometres over the poles, i.e. is a low orbit satellite whose mean distance above the earth's surface is small compared with the earth's radius. The aim of the experiment is to measure the angle defined by:

$$\theta = \Omega\, t \tag{24.14}$$

where the interval of time t is one year. The angle θ is given straightforwardly from the ECE equation (24.10), which is the Biot Savart type solution of

Eq. (24.9). It is well known that if the earth is considered as a uniform sphere of mass M, then the acceleration due to gravity is:

$$\mathbf{g} = -\frac{MG}{r^3}\mathbf{r} \tag{24.15}$$

where r is the radial coordinate. In the present context this is the distance between the earth's centre of mass and the Gravity Probe B satellite. The earth's radius is denoted R, and its mass is denoted M. Its angular momentum [4] is therefore the well known angular momentum of a sphere of uniform mass:

$$L = \frac{2}{5}MR^2\,\omega \tag{24.16}$$

where ω is the angular frequency of diurnal rotation of the earth, a well measured quantity. It is also well known that the attraction between a mass m and the earth, a sphere of mass of radius R, can be represented by the Newtonian inverse square law:

$$\mathbf{F} = m\mathbf{g} = -m\,M\,G\frac{\mathbf{r}}{r^3} \tag{24.17}$$

for all r. Now apply Eq. (24.9) to find:

$$\boldsymbol{\Omega} = \frac{mG}{c^2r^3}\mathbf{L} \tag{24.18}$$

in which the integrated angular momentum of the earth is defined as:

$$\mathbf{L} = \sum_i m\mathbf{r}_i \times \mathbf{v}_i. \tag{24.19}$$

Therefore the angular frequency in radians per second of the earth's gravitomagnetic precession is:

$$\Omega = \frac{2}{5}\frac{mG}{c^2r^3}R^2\,\omega. \tag{24.20}$$

The data relevant to Gravity Probe B are as follows, in S.I. units:

$$\left.\begin{aligned} R &= 6.37 \times 10^{6}\,\mathrm{m}, \\ r &= 7.02 \times 10^{6}\,\mathrm{m}, \\ M &= 5.98 \times 10^{24}\,\mathrm{kg}, \\ c &= 2.998 \times 10^{8}\,\mathrm{ms^{-1}}, \\ G &= 6.67 \times 10^{-11}\,\mathrm{m^{3}kg^{-1}s^{-2}}, \\ \omega &= 7.29 \times 10^{-5}\,\mathrm{rad\,s^{-1}}. \end{aligned}\right\} \tag{24.21}$$

Therefore the earth's angular momentum is:

$$L = 7.076 \times 10^{33}\,\mathrm{kg\,m^{2}s^{-1}} \tag{24.22}$$

and the earth's gravitomagnetic precession in an orbit 650 kilometres above the surface is:

$$\Omega = 1.52 \times 10^{-14}\,\mathrm{rad\,s^{-1}}. \tag{24.23}$$

One year is $3600 \times 24 \times 365.25 = 3.156 \times 10^{7}$ seconds, and in one year:

$$\theta = 4.80 \times 10^{-7}\,\mathrm{radians}. \tag{24.24}$$

Finally use:

$$1\,\mathrm{radian} = 2.06265 \times 10^{5}\,\mathrm{arcseconds} \tag{24.25}$$

so the angular change is:

$$\theta = 9.9 \times 10^{-2}\,\mathrm{arcseconds}. \tag{24.26}$$

It is not clear whether the Gravity Probe B experiment is free of artifact as claimed [16], but the observed angular change is expected to be of the order of the very simple first approximation given in this paper. No more is claimed of Eq. (24.26), because it is to be regarded as a first approximation. However, it is clear that the standard approach to the so called Lense Thirring effect is attributed [16] not to Lense and Thirring but to Einstein, whose field equation has been known to be incorrect in several ways for ninety years. The problems with this well known field equation began to emerge in 1918 [4], when Bauer and Schroedinger independently and severely criticised its energy momentum structure. Using Eq. (24.1) it becomes clear as in Section 24.3 that the equation is irretrievably self-inconsistent because of its use of a symmetric connection or zero torsion [1–12]. This should have been clear at the outset of general relativity, when the torsion was discarded in an entirely arbitrary manner. It appears that the standard theoretical approach to the Lense Thirring effect is based on metrics that involve rotation. One of these is the Kerr metric, which is shown in Section 24.3 to violate the Bianchi identity (24.1), and therefore to be incorrect geometrically. It is obvious that no physics can emerge from basically incorrect mathematics. Even worse (Section 24.3) is the failure of the Kerr Newman metric to obey the fundamental Ricci cyclic equation known in standard physics as the "first Bianchi identity". Proceeding in this way it becomes clear that no line element based on a symmetric connection can be correct mathematically, meaning that gravitational physics of the last ninety years is meaningless.

There are clear additional problems in the standard treatment [16] of the Lense Thirring effect. The usual theory of the effect is developed in a mass dipole approximation. However, a sphere of uniform mass has no multipoles except for the familar monopole - the Newtonian potential:

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi, \quad \Phi = -\frac{GM}{r}. \tag{24.27}$$

In molecular physics, it is well known that the existence of multipoles is determined by the group theory of the molecule. Cyanogen (NCCN) for example [4] has no electric dipole moment, but has a large electric quadrupole moment. The more symmetric a molecule, the higher the multipole it possesses, so sulphur hexafluoride for example only has a hexadecapole moment. A perfectly spherical molecule would not have any multipole at all. Similarly therefore a sphere of uniform mass (the earth) does not have any multipole except for the Newtonian monopole. This means that the dipole approximation used in the standard physics [16] to describe the so called Lense Thirring effect is not correct. Even if it were correct it is valid only when:

$$d << r \tag{24.28}$$

where d is the length of the dipole. It is entirely unclear whether this is the case for Gravity Probe B, and using higher order multipole terms will not cure this problem, as Pfister apparently claims. In a sphere of mass, none of these multipole terms exist, including the dipole term itself. Pfister's Eq. (24.1) [16] is therefore the result of a hypothetical dipole approximation for a sphere, a contradiction, and his result applies if and only if Eq. (24.28) of this paper is true, and if and only if the hypothetical dipole is aligned in the Z axis. Fortuitously, the first term of Pfister's result is the same as our Eq. (24.20), but this is a coincidence only. Pfister however does give a detailed history of the so called Lense - Thirring effect, and reveals several basic errors in the standard approach. The latter approach should now be regarded as entirely obsolete, and the Einstein field equation similarly abandoned.

24.3 Testing Metrics of the Einstein Equation with Eq. (24.1)

The methods used to test metrics of the Einstein field equation are a development of our previous work in papers 93 and 95 of this series (www.aias.us). In this section Eq. (24.1) is evaluated directly for the well known Kerr, Kerr Newman and Reissner Nordstrom metrics, which are all solutions of the Einstein field equation. For reference, our results for the misnamed Schwarzschild metric solution are also given. As discussed in paper 93 and elsewhere on www.aias.us, Schwarzschild in 1916 did not derive this metric. The source mass M does not appear in the two original 1916 papers by Schwarzschild, which give purely geometrical solutions of the equation:

$$G_{\mu\nu} = 0 \tag{24.29}$$

where $G_{\mu\nu}$ is the well known Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}. \tag{24.30}$$

Here $R_{\mu\nu}$ is the well known Ricci tensor, $g_{\mu\nu}$ is the symmetric metric and R is the Ricci scalar. In the standard terminology, Eq. (24.29) is known as a Ricci flat solution. Crothers has argued correctly (www.aias.us for example) that the Ricci flat solution can have no physical meaning. This should be immediately clear from the fact that the Einstein field equation is:

$$G_{\mu\nu} = k\, T_{\mu\nu} \tag{24.31}$$

where k is the Einstein constant and where $T_{\mu\nu}$ is the canonical energy momentum density of the covariant Noether theorem. A Ricci flat solution therefore has no physics in it, it assumes a priori that there is no energy density present in the calculation. Schwarzschild in 1916 produced Ricci flat solutions, a purely geometrical exercise. In particular, a Ricci flat solution is devoid of mass M by definition, because mass M is part of $T_{\mu\nu}$. Others incorrectly reinstated mass M into the geometry in order to obtain Newtonian mechanics as a limit. This is an entirely meaningless procedure. In the present context, Ricci flat solutions produce the trivial result

$$T^{\kappa\mu\nu} = 0, \quad R^{\kappa}{}_{\mu}{}^{\mu\nu} = 0 \tag{24.32}$$

simply because $R^{\kappa}{}_{\mu}{}^{\mu\nu}$ is initially set to zero and because $T^{\kappa\mu\nu}$ is also set to zero initially by use of a symmetric connection.

Table 24.1 Definition of charged/rotational metrics (Q = charge, J = angular momentum).

	Non-rotating ($J = 0$)	Rotating ($J \neq 0$)
Uncharged ($Q = 0$)	Schwarzschild	Kerr
Charged ($Q \neq 0$)	Reissner-Nordstrom	Kerr-Newman

The Kerr, Kerr-Newman and Reissner Nordstrom metrics produce the geometrically incorrect result:

$$T^{\kappa\mu\nu} = 0, \quad R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0, \tag{24.33}$$

and so cannot give meaningful physics. Our results for these four metrics are summarized in Table (24.1) and Figs. (24.1) to (24.3). These are developments of our previous results in papers 93 and 95 on www.aias.us where it was found that the Robertson Walker metric of big bang also gives the incorrect result (24.33), and indeed all solutions of the Einstein field equation. Subsequently the generally covariant equations of ECE dynamics and cosmology were developed in parallel with the ECE equations of electrodynamics, and named the ECE engineering model on www.aias.us. This is a precisely determined system of eight equations in eight unknowns and can be applied to any situation in classical physics.

24.4 Discussion of Experimental Results of Gravity Probe B

The Gravity Probe B website (http://einstein.stanford.edu) reports the well known geodetic effect to be 6.6 plus or minus 0.097 arcseconds a year. The satellite failed to measure the gravitomagnetic effect, which can therefore only be reported to be within the noise of the experiment. There is another disputed measurement [16] of the gravitomagnetic effect at 0.043 arcseconds a year from LAGEOS 1976 and 1992. Therefore the ECE result is satisfactory, because it is a first approximation that can be greatly refined. The theoretical result by Pfister [16] in his Eq. (24.1) is given in terms of a quantity which he denotes **H**. The first term of Pfister's eq. (24.1) is the same as the result in this paper, but as argued here, Pfister uses what appears to be either an incorrect or inapplicable dipole approximation. A sphere of mass density (the earth) doe not have a mass dipole or any multipole higher than the Newtonian monopole (the usual Newtonian potential), but Pfister may be using an experimentally measured mass dipole due to the well known irregularities in the earth's structure. Other sites attribute the Lense Thirring effect to the Kerr metric, which as shown in this paper is geometrically incorrect. In a first equatorial approximation the angular frequency of the Lense Thirring effect from the Kerr metric is five times greater than the result (24.26) of ECE theory.

To add to the confusion, the Gravity probe B website claims that the Lense Thirring effect is 170 times smaller than 6.6 arcseconds per year (i.e. 39 milliarcseconds a year) but does not define the gravitomagnetic effect mathematically. Pfister claims that his Eq. (24.1) is the Lense Thirring effect, but this claim can be discarded both on thereotical and experimental grounds as argued in this paper. NASA has decided to cease funding of Gravity probe B, which developed an artifact. In view of these confused and incorrect claims, and in view of the fact that the Kerr metric and Einstein field equation are geometrically incorrect, the close agreement between ECE theory and the available data is conclusive evidence in favour of ECE theory, i.e. in favour of the fact that ECE theory predicts the correct order of the earth's gravitomagnetic precession in the first approximation.

24.5 Detailed metrics

24.5.1 Schwarzschild metric

The so-called Schwarzschild metric is a pure vacuum metric. The interpretation of the parameters (M: mass, G: Newton's constant of gravitation, c: velocity of light) was added later. The Ricci tensor and Einstein tensor vanish as do the cosmological charge and current densities which are a measure for the violation of the dual Bianchi identity (24.1).

