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The Coulomb and Ampère Maxwell Laws in Generally Covariant Unified Field Theory

by

Myron W. Evans,
Alpha Institute for Advanced Study, Civil List Scientist.
(emyrone@aol.com and www.aias.us)

and

Horst Eckardt and Stephen Crothers,
Alpha Institute for Advanced Study.

Abstract

The Coulomb and Ampère Maxwell laws are calculated exactly from Einstein Cartan Evans (ECE) unified field theory. The result are given for several stationary and dynamical line elements and metrics of the Einstein Hilbert field equation, and show that in general there are relativistic corrections of the same order as those responsible for the deflection of light by gravity and perihelion advance for example. In the special relativistic limit the Coulomb and Ampère Maxwell laws of classical electrodynamics are recovered self consistently. In the stationary Schwarzschild metric there is no charge density or current density. These are finite in the dynamic Friedman Lemaître Robertson Walker metric. The laws of classical electrodynamics are investigated for the rigorously correct Crothers metric, and other metrics.

Keywords: Einstein Cartan Evans (ECE) unified field theory, exact calculation of the generally covariant laws of electrodynamics, stationary and dynamical line elements.

23.1 Introduction

In classical electrodynamics the Coulomb law and Ampère Maxwell laws are well known to be a precise laws of special relativity in Minkowski spacetime [1]. However in a generally covariant unified field theory all the laws of classical electrodynamics become unified with those of gravitation and other fundamental fields [2–9]. In previous work [2–9] these laws have been developed using the spin connection, revealing the presence of resonance phenomena that can lead to new sources of energy. A dielectric formulation of the laws of classical electrodynamics has also been given. This showed that light deflected by gravity also changes polarization, as observed for example in light deflected by white dwarf stars [10]. More generally, there are many optical and electro-dynamical changes predicted by Einstein Cartan Evans (ECE) unified field theory [2–9]. In Section 23.2 various well known line elements are used to compute the Coulomb Law and Ampère Maxwell laws, starting with the Bianchi identity of differential geometry. In Section 23.3 a discussion is given of the shortcomings of Big Bang and black hole theory, based on the rigorously correct Crothers metric. The latter is also used in Section 23.3 to develop the laws of classical electrodynamics into laws of general relativity. Appendices give sufficient mathematical detail to follow the derivation step by step.

23.2 The Generally Covariant Coulomb and Ampère Maxwell Laws

The starting point of the derivation is the Bianchi identity [11] of Cartan geometry:

$$d \wedge T^a + \omega^a{}_b \wedge T^b := R^a{}_b \wedge q^b \tag{23.1}$$

where T^a is the torsion form, $\omega^a{}_b$ is the spin connection, $d\wedge$ denotes the wedge product of differential geometry, $R^a{}_b$ is the curvature form and q^b is the tetrad form. Using the fundamental hypothesis [2–9]:

$$A^a = A^{(0)} q^a, \tag{23.2}$$

$$F^a = A^{(0)} T^a, \tag{23.3}$$

Eq. (23.1) becomes the ECE field equation:

$$d \wedge F^a = \mu_0 j^a = A^{(0)} (R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b). \tag{23.4}$$

Here A^a is the potential form, F^a is the field form, and $cA^{(0)}$ is a primordial scalar in volts. The hypothesis (23.2) has been tested experimentally in an extensive manner (www.aias.us). The field equation (23.4) is generally covariant because the Bianchi identity is generally covariant. Under the general coordinate transformation the field equation becomes:

$$(d \wedge F^a)' = (\mu_0 j^a)' \tag{23.5}$$

which is:

$$(d \wedge T^a + \omega^a{}_b \wedge T^b)' := (R^a{}_b \wedge q^b)'. \tag{23.6}$$

It retains its form under the coordinate transform because it consists of tensorial quantities. This is the essence of general relativity.

Applying the Hodge dual transform to both sides of Eq. (23.4) (Appendix (A)) the inhomogeneous ECE field equation is obtained:

$$d \wedge \widetilde{F}^a = \mu_0 J^a = A^{(0)}(\widetilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \widetilde{T}^b). \tag{23.7}$$

Here the tilde denotes Hodge transformation [2 – 9, 11]. It is seen that the same Hodge transform is applied to two-forms on both sides of the equation. The generally covariant Coulomb and Ampère Maxwell laws are part of the inhomogeneous field equation (23.7). As shown in Appendix (B), the homogeneous and inhomogeneous field equations are the tensor equations:

$$\partial_\mu \widetilde{F}^{a\mu\nu} = \mu_0 \widetilde{j}^{a\nu} \tag{23.8}$$

and

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu} \tag{23.9}$$

respectively. These tensor equations are generally covariant. They look like the Maxwell Heaviside field equations but contain more information. In the special case:

$$R^a{}_b \wedge q^b = \omega^a{}_b \wedge T^b \tag{23.10}$$

the homogeneous field equation becomes:

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \tag{23.11}$$

and the inhomogeneous equation becomes:

$$\partial_\mu F^{a\mu\nu} = -A^{(0)}(R^a{}_\mu{}^{\mu\nu})_{\text{grav}}. \tag{23.12}$$

It has been shown [2–9] that the special case (23.10) is pure rotation. A solution of Eq. (23.10) is:

$$R^a{}_b = -\frac{\kappa}{2}\epsilon^a{}_{bc}T^c,\ \omega^a{}_b = -\frac{\kappa}{2}\epsilon^a{}_{bc}q^c, \tag{23.13}$$

in which case the curvature is this well defined dual of the torsion and the spin connection is the well defined dual of the tetrad. These results are developed in all detail elsewhere [2–9]. When the connection is the Christoffel connection, however, the gravitational torsion vanishes:

$$(T^a)_{\text{grav}} = 0, \tag{23.14}$$

$$(T^\kappa{}_{\mu\nu})_{\text{grav}} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} = 0 \tag{23.15}$$

and the curvature form becomes the Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{23.16}$$

In this case the inhomogeneous equation becomes:

$$\partial_\mu F^{a\mu\nu} = -A^{(0)}R^a{}_\mu\mu\nu \tag{23.17}$$

and as shown in Appendix (C) can be written as two vector equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = (\boldsymbol{\nabla}\cdot\boldsymbol{E})^0 = -\phi^{(0)}(R^0{}_1{}^{10} + R^0{}_2{}^{20} + R^0{}_3{}^{30}) \tag{23.18}$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{J} \tag{23.19}$$

where

$$J_r = J^1_1 = -\frac{A^{(0)}}{\mu_0}(R^1{}_0{}^{10} + R^1{}_2{}^{12} + R^1{}_3{}^{13}), \tag{23.20}$$

$$J_\theta = J^2_2 = -\frac{A^{(0)}}{\mu_0}(R^2{}_0{}^{20} + R^2{}_1{}^{21} + R^2{}_3{}^{23}), \tag{23.21}$$

$$J_\phi = J^3_3 = -\frac{A^{(0)}}{\mu_0}(R^3{}_0{}^{30} + R^3{}_1{}^{31} + R^3{}_2{}^{32}). \tag{23.22}$$

Eq. (23.18) is the generally covariant Coulomb Law, and Eq. (23.19) is the generally covariant Ampère Maxwell law. As shown in Appendix (D) the index a for the Coulomb law must be zero on both sides because it is the time-like index indicating scalar quantities on both sides, and the a indices in Eqs. (23.20) to (23.22) are obtained in a well defined manner from Cartan geometry.

The generally covariant Coulomb and Ampère Maxwell laws are given by evaluating the Riemann elements on the right hand side of Eq. (23.17) for well known stationary and dynamic line elements and metric elements and the rigorously correct Crothers metric [12, 13]. The method is summarized in Appendix (E) and uses computer algebra. It consists of choosing line elements [11], evaluating the Christoffel symbols and Riemann tensor elements, and finally raising indices with the relevant metric elements. The final results are given as follows.