24.5.1.1 Coordinates

$$\mathbf{x} = \begin{pmatrix} t \\ r \\ \vartheta \\ \varphi \end{pmatrix}$$

24.5.1.2 Metric

$$g_{\mu\nu} = \begin{pmatrix} \frac{2\,G\,M}{c^2\,r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2\,G\,M}{c^2\,r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\sin^2\vartheta \end{pmatrix}$$

24.5.1.3 Contravariant Metric

$$g^{\mu\nu} = \begin{pmatrix} \frac{c^2\,r}{2\,G\,M-c^2\,r} & 0 & 0 & 0 \\ 0 & -\frac{2\,G\,M-c^2\,r}{c^2\,r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\,\sin^2\vartheta} \end{pmatrix}$$

24.5.1.4 Christoffel Connection

$$\Gamma^0{}_{01} = -\frac{G\,M}{2\,r\,G\,M - c^2\,r^2}$$

$$\Gamma^0{}_{10} = -\frac{G\,M}{2\,r\,G\,M - c^2\,r^2}$$

$$\Gamma^1{}_{00} = -\frac{2\,G^2\,M^2 - c^2\,r\,G\,M}{c^4\,r^3}$$

$$\Gamma^1{}_{11} = \frac{G\,M}{2\,r\,G\,M - c^2\,r^2}$$

$$\Gamma^1{}_{22} = \frac{2\,G\,M - c^2\,r}{c^2}$$

$$\Gamma^1{}_{33} = \frac{2\,\sin^2\vartheta\,G\,M - c^2\,r\,\sin^2\vartheta}{c^2}$$

$$\Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\vartheta\,\sin\vartheta$$

$$\Gamma^3{}_{13} = \frac{1}{r}$$

$$\Gamma^3{}_{23} = \frac{\cos\vartheta}{\sin\vartheta}$$

$$\Gamma^3{}_{31} = \frac{1}{r}$$

$$\Gamma^3{}_{32} = \frac{\cos\vartheta}{\sin\vartheta}$$

24.5.1.5 Metric Compatibility

——— o.k.

24.5.1.6 Riemann Tensor

$$R^0{}_{101} = -\frac{2\,G\,M}{2\,r^2\,G\,M - c^2\,r^3}$$

$$R^0{}_{110} = \frac{2\,G\,M}{2\,r^2\,G\,M - c^2\,r^3}$$

$$R^0{}_{202} = -\frac{G\,M}{c^2\,r}$$

$$R^0{}_{220} = \frac{G\,M}{c^2\,r}$$

$$R^0{}_{303} = -\frac{\sin^2\vartheta\,G\,M}{c^2\,r}$$

$$R^0{}_{330} = \frac{\sin^2\vartheta\, G\, M}{c^2\, r}$$

$$R^1{}_{001} = -\frac{4\, G^2\, M^2 - 2\, c^2\, r\, G\, M}{c^4\, r^4}$$

$$R^1{}_{010} = \frac{4\, G^2\, M^2 - 2\, c^2\, r\, G\, M}{c^4\, r^4}$$

$$R^1{}_{212} = -\frac{G\, M}{c^2\, r}$$

$$R^1{}_{221} = \frac{G\, M}{c^2\, r}$$

$$R^1{}_{313} = -\frac{\sin^2\vartheta\, G\, M}{c^2\, r}$$

$$R^1{}_{331} = \frac{\sin^2\vartheta\, G\, M}{c^2\, r}$$

$$R^2{}_{002} = \frac{2\, G^2\, M^2 - c^2\, r\, G\, M}{c^4\, r^4}$$

$$R^2{}_{020} = -\frac{2\, G^2\, M^2 - c^2\, r\, G\, M}{c^4\, r^4}$$

$$R^2{}_{112} = -\frac{G\, M}{2\, r^2\, G\, M - c^2\, r^3}$$

$$R^2{}_{121} = \frac{G\, M}{2\, r^2\, G\, M - c^2\, r^3}$$

$$R^2{}_{323} = \frac{2\,\sin^2\vartheta\,G\,M}{c^2\,r}$$

$$R^2{}_{332} = -\frac{2\,\sin^2\vartheta\,G\,M}{c^2\,r}$$

$$R^3{}_{003} = \frac{2\,G^2\,M^2 - c^2\,r\,G\,M}{c^4\,r^4}$$

$$R^3{}_{030} = -\frac{2\,G^2\,M^2 - c^2\,r\,G\,M}{c^4\,r^4}$$

$$R^3{}_{113} = -\frac{G\,M}{2\,r^2\,G\,M - c^2\,r^3}$$

$$R^3{}_{131} = \frac{G\,M}{2\,r^2\,G\,M - c^2\,r^3}$$

$$R^3{}_{223} = -\frac{2\,G\,M}{c^2\,r}$$

$$R^3{}_{232} = \frac{2\,G\,M}{c^2\,r}$$

24.5.1.7 Ricci Tensor

———— all elements zero

24.5.1.8 Ricci Scalar

$$R_{sc} = 0$$

24.5.1.9 Bianchi identity (Ricci cyclic equation $R^{\kappa}{}_{[\mu\nu\sigma]} = 0$)

———— o.k.

24.5.1.10 Einstein Tensor

——— all elements zero

24.5.1.11 Hodge Dual of Bianchi Identity

——— (see charge and current densities)

24.5.1.12 Scalar Charge Density ($-R^{0}{}_{i}{}^{i0}$)

$$\rho = 0$$

24.5.1.13 Current Density Class 1 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$J_1 = 0$$
$$J_2 = 0$$
$$J_3 = 0$$

24.5.1.14 Current Density Class 2 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$J_1 = 0$$
$$J_2 = 0$$
$$J_3 = 0$$

24.5.1.15 Current Density Class 3 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$J_1 = 0$$
$$J_2 = 0$$
$$J_3 = 0$$

24.5.2 Reissner-Nordstrom metric

This is a metric of a charged mass. M is a mass parameter, Q a charge parameter. Cosmological charge and current densities do exist.

24.5.2.1 Coordinates

$$\mathbf{x} = \begin{pmatrix} t \\ r \\ \vartheta \\ \varphi \end{pmatrix}$$

24.5.2.2 Metric

$$g_{\mu\nu} = \begin{pmatrix} -\frac{Q^2}{r^2} + \frac{2\,M}{r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\frac{Q^2}{r^2} - \frac{2\,M}{r} + 1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\sin^2\vartheta \end{pmatrix}$$

24.5.2.3 Contravariant Metric

$$g^{\mu\nu} = \begin{pmatrix} -\frac{r^2}{Q^2 - 2\,r\,M + r^2} & 0 & 0 & 0 \\ 0 & \frac{Q^2 - 2\,r\,M + r^2}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\,\sin^2\vartheta} \end{pmatrix}$$

24.5.2.4 Christoffel Connection

$$\Gamma^0{}_{01} = -\frac{Q^2 - r\,M}{r\,Q^2 - 2\,r^2\,M + r^3}$$

$$\Gamma^0{}_{10} = -\frac{Q^2 - r\,M}{r\,Q^2 - 2\,r^2\,M + r^3}$$

$$\Gamma^1{}_{00} = -\frac{Q^4 + \left(r^2 - 3\,r\,M\right)\,Q^2 + 2\,r^2\,M^2 - r^3\,M}{r^5}$$

$$\Gamma^1{}_{11} = \frac{Q^2 - r\,M}{r\,Q^2 - 2\,r^2\,M + r^3}$$

$$\Gamma^1{}_{22} = -\frac{Q^2 - 2\,r\,M + r^2}{r}$$

$$\Gamma^1{}_{33} = -\frac{\sin^2\vartheta\,Q^2 - 2\,r\,\sin^2\vartheta\,M + r^2\,\sin^2\vartheta}{r}$$

$$\Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\cos\vartheta\,\sin\vartheta$$

$$\Gamma^3{}_{13} = \frac{1}{r}$$

$$\Gamma^3{}_{23} = \frac{\cos\vartheta}{\sin\vartheta}$$

$$\Gamma^3{}_{31} = \frac{1}{r}$$

$$\Gamma^3{}_{32} = \frac{\cos\vartheta}{\sin\vartheta}$$

24.5.2.5 Metric Compatibility

——— o.k.

24.5.2.6 Riemann Tensor

$$R^0{}_{101} = -\frac{3\,Q^2 - 2\,r\,M}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

$$R^0{}_{110} = \frac{3\,Q^2 - 2\,r\,M}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

$$R^0{}_{202} = \frac{Q^2 - r\,M}{r^2}$$

$$R^0{}_{220} = -\frac{Q^2 - r\,M}{r^2}$$

$$R^0{}_{303} = \frac{\sin^2\vartheta\,Q^2 - r\,\sin^2\vartheta\,M}{r^2}$$

$$R^0{}_{330} = -\frac{\sin^2\vartheta\,Q^2 - r\,\sin^2\vartheta\,M}{r^2}$$

$$R^1{}_{001} = -\frac{3\,Q^4 + \left(3\,r^2 - 8\,r\,M\right)\,Q^2 + 4\,r^2\,M^2 - 2\,r^3\,M}{r^6}$$

$$R^1{}_{010} = \frac{3\,Q^4 + \left(3\,r^2 - 8\,r\,M\right)\,Q^2 + 4\,r^2\,M^2 - 2\,r^3\,M}{r^6}$$

$$R^1{}_{212} = \frac{Q^2 - r\,M}{r^2}$$

$$R^1{}_{221} = -\frac{Q^2 - r\,M}{r^2}$$

$$R^1{}_{313} = \frac{\sin^2\vartheta\,Q^2 - r\,\sin^2\vartheta\,M}{r^2}$$

$$R^1{}_{331} = -\frac{\sin^2\vartheta\,Q^2 - r\,\sin^2\vartheta\,M}{r^2}$$

$$R^2{}_{002} = \frac{Q^4 + \left(r^2 - 3\,r\,M\right)\,Q^2 + 2\,r^2\,M^2 - r^3\,M}{r^6}$$

$$R^2{}_{020} = -\frac{Q^4 + \left(r^2 - 3\,r\,M\right)\,Q^2 + 2\,r^2\,M^2 - r^3\,M}{r^6}$$

$$R^2{}_{112} = -\frac{Q^2 - r\,M}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

$$R^2{}_{121} = \frac{Q^2 - r\,M}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

$$R^2{}_{323} = -\frac{\sin^2\vartheta\,Q^2 - 2\,r\,\sin^2\vartheta\,M}{r^2}$$

$$R^2{}_{332} = \frac{\sin^2\vartheta\,Q^2 - 2\,r\,\sin^2\vartheta\,M}{r^2}$$

$$R^3{}_{003} = \frac{Q^4 + \left(r^2 - 3\,r\,M\right)\,Q^2 + 2\,r^2\,M^2 - r^3\,M}{r^6}$$

$$R^3{}_{030} = -\frac{Q^4 + \left(r^2 - 3\,r\,M\right)\,Q^2 + 2\,r^2\,M^2 - r^3\,M}{r^6}$$

$$R^3{}_{113} = -\frac{Q^2 - r\,M}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

$$R^3{}_{131} = \frac{Q^2 - r\,M}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

$$R^3{}_{223} = \frac{Q^2 - 2\,r\,M}{r^2}$$

$$R^3{}_{232} = -\frac{Q^2 - 2\,r\,M}{r^2}$$

24.5.2.7 Ricci Tensor

$$\mathrm{Ric}_{00} = \frac{Q^2\,\left(Q^2 - 2\,r\,M + r^2\right)}{r^6}$$

$$\mathrm{Ric}_{11} = -\frac{Q^2}{r^2\,\left(Q^2 - 2\,r\,M + r^2\right)}$$

$$\mathrm{Ric}_{22} = \frac{Q^2}{r^2}$$

$$\mathrm{Ric}_{33} = \frac{\sin^2\vartheta\,Q^2}{r^2}$$

24.5.2.8 Ricci Scalar

$$R_{sc} = 0$$

24.5.2.9 Bianchi identity (Ricci cyclic equation $R^{\kappa}{}_{[\mu\nu\sigma]} = 0$)

——— o.k.

24.5.2.10 Einstein Tensor

——— not zero:

$$G_{00} = \frac{Q^4 + \left(r^2 - 2\,r\,M\right)\,Q^2}{r^6}$$

$$G_{11} = -\frac{Q^2}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

$$G_{22} = \frac{Q^2}{r^2}$$

$$G_{33} = \frac{\sin^2\vartheta\,Q^2}{r^2}$$

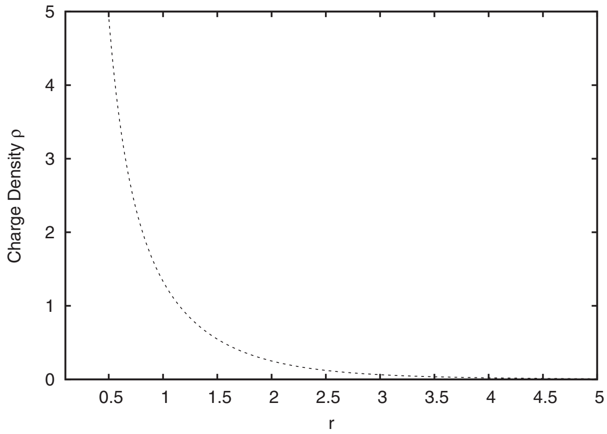

Fig. 24.1. Reissner-Nordstrom metric, cosmological charge density ρ for M = 1, Q = 2.