For the Minkowski line element of special relativity:

$$ds^2 = -c^2 dt^2 + dX^2 + dY^2 + dZ^2, \tag{23.23}$$

$$g_{00} = -1,\ g_{11} = 1,\ g_{22} = 1,\ g_{33} = 1 \tag{23.24}$$

there is no charge density and no current density, because the space-time has no curvature. So all Christoffel and Riemann elements are zero in the Minkowski space-time. This shows that Maxwell Heaviside field theory has to use charge and current densities phenomenologically, and this is neither generally covariant (objective) nor rigorously correct nor self consistent. In the stationary Schwarzschild metric as usually used:

$$ds^2 = -\left(1 - \frac{2MG}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2MG}{rc^2}\right)^{-1} r^2 + r^2 d\Omega^2 \tag{23.25}$$

$$\left.\begin{aligned} g_{00} &= -\left(1 - \frac{2MG}{rc^2}\right), \\ g_{11} &= -\left(1 - \frac{2MG}{rc^2}\right)^{-1}, \\ g_{22} &= r^2,\ g_{33} = r^2 \sin^2\theta \end{aligned}\right\} \tag{23.26}$$

there is no charge density and no current density from Eq. (23.17) because there is no canonical energy momentum density used in deriving this Schwarzschild line element. Here M is mass, G the Newton constant, c the speed of light (S.I.units are used in Eq. (23.25)) and the spherical polar coordinate system (r, θ, ϕ) is used. Therefore in both of these line elements the Coulomb and Ampère Maxwell laws are:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \tag{23.27}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{23.28}$$

The Friedman Lemaître Robertson Walker dynamical line element [13] is:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right), \tag{23.29}$$

$$g_{00} = -1,\ g_{11} = \frac{a^2(t)}{1-kr^2},\ g_{22} = a^2(t) r^2,\ g_{33} = a^2(t) r^2 \sin^2\theta \tag{23.30}$$

where a is governed by the Friedman equations. This metric is the result of homogeneity and isotropy, as is well known [14], and the Einstein Hilbert field equations are used to define the line element through the Friedman equations. Well known types of cosmologies are defined by this line element [14]. The line element (23.29) produces the Coulomb law:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= -\frac{3\phi\ddot{a}}{a} \\ &= 4\pi\phi G(\rho + 3\rho) = \frac{\rho_e}{\epsilon_0}, \end{aligned} \tag{23.31}$$

and the current density components:

$$J_r = -\frac{A^{(0)}}{\mu_0}\left(\frac{2}{a^4}(k + \ddot{a}^2)(kr^2 - 1) + \frac{\ddot{a}}{a^3}(kr^2 - 1) \right) \tag{23.32}$$

$$J_\theta = \frac{A^{(0)}}{\mu_0}\left(\frac{2}{a^4 r}(k + \ddot{a}^2) + \frac{\ddot{a}}{a^3 r^2} \right) \tag{23.33}$$

$$J_\phi = \frac{J_\theta}{\sin^2\theta}. \tag{23.34}$$

These depend on the type of universe, or cosmology, being considered [11]. The Coulomb law (23.31) depends directly on the Newton constant G and the mass density ρ, together with:

$$\rho = \frac{m}{V} \tag{23.35}$$

in the rest frame, where m is mass and V is volume. In the laboratory, Eq. (23.31) is the well tested Coulomb law of electrodynamics, one of the most precise laws of physics [1]. Eq. (23.31) is generally covariant and upon general coordinate transformation produces new physical effects. The generally covariant Ampère Maxwell law also produces new physical effects which can be looked for experimentally. Some are already known, notably the change in polarization of light deflected by gravitation [2–10]. Here, the scalar potential ϕ has the units of volts, G is the Newton constant with units of meters per kilogram, r is the radial vector of the spherical polar coordinate system (r, θ, ϕ), ρ_e is the electric charge density and ϵ_0 is the vacuum permittivity in S.I. units.

23.3 Discussion of Results and Criticisms of the Standard Model

The Coulomb and Ampère Maxwell laws in ECE theory are summarized in Table 23.1a for various metrics.