24.5.2.11 Hodge Dual of Bianchi Identity

——— (see charge and current densities)

24.5.2.12 Scalar Charge Density $(-R^{0}{}_{i}{}^{i0})$

$$\rho = \frac{Q^2}{r^2\,Q^2 - 2\,r^3\,M + r^4}$$

24.5.2.13 Current Density Class 1 $(-R^{i}{}_{\mu}{}^{\mu j})$

$$J_1 = \frac{Q^4 + \left(r^2 - 2\,r\,M\right)\,Q^2}{r^6}$$

$$J_2 = -\frac{Q^2}{r^6}$$

$$J_3 = -\frac{Q^2}{r^6\,\sin^2\vartheta}$$

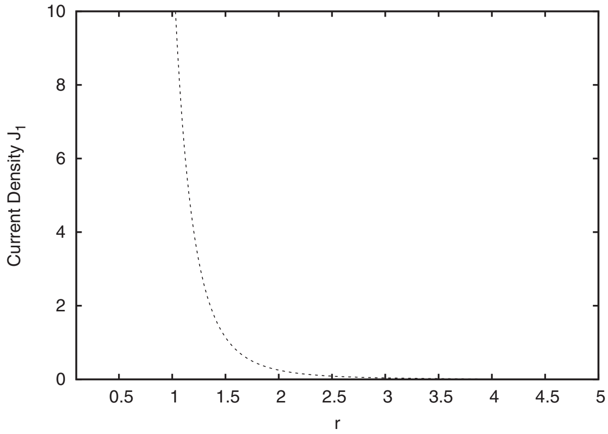

Fig. 24.2. Reissner-Nordstrom metric, cosmological current density J_r for M = 1, Q = 2.

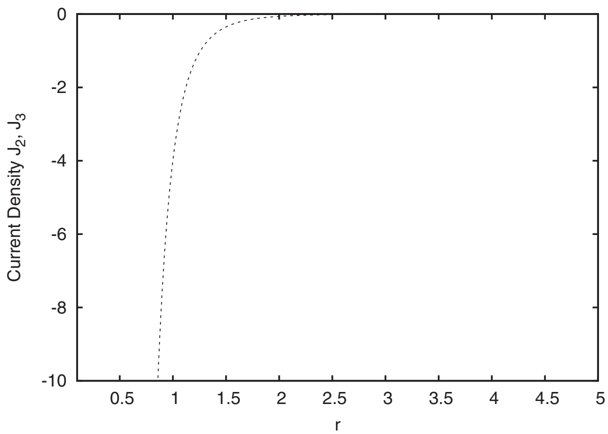

Fig. 24.3. Reissner-Nordstrom metric, cosmological current density J_θ, J_φ for M = 1, Q = 2.

24.5.2.14 Current Density Class 2 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$J_1 = 0$$

$$J_2 = 0$$

$$J_3 = 0$$

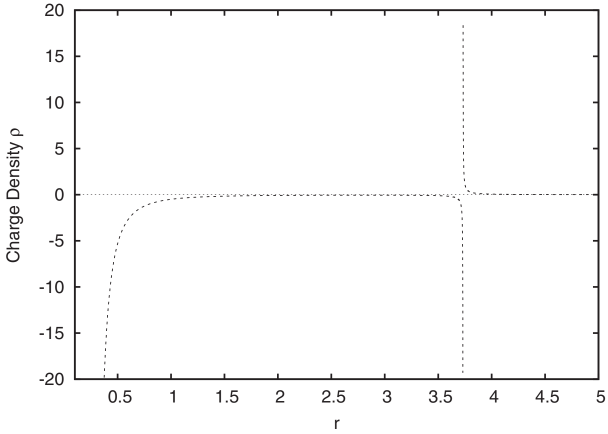

Fig. 24.4. Reissner-Nordstrom metric, cosmological charge density ρ for M=2, Q=1.

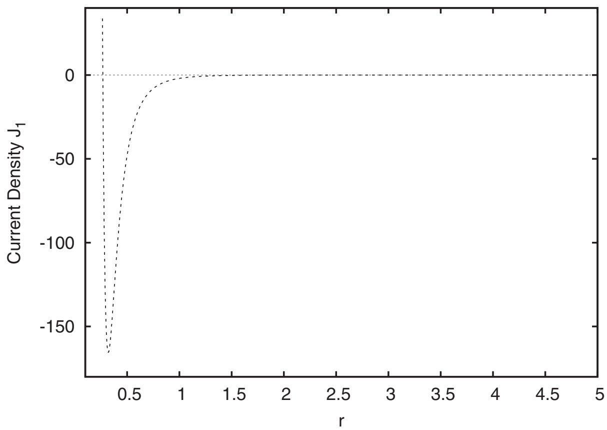

Fig. 24.5. Reissner-Nordstrom metric, cosmological current density J_r, for J_r M=2, Q=1.

24.5.2.15 Current Density Class 3 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$J_1 = 0$$

$$J_2 = 0$$

$$J_3 = 0$$

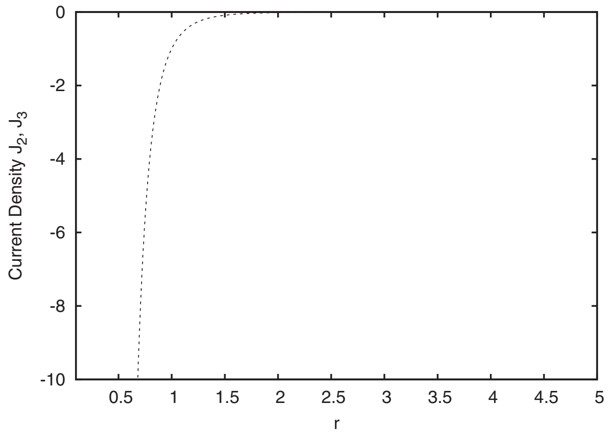

Fig. 24.6. Reissner-Nordstrom metric, cosmological current density J_θ, J_φ for M=2, Q=1.

24.5.3 Kerr metric

This metric describes a rotating mass without charge. M is the mass parameter, J the parameter of angular momentum. Cosmological charge and current densities do exist. There are horizonts (pole locations) in these quantities which give hint to an irregular behaviour of this metric.

24.5.3.1 Coordinates

$$\mathbf{x} = \begin{pmatrix} t \\ r \\ \vartheta \\ \varphi \end{pmatrix}$$

24.5.3.2 Metric

$$g_{\mu\nu} = \begin{pmatrix} \frac{2M}{r} - 1 & 0 & 0 & -\frac{4\sin^2\vartheta\, J}{r} \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\frac{4\sin^2\vartheta\, J}{r} & 0 & 0 & r^2\sin^2\vartheta \end{pmatrix}$$

24.5.3.3 Contravariant Metric

$$g^{\mu\nu} = \begin{pmatrix} \frac{r^4}{2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4} & 0 & 0 & \frac{4\,r\,J}{2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4} \\ 0 & -\frac{2\,M-r}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ \frac{4\,r\,J}{2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4} & 0 & 0 & \frac{r\,(2\,M-r)}{\sin^2\vartheta\,(2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4)} \end{pmatrix}$$

24.5.3.4 Christoffel Connection

$$\Gamma^0{}_{01} = -\frac{r^3\,M - 8\,\sin^2\vartheta\,J^2}{2\,r^4\,M - 16\,r\,\sin^2\vartheta\,J^2 - r^5}$$

$$\Gamma^0{}_{02} = -\frac{16\,\cos\vartheta\,\sin\vartheta\,J^2}{2\,r^3\,M - 16\,\sin^2\vartheta\,J^2 - r^4}$$

$$\Gamma^0{}_{10} = -\frac{r^3\,M - 8\,\sin^2\vartheta\,J^2}{2\,r^4\,M - 16\,r\,\sin^2\vartheta\,J^2 - r^5}$$

$$\Gamma^0{}_{13} = \frac{6\,r^2\,\sin^2\vartheta\,J}{2\,r^3\,M - 16\,\sin^2\vartheta\,J^2 - r^4}$$

$$\Gamma^0{}_{20} = -\frac{16\,\cos\vartheta\,\sin\vartheta\,J^2}{2\,r^3\,M - 16\,\sin^2\vartheta\,J^2 - r^4}$$

$$\Gamma^0{}_{31} = \frac{6\,r^2\,\sin^2\vartheta\,J}{2\,r^3\,M - 16\,\sin^2\vartheta\,J^2 - r^4}$$

$$\Gamma^1{}_{00} = -\frac{2\,M^2 - r\,M}{r^3}$$

$$\Gamma^1{}_{03} = \frac{4\,\sin^2\vartheta\,J\,M - 2\,r\,\sin^2\vartheta\,J}{r^3}$$

$$\Gamma^1{}_{11} = \frac{M}{2\,r\,M - r^2}$$

$$\Gamma^1{}_{22} = 2\,M - r$$

$$\Gamma^1{}_{30} = \frac{4\,\sin^2\vartheta\,J\,M - 2\,r\,\sin^2\vartheta\,J}{r^3}$$

$$\Gamma^1{}_{33} = 2\,\sin^2\vartheta\,M - r\,\sin^2\vartheta$$

$$\Gamma^2{}_{03} = \frac{4\,\cos\vartheta\,\sin\vartheta\,J}{r^3}$$

$$\Gamma^2{}_{12} = \frac{1}{r}$$

$$\Gamma^2{}_{21} = \frac{1}{r}$$

$$\Gamma^2{}_{30} = \frac{4\,\cos\vartheta\,\sin\vartheta\,J}{r^3}$$

$$\Gamma^2{}_{33} = -\cos\vartheta\,\sin\vartheta$$

$$\Gamma^3{}_{01} = -\frac{2\,J}{2\,r^3\,M - 16\,\sin^2\vartheta\,J^2 - r^4}$$

$$\Gamma^3{}_{02} = -\frac{8\,\cos\vartheta\,J\,M - 4\,r\,\cos\vartheta\,J}{2\,r^3\,\sin\vartheta\,M - 16\,\sin^3\vartheta\,J^2 - r^4\,\sin\vartheta}$$

$$\Gamma^3{}_{10} = -\frac{2\,J}{2\,r^3\,M - 16\,\sin^2\vartheta\,J^2 - r^4}$$

$$\Gamma^3{}_{13} = \frac{2\,r^3\,M + 8\,\sin^2\vartheta\,J^2 - r^4}{2\,r^4\,M - 16\,r\,\sin^2\vartheta\,J^2 - r^5}$$

$$\Gamma^3{}_{20} = -\frac{8\,\cos\vartheta\,J\,M - 4\,r\,\cos\vartheta\,J}{2\,r^3\,\sin\vartheta\,M - 16\,\sin^3\vartheta\,J^2 - r^4\,\sin\vartheta}$$

$$\Gamma^3{}_{23} = \frac{\cos\vartheta}{\sin\vartheta}$$

$$\Gamma^3{}_{31} = \frac{2\,r^3\,M + 8\,\sin^2\vartheta\,J^2 - r^4}{2\,r^4\,M - 16\,r\,\sin^2\vartheta\,J^2 - r^5}$$

$$\Gamma^3{}_{32} = \frac{\cos\vartheta}{\sin\vartheta}$$

24.5.3.5 Metric Compatibility

———— o.k.

24.5.3.6 Riemann Tensor

$$R^0{}_{003} = \frac{8\,r^3\,\sin^2\vartheta\,J\,M^2 + \left(32\,\sin^4\vartheta\,J^3 - 4\,r^4\,\sin^2\vartheta\,J\right)M + \left(48\,r\,\sin^4\vartheta - 64\,r\,\sin^2\vartheta\right)J^3}{2\,r^7\,M - 16\,r^4\,\sin^2\vartheta\,J^2 - r^8}$$

$$R^0{}_{012} = \frac{48\,r^2\,\cos\vartheta\,\sin\vartheta\,J^2\,M - 24\,r^3\,\cos\vartheta\,\sin\vartheta\,J^2}{4\,r^6\,M^2 + (-64\,r^3\,\sin^2\vartheta\,J^2 - 4\,r^7)\,M + 256\,\sin^4\vartheta\,J^4 + 32\,r^4\,\sin^2\vartheta\,J^2 + r^8}$$

$$R^0{}_{021} = -\frac{48\,r^2\,\cos\vartheta\,\sin\vartheta\,J^2\,M - 24\,r^3\,\cos\vartheta\,\sin\vartheta\,J^2}{4\,r^6\,M^2 + (-64\,r^3\,\sin^2\vartheta\,J^2 - 4\,r^7)\,M + 256\,\sin^4\vartheta\,J^4 + 32\,r^4\,\sin^2\vartheta\,J^2 + r^8}$$

$$R^0{}_{030} = -\frac{8\,r^3\,\sin^2\vartheta\,J\,M^2 + \left(32\,\sin^4\vartheta\,J^3 - 4\,r^4\,\sin^2\vartheta\,J\right)M + \left(48\,r\,\sin^4\vartheta - 64\,r\,\sin^2\vartheta\right)J^3}{2\,r^7\,M - 16\,r^4\,\sin^2\vartheta\,J^2 - r^8}$$