In this section a discussion of these results is given with fundamental criticisms of standard model cosmologies. There are various well known exact solutions of the Einstein Hilbert (EH) field equation which are considered as follows. The class of vacuum solutions assume that there is no matter or non-gravitational fields present, this is typified by what is usually called the Schwarzschild metric, which in ECE self-consistently produces no charge density and no current density, i.e. a vacuum. The class of electro-vacuum solutions solves the source free Maxwell Heaviside (MH) field equations in the given curved Lorentzian manifold, the source of the gravitational field being the electromagnetic energy-momentum. In this class, as in all classes of solution of EH, there is no Cartan torsion present. The electro-vacuum class of solutions is typified by the Reissner Nordstrom (RN) metric, the Kerr metric, and variations thereof. Einstein did not accept the RN metric as a unified field theory, because it replaces the derivative of the MH theory by a covariant derivative and so this is not an objective procedure based on geometry, it is an ad hoc fix of the phenomenological MH theory, one which assumes the presence of an electromagnetic field that has no source, a logical contradiction present in MH electrodynamics. In ECE theory the electromagnetic field is due to the electromagnetic Cartan torsion, which is missing from all Riemannian theories such as EH. So to use RN or Kerr with ECE is a contradiction in fundamental concepts.

Table 23.1a Metric for the Coulomb and Ampère Maxwell laws. In most cases $c = 1$ was assumed.

Minkowski	$ds = -c^2 dt^2 + dx^2 + dy^2 + dz^2$
Schwarzschild	$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$
General Spherical	$ds^2 = -e^{2\alpha(r,t)} c^2 dt^2 + e^{2\beta(r,t)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$
Crothers General	$ds^2 = -A\,(C(r))^{1/2}\, c^2 dt^2 + B\,(C(r))^{1/2}\, dr^2 + C(r)\left(d\theta^2 + \sin^2\theta \; d\phi^2\right)$ where $C(r) = (\lvert r - r_0\rvert^n + \alpha^n)^{2/n}$
Crothers / Original Schwarzschild	Crothers General with $n = 3, \quad r_0 = 0, \quad r > r_0$
Crothers / Schwarzschild	Crothers General with $n = 1, \quad r_0 = \alpha, \quad r > r_0$
Crothers Type 1	$ds^2 = -c^2 dt^2 + dr^2 + \lvert r - r_0\rvert^2 \left(d\theta^2 + \sin^2\theta \; d\phi^2\right)$
Goedel	$ds^2 = \frac{1}{2\omega^2}\left(-(dt + e^x dz)^2 + dx^2 + dy^2 + \frac{1}{2} e^{2x} dz^2\right)$
FLRW	$ds^2 = dt^2 - (a(t))^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2\right)$
Static de Sitter	$ds^2 = -\left(1 - \frac{r^2}{\alpha^2}\right) dt^2 + \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$
Kasner	$ds^2 = -dt^2 + \sum_{j=1}^{D-1} t^{2p_j} dx_j^2$ where $\sum_{j=1}^{D-1} p_j = 1, \quad \sum_{j=1}^{D-1} p_j^2 = 1, \quad D > 3$
Perfect Spherical Fluid	$ds^2 = -\left(1 + ar^2\right) dt^2 + \frac{(1-3ar^2)^{2/3}}{(1+3ar^2)^{2/3} - br^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$
Friedmann Dust	$ds^2 = -dt^2 + \left(\cosh\left(\frac{3t}{a} - 1\right)\right)^{2/3} \left(dx^2 + dy^2 + dz^2\right)$
Reissner-Nordstrom	$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2$

Table 23.1b Charge and current densities for the metrics of Table 1a. Factors $A^{(0)}, \mu_0$ and $\phi^{(0)}$ have been omitted.