$$R^0{}_{101} = -R^0{}_{110}$$

$$R^0{}_{102} = -\frac{48\,r^3\,\cos\vartheta\,\sin\vartheta\,J^2\,M - 384\,\cos\vartheta\,\sin^3\vartheta\,J^4 - 32\,r^4\,\cos\vartheta\,\sin\vartheta\,J^2}{4\,r^7\,M^2 + (-64\,r^4\,\sin^2\vartheta\,J^2 - 4\,r^8)\,M + 256\,r\,\sin^4\vartheta\,J^4 + 32\,r^5\,\sin^2\vartheta\,J^2 + r^9}$$

$$R^0{}_{110} = \frac{8r^6M^3 + \left(-128r^3\sin^2\vartheta J^2 - 8r^7\right)M^2 + \left(512\sin^4\vartheta J^4 + 112r^4\sin^2\vartheta J^2 + 2r^8\right)M - 192r\sin^4\vartheta J^4 - 28r^5\sin^2\vartheta J^2}{8r^8M^3 + \left(-128r^5\sin^2\vartheta J^2 - 12r^9\right)M^2 + \left(512r^2\sin^4\vartheta J^4 + 128r^6\sin^2\vartheta J^2 + 6r^{10}\right)M - 256r^3\sin^4\vartheta J^4 - 32r^7\sin^2\vartheta J^2 - r^{11}}$$

$$R^0{}_{113} = -\frac{24r^4 \sin^2 \vartheta J M^2 + \left(96 r \sin^4 \vartheta J^3 - 24 r^5 \sin^2 \vartheta J\right) M - 96 r^2 \sin^4 \vartheta J^3 + 6 r^6 \sin^2 \vartheta J}{8 r^6 M^3 + \left(-128 r^3 \sin^2 \vartheta J^2 - 12 r^7\right) M^2 + \left(512 \sin^4 \vartheta J^4 + 128 r^4 \sin^2 \vartheta J^2 + 6 r^8\right) M - 256 r \sin^4 \vartheta J^4 - 32 r^5 \sin^2 \vartheta J^2 - r^9}$$

$$R^0{}_{120} = \frac{48\, r^3 \cos\vartheta \sin\vartheta\, J^2 M - 384 \cos\vartheta \sin^3\vartheta\, J^4 - 32\, r^4 \cos\vartheta \sin\vartheta\, J^2}{4\, r^7 M^2 + (-64\, r^4 \sin^2\vartheta\, J^2 - 4\, r^8)\, M + 256\, r \sin^4\vartheta\, J^4 + 32\, r^5 \sin^2\vartheta\, J^2 + r^9}$$

$$R^0{}_{123} = \frac{24\, r^5 \cos\vartheta \sin\vartheta\, J\, M - 96\, r^2 \cos\vartheta \sin^3\vartheta\, J^3 - 12\, r^6 \cos\vartheta \sin\vartheta\, J}{4\, r^6 M^2 + (-64\, r^3 \sin^2\vartheta\, J^2 - 4\, r^7)\, M + 256 \sin^4\vartheta\, J^4 + 32\, r^4 \sin^2\vartheta\, J^2 + r^8}$$

$$R^0{}_{131} = \frac{24r^4 \sin^2 \vartheta J M^2 + \left(96 r \sin^4 \vartheta J^3 - 24 r^5 \sin^2 \vartheta J\right) M - 96 r^2 \sin^4 \vartheta J^3 + 6 r^6 \sin^2 \vartheta J}{8 r^6 M^3 + \left(-128 r^3 \sin^2 \vartheta J^2 - 12 r^7\right) M^2 + \left(512 \sin^4 \vartheta J^4 + 128 r^4 \sin^2 \vartheta J^2 + 6 r^8\right) M - 256 r \sin^4 \vartheta J^4 - 32 r^5 \sin^2 \vartheta J^2 - r^9}$$

$$R^0{}_{132} = -\frac{24\, r^5 \cos\vartheta \sin\vartheta\, J\, M - 96\, r^2 \cos\vartheta \sin^3\vartheta\, J^3 - 12\, r^6 \cos\vartheta \sin\vartheta\, J}{4\, r^6 M^2 + (-64\, r^3 \sin^2\vartheta\, J^2 - 4\, r^7)\, M + 256 \sin^4\vartheta\, J^4 + 32\, r^4 \sin^2\vartheta\, J^2 + r^8}$$

$$R^0{}_{201} = -\frac{96\, r^3 \cos\vartheta \sin\vartheta\, J^2 M - 384 \cos\vartheta \sin^3\vartheta\, J^4 - 56\, r^4 \cos\vartheta \sin\vartheta\, J^2}{4\, r^7 M^2 + (-64\, r^4 \sin^2\vartheta\, J^2 - 4\, r^8)\, M + 256\, r \sin^4\vartheta\, J^4 + 32\, r^5 \sin^2\vartheta\, J^2 + r^9}$$

$$N = 4\, r^7 M^2 - 64\, r^4 \sin^2\vartheta\, J^2 M - 4\, r^8 M + 256\, r \sin^4\vartheta\, J^4 + 32\, r^5 \sin^2\vartheta\, J^2 + r^9$$

$$R^0{}_{202} = \frac{-4\, r^6 M^3 + 64\, r^3 \sin^2\vartheta\, J^2 M^2 + 4\, r^7 M^2 - 256 \sin^4\vartheta\, J^4 M - 112\, r^4 \sin^2\vartheta\, J^2 M}{N} + \frac{32\, r^4 J^2 M - r^8 M + 384\, r \sin^4\vartheta\, J^4 + 40\, r^5 \sin^2\vartheta\, J^2 - 16\, r^5 J^2}{N}$$

$$R^0{}_{210} = \frac{96\, r^3 \cos\vartheta \sin\vartheta\, J^2 M - 384 \cos\vartheta \sin^3\vartheta\, J^4 - 56\, r^4 \cos\vartheta \sin\vartheta\, J^2}{4\, r^7 M^2 + (-64\, r^4 \sin^2\vartheta\, J^2 - 4\, r^8)\, M + 256\, r \sin^4\vartheta\, J^4 + 32\, r^5 \sin^2\vartheta\, J^2 + r^9}$$

$$R^0{}_{213} = \frac{6\, r^2 \cos\vartheta \sin\vartheta\, J}{2\, r^3 M - 16 \sin^2\vartheta\, J^2 - r^4}$$

$$R^0{}_{220} = -R^0{}_{202}$$

$$R^0{}_{223} = -\frac{12\, r^2 \sin^2\vartheta\, J\, M - 6\, r^3 \sin^2\vartheta\, J}{2\, r^3 M - 16 \sin^2\vartheta\, J^2 - r^4}$$

$$R^0{}_{231} = -\frac{6\,r^2\,\cos\vartheta\,\sin\vartheta\,J}{2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4}$$

$$R^0{}_{232} = \frac{12\,r^2\,\sin^2\vartheta\,J\,M-6\,r^3\,\sin^2\vartheta\,J}{2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4}$$

$$R^0{}_{303} = -\frac{2\,r^3\,\sin^2\vartheta\,M^2+\left(8\,\sin^4\vartheta\,J^2-r^4\,\sin^2\vartheta\right)M+\left(12\,r\,\sin^4\vartheta-16\,r\,\sin^2\vartheta\right)J^2}{2\,r^4\,M-16\,r\,\sin^2\vartheta\,J^2-r^5}$$

$$R^0{}_{312} = -\frac{12\,r^5\,\cos\vartheta\,\sin\vartheta\,J\,M-6\,r^6\,\cos\vartheta\,\sin\vartheta\,J}{4\,r^6\,M^2+(-64\,r^3\,\sin^2\vartheta\,J^2-4\,r^7)\,M+256\,\sin^4\vartheta\,J^4+32\,r^4\,\sin^2\vartheta\,J^2+r^8}$$

$$R^0{}_{321} = \frac{12\,r^5\,\cos\vartheta\,\sin\vartheta\,J\,M-6\,r^6\,\cos\vartheta\,\sin\vartheta\,J}{4\,r^6\,M^2+(-64\,r^3\,\sin^2\vartheta\,J^2-4\,r^7)\,M+256\,\sin^4\vartheta\,J^4+32\,r^4\,\sin^2\vartheta\,J^2+r^8}$$

$$R^0{}_{330} = \frac{2\,r^3\,\sin^2\vartheta\,M^2+\left(8\,\sin^4\vartheta\,J^2-r^4\,\sin^2\vartheta\right)M+\left(12\,r\,\sin^4\vartheta-16\,r\,\sin^2\vartheta\right)J^2}{2\,r^4\,M-16\,r\,\sin^2\vartheta\,J^2-r^5}$$

$$R^1{}_{001} = -\frac{8\,r^3\,M^3+\left(-64\,\sin^2\vartheta\,J^2-8\,r^4\right)M^2+\left(32\,r\,\sin^2\vartheta\,J^2+2\,r^5\right)M-4\,r^2\,\sin^2\vartheta\,J^2}{2\,r^7\,M-16\,r^4\,\sin^2\vartheta\,J^2-r^8}$$

$$R^1{}_{002} = \frac{16\,\cos\vartheta\,\sin\vartheta\,J^2\,M-8\,r\,\cos\vartheta\,\sin\vartheta\,J^2}{2\,r^5\,M-16\,r^2\,\sin^2\vartheta\,J^2-r^6}$$

$$R^1{}_{010} = \frac{8\,r^3\,M^3+\left(-64\,\sin^2\vartheta\,J^2-8\,r^4\right)M^2+\left(32\,r\,\sin^2\vartheta\,J^2+2\,r^5\right)M-4\,r^2\,\sin^2\vartheta\,J^2}{2\,r^7\,M-16\,r^4\,\sin^2\vartheta\,J^2-r^8}$$

$$R^1{}_{013} = -\frac{16\,r^3\,\sin^2\vartheta\,J\,M^2+\left(-128\,\sin^4\vartheta\,J^3-20\,r^4\,\sin^2\vartheta\,J\right)M+48\,r\,\sin^4\vartheta\,J^3+6\,r^5\,\sin^2\vartheta\,J}{2\,r^7\,M-16\,r^4\,\sin^2\vartheta\,J^2-r^8}$$

$$R^1{}_{020} = -\frac{16\,\cos\vartheta\,\sin\vartheta\,J^2\,M-8\,r\,\cos\vartheta\,\sin\vartheta\,J^2}{2\,r^5\,M-16\,r^2\,\sin^2\vartheta\,J^2-r^6}$$

$$R^1{}_{023} = \frac{48r^3\cos\vartheta\sin\vartheta\,J\,M^2+\left(-192\cos\vartheta\sin^3\vartheta\,J^3-48r^4\cos\vartheta\sin\vartheta\,J\right)M+96r\cos\vartheta\sin^3\vartheta\,J^3+12r^5\cos\vartheta\sin\vartheta\,J}{2r^6M-16r^3\sin^2\vartheta\,J^2-r^7}$$

$$R^1{}_{031} = \frac{16\,r^3\,\sin^2\vartheta\,J\,M^2+\left(-128\,\sin^4\vartheta\,J^3-20\,r^4\,\sin^2\vartheta\,J\right)M+48\,r\,\sin^4\vartheta\,J^3+6\,r^5\,\sin^2\vartheta\,J}{2\,r^7\,M-16\,r^4\,\sin^2\vartheta\,J^2-r^8}$$

$$R^1{}_{032} = -\frac{48r^3\cos\vartheta\sin\vartheta\,J\,M^2+\left(-192\cos\vartheta\sin^3\vartheta\,J^3-48r^4\cos\vartheta\sin\vartheta\,J\right)M+96r\cos\vartheta\sin^3\vartheta\,J^3+12r^5\cos\vartheta\sin\vartheta\,J}{2r^6M-16r^3\sin^2\vartheta\,J^2-r^7}$$

$$R^1{}_{203} = \frac{24\,\cos\vartheta\,\sin\vartheta\,J\,M^2-24\,r\,\cos\vartheta\,\sin\vartheta\,J\,M+6\,r^2\,\cos\vartheta\,\sin\vartheta\,J}{2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4}$$

$$R^1{}_{212} = -\frac{M}{r}$$

$$R^1{}_{221} = \frac{M}{r}$$

$$R^1{}_{230} = -\frac{24\cos\vartheta\sin\vartheta\, J\, M^2 - 24\, r\cos\vartheta\sin\vartheta\, J\, M + 6\, r^2\cos\vartheta\sin\vartheta\, J}{2\, r^3\, M - 16\sin^2\vartheta\, J^2 - r^4}$$

$$R^1{}_{301} = \frac{16\, r^3\sin^2\vartheta\, J\, M^2 + \left(-128\sin^4\vartheta\, J^3 - 20\, r^4\sin^2\vartheta\, J\right) M + 48\, r\sin^4\vartheta\, J^3 + 6\, r^5\sin^2\vartheta\, J}{2\, r^7\, M - 16\, r^4\sin^2\vartheta\, J^2 - r^8}$$

$$R^1{}_{302} = -\frac{12\cos\vartheta\sin\vartheta\, J\, M - 6\, r\cos\vartheta\sin\vartheta\, J}{r^3}$$

$$R^1{}_{310} = -\frac{16r^3\sin^2\vartheta J M^2 + \left(-128\sin^4\vartheta J^3 - 20r^4\sin^2\vartheta J\right) M + 48r\sin^4\vartheta J^3 + 6r^5\sin^2\vartheta J}{2r^7 M - 16r^4\sin^2\vartheta J^2 - r^8}$$

$$R^1{}_{313} = -\frac{2\, r^3\sin^2\vartheta\, M^2 + \left(56\sin^4\vartheta\, J^2 - r^4\sin^2\vartheta\right) M - 36\, r\sin^4\vartheta\, J^2}{2\, r^4\, M - 16\, r\sin^2\vartheta\, J^2 - r^5}$$

$$R^1{}_{320} = \frac{12\cos\vartheta\sin\vartheta\, J\, M - 6\, r\cos\vartheta\sin\vartheta\, J}{r^3}$$

$$R^1{}_{331} = \frac{2\, r^3\sin^2\vartheta\, M^2 + \left(56\sin^4\vartheta\, J^2 - r^4\sin^2\vartheta\right) M - 36\, r\sin^4\vartheta\, J^2}{2\, r^4\, M - 16\, r\sin^2\vartheta\, J^2 - r^5}$$

$$R^2{}_{001} = -\frac{8\cos\vartheta\sin\vartheta\, J^2}{2\, r^6\, M - 16\, r^3\sin^2\vartheta\, J^2 - r^7}$$

$$R^2{}_{002} = \frac{4r^3 M^3 + \left(-32\sin^2\vartheta J^2 - 4r^4\right) M^2 + \left(\left(48r\sin^2\vartheta - 32r\right) J^2 + r^5\right) M + \left(16r^2 - 16r^2\sin^2\vartheta\right) J^2}{2r^7 M - 16r^4\sin^2\vartheta J^2 - r^8}$$

$$R^2{}_{010} = \frac{8\cos\vartheta\sin\vartheta\, J^2}{2\, r^6\, M - 16\, r^3\sin^2\vartheta\, J^2 - r^7}$$

$$R^2{}_{013} = -\frac{6\cos\vartheta\sin\vartheta\, J}{r^4}$$

$$R^2{}_{020} = -\frac{4r^3 M^3 + \left(-32\sin^2\vartheta J^2 - 4r^4\right) M^2 + \left(\left(48r\sin^2\vartheta - 32r\right) J^2 + r^5\right) M + \left(16r^2 - 16r^2\sin^2\vartheta\right) J^2}{2\, r^7\, M - 16r^4\sin^2\vartheta J^2 - r^8}$$

$$R^2{}_{023} = \frac{4\sin^2\vartheta\, J\, M - 6\, r\sin^2\vartheta\, J}{r^4}$$

$$R^2{}_{031} = \frac{6\,\cos\vartheta\,\sin\vartheta\,J}{r^4}$$

$$R^2{}_{032} = -\frac{4\,\sin^2\vartheta\,J\,M - 6\,r\,\sin^2\vartheta\,J}{r^4}$$

$$R^2{}_{103} = \frac{12\,\cos\vartheta\,\sin\vartheta\,J\,M - 6\,r\,\cos\vartheta\,\sin\vartheta\,J}{2\,r^4\,M - 16\,r\,\sin^2\vartheta\,J^2 - r^5}$$

$$R^2{}_{112} = -\frac{M}{2\,r^2\,M - r^3}$$

$$R^2{}_{121} = \frac{M}{2\,r^2\,M - r^3}$$

$$R^2{}_{130} = -\frac{12\,\cos\vartheta\,\sin\vartheta\,J\,M - 6\,r\,\cos\vartheta\,\sin\vartheta\,J}{2\,r^4\,M - 16\,r\,\sin^2\vartheta\,J^2 - r^5}$$

$$R^2{}_{301} = \frac{24\,r^3\,\cos\vartheta\,\sin\vartheta\,J\,M - 96\,\cos\vartheta\,\sin^3\vartheta\,J^3 - 12\,r^4\,\cos\vartheta\,\sin\vartheta\,J}{2\,r^7\,M - 16\,r^4\,\sin^2\vartheta\,J^2 - r^8}$$

$$R^2{}_{302} = -\frac{4\,\sin^2\vartheta\,J\,M - 6\,r\,\sin^2\vartheta\,J}{r^4}$$

$$R^2{}_{310} = -\frac{24\,r^3\,\cos\vartheta\,\sin\vartheta\,J\,M - 96\,\cos\vartheta\,\sin^3\vartheta\,J^3 - 12\,r^4\,\cos\vartheta\,\sin\vartheta\,J}{2\,r^7\,M - 16\,r^4\,\sin^2\vartheta\,J^2 - r^8}$$

$$R^2{}_{320} = \frac{4\,\sin^2\vartheta\,J\,M - 6\,r\,\sin^2\vartheta\,J}{r^4}$$

$$R^2{}_{323} = \frac{2\,\sin^2\vartheta\,M}{r}$$

$$R^2{}_{332} = -\frac{2\,\sin^2\vartheta\,M}{r}$$

$$R^3{}_{003} = \frac{4r^3M^3 + \left(16\sin^2\vartheta J^2 - 4r^4\right)M^2 + \left(\left(16r\sin^2\vartheta - 32r\right)J^2 + r^5\right)M + \left(16r^2 - 12r^2\sin^2\vartheta\right)J^2}{2r^7M - 16r^4\sin^2\vartheta J^2 - r^8}$$

$$R^3{}_{012} = \frac{24r^2\cos\vartheta JM^2 - 24r^3\cos\vartheta JM + 6r^4\cos\vartheta J}{4r^6\sin\vartheta M^2 + \left(-64r^3\sin^3\vartheta J^2 - 4r^7\sin\vartheta\right)M + 256\sin^5\vartheta J^4 + 32r^4\sin^3\vartheta J^2 + r^8\sin\vartheta}$$

$$R^3{}_{021} = -\frac{24r^2\cos\vartheta JM^2 - 24r^3\cos\vartheta JM + 6r^4\cos\vartheta J}{4r^6\sin\vartheta M^2 + \left(-64r^3\sin^3\vartheta J^2 - 4r^7\sin\vartheta\right)M + 256\sin^5\vartheta J^4 + 32r^4\sin^3\vartheta J^2 + r^8\sin\vartheta}$$

$$R^3{}_{030} = -\frac{4^3 M^3 + \left(16 \sin^2\vartheta J^2 - 4r^4\right) M^2 + \left(\left(16r \sin^2\vartheta - 32r\right) J^2 + r^5\right) M + \left(16r^2 - 12r^2 \sin^2\vartheta\right) J^2}{2r^7 M - 16r^4 \sin^2\vartheta J^2 - r^8}$$

$$R^3{}_{101} = -\frac{24r^3 J M^2 + \left(-96 \sin^2\vartheta J^3 - 24r^4 J\right) M + 32r \sin^2\vartheta J^3 + 6r^5 J}{8r^7 M^3 + \left(-128r^4 \sin^2\vartheta J^2 - 12r^8\right) M^2 + \left(512r \sin^4\vartheta J^4 + 128r^5 \sin^2\vartheta J^2 + 6r^9\right) M - 256r^2 \sin^4\vartheta J^4 - 32r^6 \sin^2\vartheta J^2 - r^{10}}$$

$$R^3{}_{102} = -\frac{24r^3 \cos\vartheta J M^2 + \left(-192 \cos\vartheta \sin^2\vartheta J^3 - 24r^4 \cos\vartheta J\right) M + 64r \cos\vartheta \sin^2\vartheta J^3 + 6r^5 \cos\vartheta J}{4r^7 \sin\vartheta M^2 + (-64r^4 \sin^3\vartheta J^2 - 4r^8 \sin\vartheta) M + 256r \sin^5\vartheta J^4 + 32r^5 \sin^3\vartheta J^2 + r^9 \sin\vartheta}$$

$$R^3{}_{110} = \frac{24r^3 J M^2 + \left(-96 \sin^2\vartheta J^3 - 24r^4 J\right) M + 32r \sin^2\vartheta J^3 + 6r^5 J}{8r^7 M^3 + \left(-128r^4 \sin^2\vartheta J^2 - 12r^8\right) M^2 + \left(512r \sin^4\vartheta J^4 + 128r^5 \sin^2\vartheta J^2 + 6r^9\right) M - 256r^2 \sin^4\vartheta J^4 - 32r^6 \sin^2\vartheta J^2 - r^{10}}$$

$$R^3{}_{113} = -R^3{}_{131}$$

$$R^3{}_{120} = \frac{24r^3 \cos\vartheta J M^2 + \left(-192 \cos\vartheta \sin^2\vartheta J^3 - 24r^4 \cos\vartheta J\right) M + 64r \cos\vartheta \sin^2\vartheta J^3 + 6r^5 \cos\vartheta J}{4r^7 \sin\vartheta M^2 + (-64r^4 \sin^3\vartheta J^2 - 4r^8 \sin\vartheta) M + 256r \sin^5\vartheta J^4 + 32r^5 \sin^3\vartheta J^2 + r^9 \sin\vartheta}$$

$$R^3{}_{123} = \frac{96\, r^3 \cos\vartheta \sin\vartheta J^2 M - 384 \cos\vartheta \sin^3\vartheta J^4 - 48\, r^4 \cos\vartheta \sin\vartheta J^2}{4\, r^7 M^2 + (-64\, r^4 \sin^2\vartheta J^2 - 4\, r^8) M + 256\, r \sin^4\vartheta J^4 + 32\, r^5 \sin^2\vartheta J^2 + r^9}$$

$$R^3{}_{131} = \frac{4r^6 M^3 + \left(176r^3 \sin^2\vartheta J^2 - 4r^7\right) M^2 + \left(-512 \sin^4\vartheta J^4 - 208r^4 \sin^2\vartheta J^2 + r^8\right) M + 192r \sin^4\vartheta J^4 + 60r^5 \sin^2\vartheta J^2}{8r^8 M^3 + \left(-128r^5 \sin^2\vartheta J^2 - 12r^9\right) M^2 + \left(512r^2 \sin^4\vartheta J^4 + 128r^6 \sin^2\vartheta J^2 + 6r^{10}\right) M - 256r^3 \sin^4\vartheta J^4 - 32r^7 \sin^2\vartheta J^2 - r^{11}}$$

$$R^3{}_{132} = -\frac{96\, r^3 \cos\vartheta \sin\vartheta J^2 M - 384 \cos\vartheta \sin^3\vartheta J^4 - 48\, r^4 \cos\vartheta \sin\vartheta J^2}{4\, r^7 M^2 + (-64\, r^4 \sin^2\vartheta J^2 - 4\, r^8) M + 256\, r \sin^4\vartheta J^4 + 32\, r^5 \sin^2\vartheta J^2 + r^9}$$

$$R^3{}_{201} = -\frac{48r^3 \cos\vartheta J M^2 + \left(-192 \cos\vartheta \sin^2\vartheta J^3 - 48r^4 \cos\vartheta J\right) M + 64r \cos\vartheta \sin^2\vartheta J^3 + 12r^5 \cos\vartheta J}{4r^7 \sin\vartheta M^2 + \left(-64r^4 \sin^3\vartheta J^2 - 4r^8 \sin\vartheta\right) M + 256r \sin^5\vartheta J^4 + 32r^5 \sin^3\vartheta J^2 + r^9 \sin\vartheta}$$

$$R^3{}_{202} = -\frac{24\, r^3 J M^2 + \left(\left(-64 \sin^2\vartheta - 128\right) J^3 - 24\, r^4 J\right) M + \left(32\, r \sin^2\vartheta + 64\, r\right) J^3 + 6\, r^5 J}{4\, r^6 M^2 + (-64\, r^3 \sin^2\vartheta J^2 - 4\, r^7) M + 256 \sin^4\vartheta J^4 + 32\, r^4 \sin^2\vartheta J^2 + r^8}$$

$$R^3{}_{210} = \frac{48\, r^3 \cos\vartheta J M^2 + \left(-192 \cos\vartheta \sin^2\vartheta J^3 - 48\, r^4 \cos\vartheta J\right) M + 64\, r \cos\vartheta \sin^2\vartheta J^3 + 12\, r^5 \cos\vartheta J}{4\, r^7 \sin\vartheta M^2 + \left(-64\, r^4 \sin^3\vartheta J^2 - 4\, r^8 \sin\vartheta\right) M + 256\, r \sin^5\vartheta J^4 + 32\, r^5 \sin^3\vartheta J^2 + r^9 \sin\vartheta}$$

$$R^3{}_{213} = \frac{24 \cos\vartheta \sin\vartheta J^2}{2\, r^4 M - 16\, r \sin^2\vartheta J^2 - r^5}$$

$$R^3{}_{220} = \frac{24\, r^3 J M^2 + \left(\left(-64 \sin^2\vartheta - 128\right) J^3 - 24\, r^4 J\right) M + \left(32\, r \sin^2\vartheta + 64\, r\right) J^3 + 6\, r^5 J}{4\, r^6 M^2 + (-64\, r^3 \sin^2\vartheta J^2 - 4\, r^7) M + 256 \sin^4\vartheta J^4 + 32\, r^4 \sin^2\vartheta J^2 + r^8}$$