Minkowski	$\rho = 0, \quad J_r = 0, \quad J_\theta = 0, \quad J_\phi = 0$
Scharzschild	$\rho = 0, \quad J_r = 0, \quad J_\theta = 0, \quad J_\phi = 0$
General Spherical	$\rho = \frac{2\,\left(\frac{d}{d\,r}\,\alpha\right)\,e^{-2\,\beta-2\,\alpha}}{r} - e^{-4\,\alpha}\,\left(\frac{d^2}{d\,t^2}\,\beta\right) - e^{-4\,\alpha}\,\left(\frac{d}{d\,t}\,\beta\right)^2$ $+e^{-4\,\alpha}\,\left(\frac{d}{d\,t}\,\alpha\right)\,\left(\frac{d}{d\,t}\,\beta\right) - \frac{d}{d\,r}\,\alpha\,e^{-2\,\beta-2\,\alpha}\,\left(\frac{d}{d\,r}\,\beta\right) + \frac{d^2}{d\,r^2}\,\alpha\,e^{-2\,\beta-2\,\alpha} + \left(\frac{d}{d\,r}\,\alpha\right)^2\,e^{-2\,\beta-2\,\alpha}$ $J_r = -\frac{2\,e^{-4\,\beta}\,\left(\frac{d}{d\,r}\,\beta\right)}{r} - e^{-2\,\beta-2\,\alpha}\,\left(\frac{d^2}{d\,t^2}\,\beta\right) - e^{-2\,\beta-2\,\alpha}\,\left(\frac{d}{d\,t}\,\beta\right)^2$ $+\frac{d}{d\,t}\,\alpha\,e^{-2\,\beta-2\,\alpha}\,\left(\frac{d}{d\,t}\,\beta\right) - \frac{d}{d\,r}\,\alpha\,e^{-4\,\beta}\,\left(\frac{d}{d\,r}\,\beta\right) + \frac{d^2}{d\,r^2}\,\alpha\,e^{-4\,\beta} + \left(\frac{d}{d\,r}\,\alpha\right)^2\,e^{-4\,\beta}$ $J_\theta = -\frac{e^{-2\,\beta}\,\left(\frac{d}{d\,r}\,\beta\right)}{r^3} + \frac{\frac{d}{d\,r}\,\alpha\,e^{-2\,\beta}}{r^3} + \frac{e^{-2\,\beta}}{r^4} - \frac{1}{r^4}$ $J_\phi = -\frac{e^{-2\,\beta}\,\left(\frac{d}{d\,r}\,\beta\right)}{r^3\,\sin^2\vartheta} + \frac{\frac{d}{d\,r}\,\alpha\,e^{-2\,\beta}}{r^3\,\sin^2\vartheta} + \frac{e^{-2\,\beta}}{r^4\,\sin^2\vartheta} - \frac{1}{r^4\,\sin^2\vartheta}$
Crothers General	$\rho = \frac{\frac{d^2}{d\,r^2}\,C}{4\,A\,B\,C^2}$ $J_r = \frac{5\,C\,\left(\frac{d^2}{d\,r^2}\,C\right) - 4\,\left(\frac{d}{d\,r}\,C\right)^2}{4\,B^2\,C^3}$ $J_\theta = -\frac{2\,B\,\sqrt{C} - \frac{d^2}{d\,r^2}\,C}{2\,B\,C^{\frac{5}{2}}}$ $J_\phi = -\frac{2\,B\,\sqrt{C} - \frac{d^2}{d\,r^2}\,C}{2\,\sin^2\vartheta\,B\,C^{\frac{5}{2}}}$
Crothers / Schwarzschild	$\rho = 0, \quad J_r = 0, \quad J_\theta = 0, \quad J_\phi = 0$
Crohters with defined C(r)	$\rho = \frac{\|r_0-r\|^{2\,n}+(\alpha^n\,n-\alpha^n)\,\|r_0-r\|^n}{(\|r_0-r\|^n+\alpha^n)^{\frac{2}{n}}\,\left(\left(2\,r_0{}^2-4\,r\,r_0+2\,r^2\right)\,\|r_0-r\|^{2\,n}+\left(4\,\alpha^n\,r_0{}^2-8\,\alpha^n\,r\,r_0+4\,\alpha^n\,r^2\right)\,\|r_0-r\|^n+2\,\alpha^{2\,n}\,r_0{}^2-4\,\alpha^{2\,n}\,r\,r_0+2\,\alpha^{2\,n}\,r^2\right)\,A\,B}$ $J_r = -\frac{3\,\|r_0-r\|^{2\,n}+(5\,\alpha^n-5\,\alpha^n\,n)\,\|r_0-r\|^n}{(\|r_0-r\|^n+\alpha^n)^{\frac{2}{n}}\,\left(\left(2\,r_0{}^2-4\,r\,r_0+2\,r^2\right)\,\|r_0-r\|^{2\,n}+\left(4\,\alpha^n\,r_0{}^2-8\,\alpha^n\,r\,r_0+4\,\alpha^n\,r^2\right)\,\|r_0-r\|^n+2\,\alpha^{2\,n}\,r_0{}^2-4\,\alpha^{2\,n}\,r\,r_0+2\,\alpha^{2\,n}\,r^2\right)\,B^2}$ $J_\theta = \frac{\left(\left(-r_0{}^2+2\,r\,r_0-r^2\right)\,\|r_0-r\|^{2\,n}+\left(-2\,\alpha^n\,r_0{}^2+4\,\alpha^n\,r\,r_0-2\,\alpha^n\,r^2\right)\,\|r_0-r\|^n-\alpha^{2\,n}\,r_0{}^2+2\,\alpha^{2\,n}\,r\,r_0-\alpha^{2\,n}\,r^2\right)\,B+\sqrt{(\|r_0-r\|^n+\alpha^n)^{\frac{2}{n}}}\,\left(\|r_0-r\|^{2\,n}+(\alpha^n\,n-\alpha^n)\,\|r_0-r\|^n\right)}{(\|r_0-r\|^n+\alpha^n)^{\frac{4}{n}}\,\left(\left(r_0{}^2-2\,r\,r_0+r^2\right)\,\|r_0-r\|^{2\,n}+\left(2\,\alpha^n\,r_0{}^2-4\,\alpha^n\,r\,r_0+2\,\alpha^n\,r^2\right)\,\|r_0-r\|^n+\alpha^{2\,n}\,r_0{}^2-2\,\alpha^{2\,n}\,r\,r_0+\alpha^{2\,n}\,r^2\right)\,B}$ $J_\phi = \frac{\left(\left(-r_0{}^2+2\,r\,r_0-r^2\right)\,\|r_0-r\|^{2\,n}+\left(-2\,\alpha^n\,r_0{}^2+4\,\alpha^n\,r\,r_0-2\,\alpha^n\,r^2\right)\,\|r_0-r\|^n-\alpha^{2\,n}\,r_0{}^2+2\,\alpha^{2\,n}\,r\,r_0-\alpha^{2\,n}\,r^2\right)\,B+\sqrt{(\|r_0-r\|^n+\alpha^n)^{\frac{2}{n}}}\,\left(\|r_0-r\|^{2\,n}+(\alpha^n\,n-\alpha^n)\,\|r_0-r\|^n\right)}{(\|r_0-r\|^n+\alpha^n)^{\frac{4}{n}}\,\left(\left(r_0{}^2-2\,r\,r_0+r^2\right)\,\|r_0-r\|^{2\,n}+\left(2\,\alpha^n\,r_0{}^2-4\,\alpha^n\,r\,r_0+2\,\alpha^n\,r^2\right)\,\|r_0-r\|^n+\alpha^{2\,n}\,r_0{}^2-2\,\alpha^{2\,n}\,r\,r_0+\alpha^{2\,n}\,r^2\right)\,\sin^2\vartheta\,B}$