$$R^3{}_{223} = -\frac{4\,r^3\,M^2+\left(16\,\sin^2\vartheta\,J^2-2\,r^4\right)M-24\,r\,\sin^2\vartheta\,J^2}{2\,r^4\,M-16\,r\,\sin^2\vartheta\,J^2-r^5}$$

$$R^3{}_{231} = -\frac{24\,\cos\vartheta\,\sin\vartheta\,J^2}{2\,r^4\,M-16\,r\,\sin^2\vartheta\,J^2-r^5}$$

$$R^3{}_{232} = \frac{4\,r^3\,M^2+\left(16\,\sin^2\vartheta\,J^2-2\,r^4\right)M-24\,r\,\sin^2\vartheta\,J^2}{2\,r^4\,M-16\,r\,\sin^2\vartheta\,J^2-r^5}$$

$$R^3{}_{303} = -\frac{8\,r^3\,\sin^2\vartheta\,J\,M^2+\left(32\,\sin^4\vartheta\,J^3-4\,r^4\,\sin^2\vartheta\,J\right)M+\left(48\,r\,\sin^4\vartheta-64\,r\,\sin^2\vartheta\right)J^3}{2\,r^7\,M-16\,r^4\,\sin^2\vartheta\,J^2-r^8}$$

$$R^3{}_{312} = -\frac{48\,r^2\,\cos\vartheta\,\sin\vartheta\,J^2\,M-24\,r^3\,\cos\vartheta\,\sin\vartheta\,J^2}{4\,r^6\,M^2+(-64\,r^3\,\sin^2\vartheta\,J^2-4\,r^7)\,M+256\,\sin^4\vartheta\,J^4+32\,r^4\,\sin^2\vartheta\,J^2+r^8}$$

$$R^3{}_{321} = \frac{48\,r^2\,\cos\vartheta\,\sin\vartheta\,J^2\,M-24\,r^3\,\cos\vartheta\,\sin\vartheta\,J^2}{4\,r^6\,M^2+(-64\,r^3\,\sin^2\vartheta\,J^2-4\,r^7)\,M+256\,\sin^4\vartheta\,J^4+32\,r^4\,\sin^2\vartheta\,J^2+r^8}$$

$$R^3{}_{330} = \frac{8\,r^3\,\sin^2\vartheta\,J\,M^2+\left(32\,\sin^4\vartheta\,J^3-4\,r^4\,\sin^2\vartheta\,J\right)M+\left(48\,r\,\sin^4\vartheta-64\,r\,\sin^2\vartheta\right)J^3}{2\,r^7\,M-16\,r^4\,\sin^2\vartheta\,J^2-r^8}$$

24.5.3.7 Ricci Tensor

$$\mathrm{Ric}_{00} = \frac{8\,J^2\,\left(6\,\cos^2\vartheta\,M^2-6\,M^2+4\,r\,\cos^2\vartheta\,M+4\,r\,M-3\,r^2\,\cos^2\vartheta-r^2\right)}{r^4\,\left(2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4\right)}$$

$$\mathrm{Ric}_{03} = \frac{32\,\sin^2\vartheta\,J^3\,\left(3\,\sin^2\vartheta\,M+3\,r\,\sin^2\vartheta-2\,r\right)}{r^4\,\left(2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4\right)}$$

$$\mathrm{Ric}_{11} = \frac{8\sin^2\vartheta J^2\left(30r^3M^2-96\sin^2\vartheta J^2M-36r^4M+48r\sin^2\vartheta J^2+11r^5\right)}{r^2\,(2M-r)\left(2r^3M-16\sin^2\vartheta J^2-r^4\right)^2}$$

$$\mathrm{Ric}_{12} = -\frac{16\,\cos\vartheta\,\sin\vartheta\,J^2\,\left(9\,r^3\,M-48\,\sin^2\vartheta\,J^2-5\,r^4\right)}{r\,\left(2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4\right)^2}$$

$$\mathrm{Ric}_{21} = -\frac{16\,\cos\vartheta\,\sin\vartheta\,J^2\,\left(9\,r^3\,M-48\,\sin^2\vartheta\,J^2-5\,r^4\right)}{r\,\left(2\,r^3\,M-16\,\sin^2\vartheta\,J^2-r^4\right)^2}$$

$$\mathrm{Ric}_{22} = \frac{16J^2 \left(6r^3 \sin^2 \vartheta M^2 - 48 \sin^4 \vartheta J^2 M - 11r^4 \sin^2 \vartheta M + 2r^4 M + 48r \sin^4 \vartheta J^2 + 4r^5 \sin^2 \vartheta - r^5\right)}{r \left(2r^3 M - 16 \sin^2 \vartheta J^2 - r^4\right)^2}$$

$$\mathrm{Ric}_{30} = \frac{32 \sin^2 \vartheta \, J^3 \, \left(3 \sin^2 \vartheta \, M + 3\, r \sin^2 \vartheta - 2\, r\right)}{r^4 \, \left(2\, r^3 \, M - 16 \sin^2 \vartheta \, J^2 - r^4\right)}$$

$$\mathrm{Ric}_{33} = -\frac{8 \sin^2 \vartheta \, J^2 \, \left(12 \sin^2 \vartheta \, M - 3\, r \sin^2 \vartheta - 2\, r\right)}{r \, \left(2\, r^3 \, M - 16 \sin^2 \vartheta \, J^2 - r^4\right)}$$

24.5.3.8 Ricci Scalar

$$N = 4r^9 M^2 - 64r^6 \sin^2 \vartheta J^2 M - 4r^{10} M + 256r^3 \sin^4 \vartheta J^4 + 32r^7 \sin^2 \vartheta J^2 + r^{11}$$

$$\begin{aligned} R_{sc} = & \frac{-384r^3 \sin^2 \vartheta J^2 M^2 + 768 \sin^4 \vartheta J^4 M + 224r^4 \sin^2 \vartheta J^2 M + 128r^4 J^2 M}{N} \\ & + \frac{1152r \sin^4 \vartheta J^4 - 512r \sin^2 \vartheta J^4 - 24r^5 \sin^2 \vartheta J^2 - 64r^5 J^2}{N} \end{aligned}$$

24.5.3.9 Bianchi identity (Ricci cyclic equation $R^{\kappa}{}_{[\mu\nu\sigma]} = 0$)

———— o.k.

24.5.3.10 Einstein Tensor

———— not zero:

$$N = 4r^9 M^2 - 64r^6 \sin^2 \vartheta J^2 M - 4r^{10} M + 256r^3 \sin^4 \vartheta J^4 + 32r^7 \sin^2 \vartheta J^2 + r^{11}$$

$$G_{00} = \frac{288r^2 \sin^2 \vartheta J^2 M^3 - 432r^3 \sin^2 \vartheta J^2 M^2 - 512 \sin^2 \vartheta J^4 M + 216r^4 \sin^2 \vartheta J^2 M + 192r \sin^4 \vartheta J^4 + 256r \sin^2 \vartheta J^4 - 36r^5 \sin^2 \vartheta J^2}{N}$$

$$G_{03} = -\frac{576r^2 \sin^4 \vartheta J^3 M^2 + \left(-544r^3 \sin^4 \vartheta - 128r^3 \sin^2 \vartheta\right) J^3 M - 768 \sin^6 \vartheta J^5 + \left(144r^4 \sin^4 \vartheta + 64r^4 \sin^2 \vartheta\right) J^3}{4r^9 M^2 + \left(-64r^6 \sin^2 \vartheta J^2 - 4r^{10}\right) M + 256r^3 \sin^4 \vartheta J^4 + 32r^7 \sin^2 \vartheta J^2 + r^{11}}$$

$$G_{11} = \frac{48r^3 \sin^2 \vartheta J^2 M^2 + \left(\left(64r^4 - 176r^4 \sin^2 \vartheta\right) J^2 - 384 \sin^4 \vartheta J^4\right) M + \left(960r \sin^4 \vartheta - 256r \sin^2 \vartheta\right) J^4 + \left(76r^5 \sin^2 \vartheta - 32r^5\right) J^2}{8r^8 M^3 + \left(-128r^5 \sin^2 \vartheta J^2 - 12r^9\right) M^2 + \left(512r^2 \sin^4 \vartheta J^4 + 128r^6 \sin^2 \vartheta J^2 + 6r^{10}\right) M - 256r^3 \sin^4 \vartheta J^4 - 32r^7 \sin^2 \vartheta J^2 - r^{11}}$$

$$G_{12} = -\frac{144\,r^3\,\cos\vartheta\,\sin\vartheta\,J^2\,M - 768\,\cos\vartheta\,\sin^3\vartheta\,J^4 - 80\,r^4\,\cos\vartheta\,\sin\vartheta\,J^2}{4\,r^7\,M^2 + (-64\,r^4\,\sin^2\vartheta\,J^2 - 4\,r^8)\,M + 256\,r\,\sin^4\vartheta\,J^4 + 32\,r^5\,\sin^2\vartheta\,J^2 + r^9}$$

$$G_{21} = -\frac{144\,r^3\,\cos\vartheta\,\sin\vartheta\,J^2\,M - 768\,\cos\vartheta\,\sin^3\vartheta\,J^4 - 80\,r^4\,\cos\vartheta\,\sin\vartheta\,J^2}{4\,r^7\,M^2 + (-64\,r^4\,\sin^2\vartheta\,J^2 - 4\,r^8)\,M + 256\,r\,\sin^4\vartheta\,J^4 + 32\,r^5\,\sin^2\vartheta\,J^2 + r^9}$$

$$G_{22} = \frac{288r^3\sin^2\vartheta J^2 M^2 + \left(\left(-288r^4\sin^2\vartheta - 32r^4\right)J^2 - 1152\sin^4\vartheta J^4\right)M + \left(192r\sin^4\vartheta + 256r\sin^2\vartheta\right)J^4 + \left(76r^5\sin^2\vartheta + 16r^5\right)J^2}{4r^7M^2 + \left(-64r^4\sin^2\vartheta J^2 - 4r^8\right)M + 256r\sin^4\vartheta J^4 + 32r^5\sin^2\vartheta J^2 + r^9}$$

$$G_{30} = -\frac{576r^2\sin^4\vartheta J^3M^2 + \left(-544r^3\sin^4\vartheta - 128r^3\sin^2\vartheta\right)J^3M - 768\sin^6\vartheta J^5 + \left(144r^4\sin^4\vartheta + 64r^4\sin^2\vartheta\right)J^3}{4r^9M^2 + \left(-64r^6\sin^2\vartheta J^2 - 4r^{10}\right)M + 256r^3\sin^4\vartheta J^4 + 32r^7\sin^2\vartheta J^2 + r^{11}}$$

$$G_{33} = \frac{\left(1152\sin^6\vartheta J^4 + \left(32r^4\sin^4\vartheta - 32r^4\sin^2\vartheta\right)J^2\right)M - 960r\sin^6\vartheta J^4 + \left(16r^5\sin^2\vartheta - 12r^5\sin^4\vartheta\right)J^2}{4r^7M^2 + \left(-64r^4\sin^2\vartheta J^2 - 4r^8\right)M + 256r\sin^4\vartheta J^4 + 32r^5\sin^2\vartheta J^2 + r^9}$$

24.5.3.11 Hodge Dual of Bianchi Identity

——— (see charge and current densities)

24.5.3.12 Scalar Charge Density $(-R^{0\ \,i0}_{\ \,i})$

$$N = 8r^9M^3 + \left(-192r^6\sin^2\vartheta J^2 - 12r^{10}\right)M^2 + \left(1536r^3\sin^4\vartheta J^4 + 192r^7\sin^2\vartheta J^2 + 6r^{11}\right)M$$
$$- 4096\sin^6\vartheta J^6 - 768r^4\sin^4\vartheta J^4 - 48r^8\sin^2\vartheta J^2 - r^{12}$$

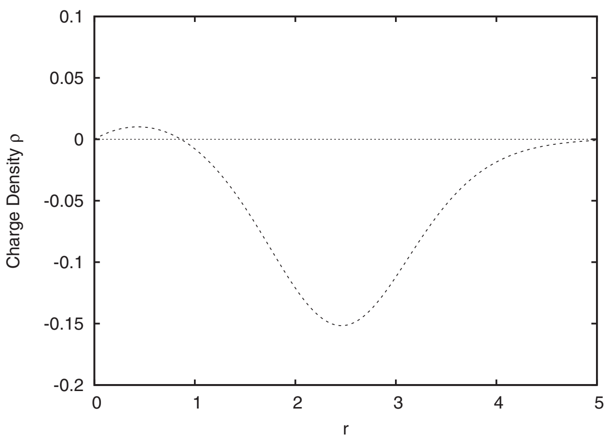

Fig. 24.7. Kerr metric, cosmological charge density ρ for M=1, J=2.

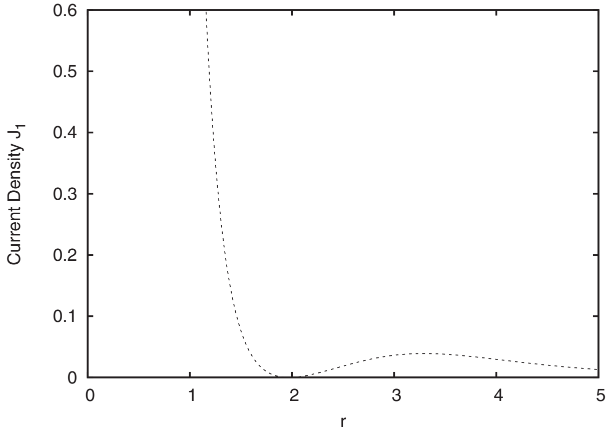

Fig. 24.8. Kerr metric, cosmological current density J_r for M=1, J=2.

$$\rho = \frac{\left(48r^4\cos^2\vartheta - 48r^4\right)J^2M^2 + \left(\left(-768r\cos^4\vartheta + 1536r\cos^2\vartheta - 768r\right)J^4 + \left(32r^5\cos^2\vartheta + 32r^5\right)J^2\right)M}{N} + \frac{\left(1152r^2\cos^4\vartheta - 2048r^2\cos^2\vartheta + 896r^2\right)J^4 + \left(-24r^6\cos^2\vartheta - 8r^6\right)J^2}{N}$$

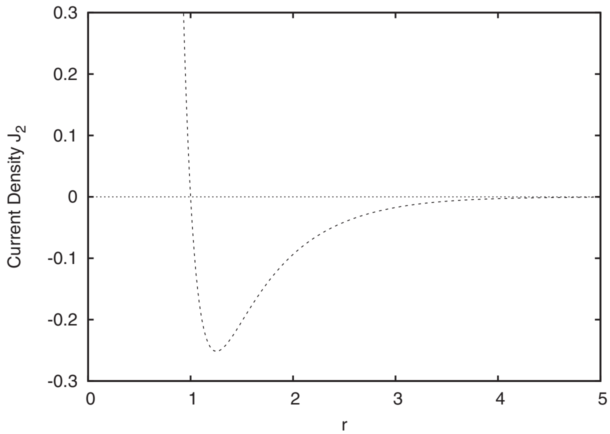

Fig. 24.9. Kerr metric, cosmological current density J_θ for M=1, J=2.

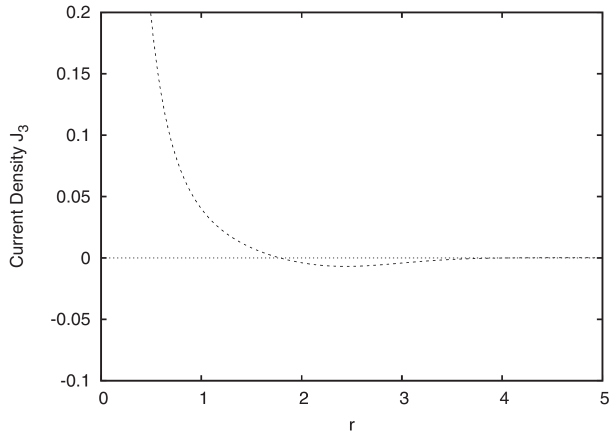

Fig. 24.10. Kerr metric, cosmological current density J_φ for M=1, J=2.

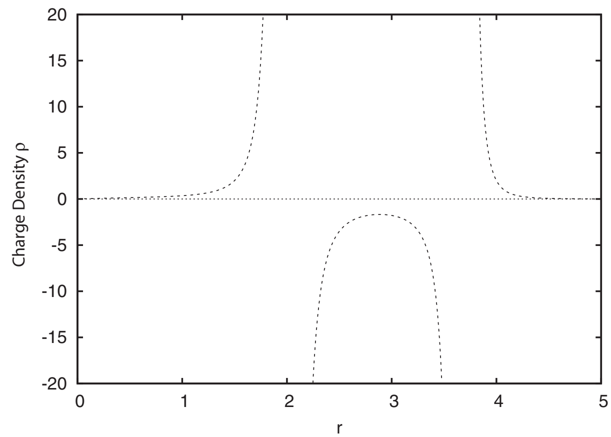

Fig. 24.11. Kerr metric, cosmological charge density ρ for M=2, J=1.

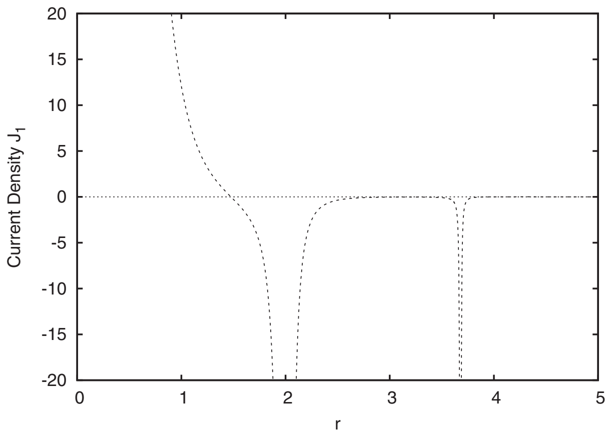

Fig. 24.12. Kerr metric, cosmological current density J_r for M=2, J=1.

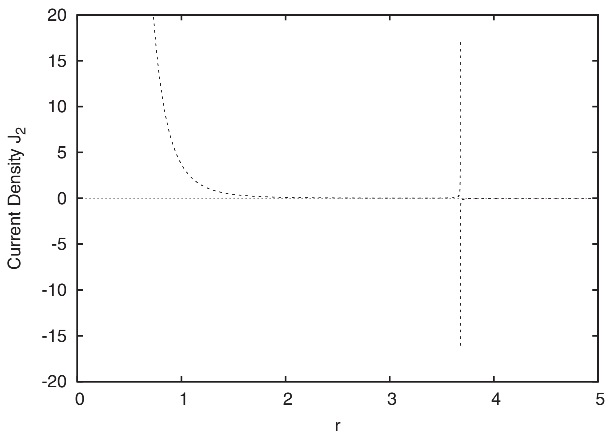

Fig. 24.13. Kerr metric, cosmological current density J_θ for M=2, J=1.

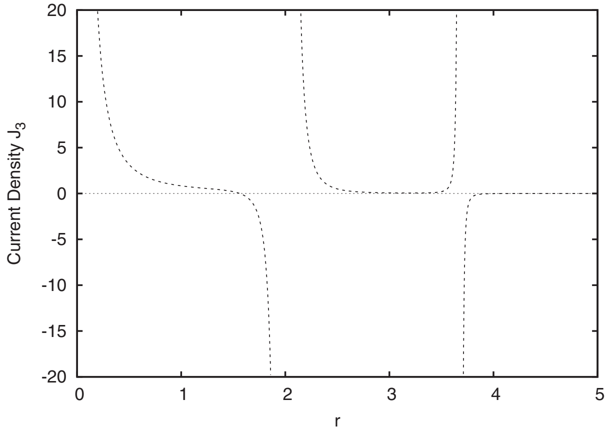

Fig. 24.14. Kerr metric, cosmological current density J_φ for M=2, J=1.

24.5.3.13 Current Density Class 1 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$N = 4r^{10}M^2 + \left(-64r^7 \sin^2\vartheta J^2 - 4r^{11}\right) M + 256r^4 \sin^4\vartheta J^4 + 32r^8 \sin^2\vartheta J^2 + r^{12}$$

$$J_1 = -\frac{480r^3 \sin^2\vartheta J^2 M^3 + \left(-1536 \sin^4\vartheta J^4 - 816r^4 \sin^2\vartheta J^2\right) M^2 + \left(1536r \sin^4\vartheta J^4 + 464r^5 \sin^2\vartheta J^2\right) M}{N} - \frac{384r^2 \sin^4\vartheta J^4 - 88r^6 \sin^2\vartheta J^2}{N}$$

$$J_2 = -\frac{96r^3 \sin^2\vartheta J^2 M^2 + \left(\left(32r^4 - 176r^4 \sin^2\vartheta\right) J^2 - 768 \sin^4\vartheta J^4\right) M + 768r \sin^4\vartheta J^4 + \left(64r^5 \sin^2\vartheta - 16r^5\right) J^2}{4r^{11} M^2 + \left(-64r^8 \sin^2\vartheta J^2 - 4r^{12}\right) M + 256r^5 \sin^4\vartheta J^4 + 32r^9 \sin^2\vartheta J^2 + r^{13}}$$

$$\begin{aligned} N = {} & 8\, r^{11} \sin^2\vartheta\, M^3 + \left(-192\, r^8 \sin^4\vartheta\, J^2 - 12\, r^{12} \sin^2\vartheta\right) M^2 \\ & + \left(1536\, r^5 \sin^6\vartheta\, J^4 + 192\, r^9 \sin^4\vartheta\, J^2 + 6\, r^{13} \sin^2\vartheta\right) M \\ & - 4096\, r^2 \sin^8\vartheta\, J^6 - 768\, r^6 \sin^6\vartheta\, J^4 - 48\, r^{10} \sin^4\vartheta\, J^2 - r^{14} \sin^2\vartheta \end{aligned}$$

$$J_3 = \frac{384\, r^3 \sin^2\vartheta\, J^2 M^3 + \left(\left(-480\, r^4 \sin^2\vartheta - 64\, r^4\right) J^2 - 768 \sin^4\vartheta\, J^4\right) M^2}{N} + \frac{\left(\left(192r^5 \sin^2\vartheta + 64r^5\right) J^2 - 256r \sin^4\vartheta J^4\right) M + 384r^2 \sin^4\vartheta J^4 + \left(-24r^6 \sin^2\vartheta - 16r^6\right) J^2}{N}$$

24.5.3.14 Current Density Class 2 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$J_1 = 0$$

$$J_2 = -\frac{288r^3 \cos\vartheta \sin\vartheta J^2 M^2 + \left(-1536 \cos\vartheta \sin^3\vartheta J^4 - 304r^4 \cos\vartheta \sin\vartheta J^2\right) M + 768r \cos\vartheta \sin^3\vartheta J^4 + 80r^5 \cos\vartheta \sin\vartheta J^2}{4r^{10} M^2 + \left(-64r^7 \sin^2\vartheta J^2 - 4r^{11}\right) M + 256r^4 \sin^4\vartheta J^4 + 32r^8 \sin^2\vartheta J^2 + r^{12}}$$

$$J_3 = 0$$

24.5.3.15 Current Density Class 3 ($-R^{i}{}_{\mu}{}^{\mu j}$)

$$J_1 = -\frac{288r^3 \cos\vartheta \sin\vartheta J^2 M^2 + \left(-1536 \cos\vartheta \sin^3\vartheta J^4 - 304r^4 \cos\vartheta \sin\vartheta J^2\right) M + 768r \cos\vartheta \sin^3\vartheta J^4 + 80r^5 \cos\vartheta \sin\vartheta J^2}{4r^{10} M^2 + \left(-64r^7 \sin^2\vartheta J^2 - 4r^{11}\right) M + 256r^4 \sin^4\vartheta J^4 + 32r^8 \sin^2\vartheta J^2 + r^{12}}$$

$$J_2 = 0$$

$$J_3 = 0$$

24.5.4 Kerr-Newman metric

This is the most complex metric of this group for a charged mass with rotation. The functions occuring in the metric are defined as follows:

$$\rho^2 = r^2 + a^2 \cos^2\theta$$
$$\Delta = r^2 - 2Mr + a^2 + Q^2$$
$$a = \frac{J}{M}$$

where a is the angular momentum per unit mass. The inverse metric is highly complex and not shown. The same holds for most of the derived quantities like Christoffel symbols, Riemann, Ricci and Einstein tensors and cosmological charge and current density. In particular the charge and current density are not zero.

24.5.4.1 Coordinates

$$\mathbf{x} = \begin{pmatrix} t \\ r \\ \vartheta \\ \varphi \end{pmatrix}$$

24.5.4.2 Metric

$$g_{\mu\nu} = \begin{pmatrix} \frac{2rM-Q^2}{a^2\cos^2\vartheta+r^2} - 1 & 0 & 0 & -\frac{a\sin^2\vartheta\left(4rM-2Q^2\right)}{a^2\cos^2\vartheta+r^2} \\ 0 & \frac{a^2\cos^2\vartheta+r^2}{Q^2-2rM+r^2+a^2} & 0 & 0 \\ 0 & 0 & a^2\cos^2\vartheta + r^2 & 0 \\ -\frac{a\sin^2\vartheta\left(4rM-2Q^2\right)}{a^2\cos^2\vartheta+r^2} & 0 & 0 & \sin^2\vartheta\left(\frac{a^2\sin^2\vartheta\left(2rM-Q^2\right)}{a^2\cos^2\vartheta+r^2} + r^2 + a^2\right) \end{pmatrix}$$

24.5.4.3 Christoffel Connection

$$\Gamma^0{}_{01} \neq 0$$

$$\Gamma^0{}_{02} \neq 0$$

$$\Gamma^0{}_{10} \neq 0$$

$$\Gamma^0{}_{13} \neq 0$$

$$\Gamma^0{}_{20} \neq 0$$

$$\Gamma^0{}_{23} \neq 0$$

$$\Gamma^0{}_{31} \neq 0$$

$$\Gamma^0{}_{32} \neq 0$$

$$\Gamma^1{}_{00} = -\frac{r\,Q^4 + \left(\left(a^2\cos^2\vartheta - 3\,r^2\right)M + r^3 + a^2\,r\right)Q^2 + \left(2\,r^3 - 2\,a^2\,r\cos^2\vartheta\right)M^2 + \left(\left(a^2\,r^2 + a^4\right)\cos^2\vartheta - r^4 - a^2\,r^2\right)M}{a^6\cos^6\vartheta + 3\,a^4\,r^2\cos^4\vartheta + 3\,a^2\,r^4\cos^2\vartheta + r^6}$$

$$\Gamma^1{}_{03} \neq 0$$

$$\Gamma^1{}_{11} = \frac{r\,Q^2 + \left(-a^2\sin^2\vartheta - r^2 + a^2\right)M + a^2\,r\sin^2\vartheta}{\left(a^2\cos^2\vartheta + r^2\right)Q^2 + \left(-2\,a^2\,r\cos^2\vartheta - 2\,r^3\right)M + \left(a^2\,r^2 + a^4\right)\cos^2\vartheta + r^4 + a^2\,r^2}$$

$$\Gamma^1{}_{12} = -\frac{a^2\cos\vartheta\sin\vartheta}{a^2\cos^2\vartheta + r^2}$$

$$\Gamma^1{}_{21} = -\frac{a^2\,\cos\vartheta\,\sin\vartheta}{a^2\,\cos^2\vartheta + r^2}$$

$$\Gamma^1{}_{22} = -\frac{r\,Q^2 - 2\,r^2\,M + r^3 + a^2\,r}{a^2\,\cos^2\vartheta + r^2}$$

$$\Gamma^1{}_{30} \neq 0$$

$$\Gamma^1{}_{33} \neq 0$$

$$\Gamma^2{}_{00} = \frac{a^2\,\cos\vartheta\,\sin\vartheta\,Q^2 - 2\,a^2\,r\,\cos\vartheta\,\sin\vartheta\,M}{a^6\,\cos^6\vartheta + 3\,a^4\,r^2\,\cos^4\vartheta + 3\,a^2\,r^4\,\cos^2\vartheta + r^6}$$

$$\Gamma^2{}_{03} = -\frac{\left(2\,a\,r^2 + 2\,a^3\right)\,\cos\vartheta\,\sin\vartheta\,Q^2 + \left(-4\,a\,r^3 - 4\,a^3\,r\right)\,\cos\vartheta\,\sin\vartheta\,M}{a^6\,\cos^6\vartheta + 3\,a^4\,r^2\,\cos^4\vartheta + 3\,a^2\,r^4\,\cos^2\vartheta + r^6}$$

$$\Gamma^2{}_{11} = \frac{a^2\,\cos\vartheta\,\sin\vartheta}{\left(a^2\,\cos^2\vartheta + r^2\right)\,Q^2 + \left(-2\,a^2\,r\,\cos^2\vartheta - 2\,r^3\right)\,M + \left(a^2\,r^2 + a^4\right)\,\cos^2\vartheta + r^4 + a^2\,r^2}$$

$$\Gamma^2{}_{12} = \frac{r}{a^2\,\cos^2\vartheta + r^2}$$

$$\Gamma^2{}_{21} = \frac{r}{a^2\,\cos^2\vartheta + r^2}$$

$$\Gamma^2{}_{22} = -\frac{a^2\,\cos\vartheta\,\sin\vartheta}{a^2\,\cos^2\vartheta + r^2}$$

$$\Gamma^2{}_{30} = -\frac{\left(2\,a\,r^2 + 2\,a^3\right)\,\cos\vartheta\,\sin\vartheta\,Q^2 + \left(-4\,a\,r^3 - 4\,a^3\,r\right)\,\cos\vartheta\,\sin\vartheta\,M}{a^6\,\cos^6\vartheta + 3\,a^4\,r^2\,\cos^4\vartheta + 3\,a^2\,r^4\,\cos^2\vartheta + r^6}$$

$$\Gamma^2{}_{33} \neq 0$$

$$\Gamma^3{}_{01} \neq 0$$

$$\Gamma^3{}_{02} \neq 0$$

$$\Gamma^3{}_{10} \neq 0$$

$$\Gamma^3{}_{13} \neq 0$$

$$\Gamma^3{}_{20} \neq 0$$

$$\Gamma^3{}_{23} \neq 0$$

$$\Gamma^3{}_{31} \neq 0$$

$$\Gamma^3{}_{32} \neq 0$$

24.5.4.4 Metric Compatibility

——— o.k.

24.5.4.5 Bianchi identity (Ricci cyclic equation $R^\kappa{}_{[\mu\nu\sigma]} = 0$)

——— o.k.

24.5.4.6 Einstein Tensor

——— not zero:

$$G_{00} \neq 0$$

$$G_{03} \neq 0$$

$$G_{11} \neq 0$$

$$G_{12} \neq 0$$

$$G_{21} \neq 0$$

$$G_{22} \neq 0$$

$$\ldots$$

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A

Appendix 1: Validity of the Dipole Approximation

The ECE theory of gravitomagnetism is based on the ECE theory of magnetostatics, which can be built from electrostatics as is well known [17]. It is sufficient for the purposes of this appendix to consider the static electric field without spin connection:

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi. \tag{A.1}$$

The multipole expansion used in electrostatics [17] is:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}. \tag{A.2}$$

The choice of constant coefficients is a convention. A localized distribution of charge is described by the charge density $\rho(\mathbf{x}')$, which is non-vanishing only inside a sphere of radius R, defined around an origin. The sphere is a concept used only to divide space into regions with and without charge. The multipole expansion is valid if and only if the charge density falls off with distance faster than any power of r.

Under these assumptions:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + ...\right) \tag{A.3}$$

where the integrated charge or monopole moment is:

$$q = \int \rho(\mathbf{x}')d^3x' \tag{A.4}$$

and where the electric dipole moment is:

$$\mathbf{p} = \int \mathbf{x}\,\rho(\mathbf{x}')d^3x'. \tag{A.5}$$

In spherical polar coordinates, the electric field strength components for a dipole aligned in the Z axis are:

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0\,r^3}, \qquad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0\,r^3}, \qquad E_\phi = 0 \tag{A.6}$$

and the total field vector is:

$$\mathbf{E} = E_r\mathbf{e}_r + E_\theta\mathbf{e}_\theta + E_\phi\mathbf{e}_\phi \tag{A.7}$$

where:

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k},\\
\mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k},\\
\mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.
\end{aligned}$$

Therefore:

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0\,r^3}\left(3\sin\theta\cos\theta\cos\phi\,\mathbf{i} + 3\sin\theta\cos\theta\sin\phi\,\mathbf{j} + \left(2\cos^2\theta - \sin^2\theta\right)\,\mathbf{k}\right) \tag{A.8}$$

where:

$$\begin{aligned}
\sin\theta\cos\phi &= \frac{x}{r},\\
\sin\theta\sin\phi &= \frac{y}{r},\\
\cos\theta &= \frac{z}{r},\\
\sin\theta &= \left(1 - \frac{z^2}{r^2}\right)^{1/2}.
\end{aligned}$$

The dipole field is therefore:

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0\,r^3}\left(\frac{3z}{r^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) - \mathbf{k}\right). \tag{A.9}$$

This result is denoted in ref. [17], eq. (4.13), as:

$$\mathbf{E}(\mathbf{x}) = \frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_0|^3} \tag{A.10}$$

where $\mathbf{n}$ is a unit vector directed from $\mathbf{x}_0$ to $\mathbf{x}$. Thus:

$$\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}, \tag{A.11}$$

$$r = |\mathbf{x}-\mathbf{x}_0|, \tag{A.12}$$

$$\mathbf{n}\,(\mathbf{p}\cdot\mathbf{n}) = \frac{p\,z}{r^2}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\,. \tag{A.13}$$

The electric dipole moment (A.5) is defined in the range of validity:

$$|\mathbf{x}'| << |\mathbf{x}-\mathbf{x}_0| \tag{A.14}$$

where $|\mathbf{x}'|$ is the distance between the two charges of the dipole. By reference to Fig. A1, the exact solution:

$$\Phi = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\left(\left(z-\frac{d}{2}\right)^2 + x^2 + y^2\right)^{1/2}} - \frac{q}{\left(\left(z+\frac{d}{2}\right)^2 + x^2 + y^2\right)^{1/2}}\right) \tag{A.15}$$

should be used. Eq. (A.9) is obtained only if:

$$d << |\mathbf{r}| \tag{A.16}$$

when:

$$\left(z-\frac{d}{2}\right)^2 \approx z^2 - z\,d \tag{A.17}$$

and

$$\left(1-\frac{zd}{r^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{z\,d}{r^2}. \tag{A.18}$$

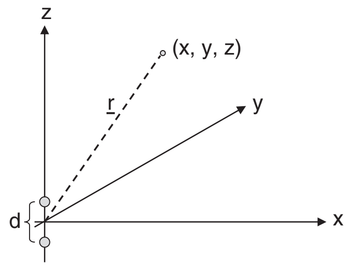

Fig. A.1. Position of dipole charges in the coordinate system.

If

$$p = q\,d \tag{A.19}$$

then

$$\Phi \approx \frac{1}{4\pi\epsilon_0}\frac{zqd}{r^3}. \tag{A.20}$$

Now use:

$$\cos\theta = \frac{z}{r} \tag{A.21}$$

so

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}. \tag{A.22}$$

The vector **p** is defined as along the Z axis from −q to q, with

$$|\mathbf{p}| = q\,d. \tag{A.23}$$

Therefore:

$$\mathbf{p}\cdot\mathbf{r} = r\,p\,\cos\theta \tag{A.24}$$

i.e.

$$p\,\cos\theta = \frac{\mathbf{p}\cdot\mathbf{r}}{r} \tag{A.25}$$

and

$$\Phi = \frac{1}{4\pi\epsilon_0\, r^3}\,\mathbf{p}\cdot\mathbf{r}. \tag{A.26}$$

The magnetic flux density from the electric dipole field can be found in the first approximation from Eq. (24.6) of the text, i.e. from:

$$\mathbf{B}(\text{dipole}) = -\frac{1}{c^2}\mathbf{v}\times\mathbf{E}(\text{dipole}) \tag{A.27}$$

and the rigorously correct method to use is from Eq. (A.15).

In Gravity Probe B the satellite orbits a distance r from the centre of the earth of radius R, where r is of the order of R. Newton's law holds accurately for this situation because the mass M of the earth can be considered as being at its centre of mass, which is the centre of the earth. So:

$$\mathbf{F} = m\mathbf{g} = -\frac{mMG}{r^2}\mathbf{e}_r. \tag{A.28}$$

The acceleration due to gravity is:

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi = -\frac{MG}{r^2}\mathbf{e}_r \tag{A.29}$$

where only one (monopole) potential is needed:

$$\Phi = -\frac{MG}{r}. \tag{A.30}$$

The mass is the integral of the mass density:

$$M = \int \rho_m(\mathbf{r}')d^3r' \tag{A.31}$$

and:

$$\boldsymbol{\nabla}\cdot\mathbf{g} = 4\pi G\rho_m. \tag{A.32}$$

Comparing Eq. (A.30) to (A.3) it is seen that only the first term of the multipole expansion is needed for an accurate description of the Newtonian attraction between the satellite and the earth if the latter were a perfect sphere. The gravitational potential at any point outside a spherical distribution of matter, a solid or a shell, is independent of the size of the distribution as is well known. However the earth is not a perfect sphere, and gravitational multipoles of the earth are experimentally observable. The expression given by Pfister is the dipole approximation:

$$\boldsymbol{\Omega} = \frac{2}{5}\frac{MGR^2}{c^2r^3}\left(\boldsymbol{\omega} - 3\mathbf{n}(\boldsymbol{\omega}\cdot\mathbf{n})\right) \tag{A.33}$$

which if valid, corrects our Eq. (24.20). Its analogue in magnetostatics is the magnetic flux density in the dipole approximation [17]:

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3}\left(\mathbf{m} - 3\mathbf{n}(\mathbf{m}\cdot\mathbf{n})\right) \tag{A.34}$$

where **m** is the magnetic dipole moment. Eq. (A.33) is a solution of Eq. (24.10) in the far field approximation (A.15). Ref. [16] claims to have verified Eq. (A.33) experimentally, and if this claim is accepted, our Eq. (24.10) is also verified experimentally.

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