Table 23.1b Continued

Goedel	$\rho = \frac{64\,\omega^8}{1024\,\omega^8-64\,\omega^4+1}$ $J_x = \frac{64\,\omega^8-4\,\omega^4}{32\,\omega^4-1}$ $J_y = 0$ $J_z = \frac{\left(128\,\omega^8-8\,\omega^4\right)\,e^{-2\,x}}{1024\,\omega^8-64\,\omega^4+1}$
FLRW	$\rho = -\frac{3\left(\frac{d^2}{d\,t^2}\,a\right)}{a}$ $J_r = \frac{\left(2\,k^2+\left(a\left(\frac{d^2}{d\,t^2}\,a\right)+2\left(\frac{d}{d\,t}\,a\right)^2\right)\,k\right)\,r^2-2\,k-a\left(\frac{d^2}{d\,t^2}\,a\right)-2\left(\frac{d}{d\,t}\,a\right)^2}{a^4}$ $J_\theta = -\frac{2\,k+a\left(\frac{d^2}{d\,t^2}\,a\right)+2\left(\frac{d}{d\,t}\,a\right)^2}{a^4\,r^2}$ $J_\phi = -\frac{2\,k+a\left(\frac{d^2}{d\,t^2}\,a\right)+2\left(\frac{d}{d\,t}\,a\right)^2}{a^4\,r^2\,\sin^2\vartheta}$
Static de Sitter	$\rho = \frac{3}{r^2-\alpha^2}$ $J_r = \frac{3\,r^2-3\,\alpha^2}{\alpha^4}$ $J_\theta = -\frac{3}{\alpha^2\,r^2}$ $J_\phi = -\frac{3}{\alpha^2\,r^2\,\sin^2\vartheta}$
Kasner	$\rho = -\frac{{p_3}^2-p_3+{p_2}^2-p_2+{p_1}^2-p_1}{t^2}$ $J_1 = -\left(p_1\,p_3+p_1\,p_2+{p_1}^2-p_1\right)\,t^{-2\,p_1-2}$ $J_2 = -\left(p_2\,p_3+{p_2}^2+(p_1-1)\,p_2\right)\,t^{-2\,p_2-2}$ $J_3 = -\left({p_3}^2+(p_2+p_1-1)\,p_3\right)\,t^{-2\,p_3-2}$
Perfect Spherical Fluid	extremely complex (dependencies up to r^{16})
Friedmann Dust	$\rho = -\frac{3\,\cosh^2\left(\frac{3\,t-a}{a}\right)+6}{a^2\,\cosh^2\left(\frac{3\,t-a}{a}\right)}$ $J_x = -\frac{3}{a^2\left(\cosh\left(\frac{3\,t-a}{a}\right)\right)^{\frac{2}{3}}}$ $J_y = -\frac{3}{a^2\left(\cosh\left(\frac{3\,t-a}{a}\right)\right)^{\frac{2}{3}}}$ $J_z = -\frac{3}{a^2\left(\cosh\left(\frac{3\,t-a}{a}\right)\right)^{\frac{2}{3}}}$
Reissner-Nordstrom	$\rho = \frac{Q^2}{r^2\,Q^2-2\,r^3\,M+r^4}$ $J_r = \frac{Q^4+\left(r^2-2\,r\,M\right)\,Q^2}{r^6}$ $J_\theta = -\frac{Q^2}{r^6}$ $J_\phi = -\frac{Q^2}{r^6\,\sin^2\vartheta}$

The third class of EH solutions is the null dust class which assumes that the source of the gravitational field is an incoherent electromagnetic field with no source. This again has the weaknesses just discussed of the electro-vacuum class of EH solutions. The class of fluid solutions assumes that the canonical energy momentum density of EH comes from the stress-energy tensor of a fluid, and that this is the only source of the gravitational field. This class of solutions assumes isotropy and homogeneity, and the FLRW metric is an example. In ECE, the FLRW metric gives a finite charge and current density, and precisely the correct form of the Coulomb Law (see Table 23.1a). In other words, ECE identifies the source of the Coulomb law as mass density, to which charge density is directly proportional. If there is mass density present anywhere in the universe, there is a source for the electromagnetic field. This cures the logical inconsistency in MH of having a field without a source. However, there are fundamental geometrical difficulties associated with this class of solutions, and these are discussed later in this Section. It seems that the rigorously correct Coulomb and Ampère Maxwell laws are given by a new class of solutions of EH deduced by Crothers [15]. This class also gives a finite charge density and current density given a finite mass density.

There are more exotic classes of exact EH solutions, for example the scalar field solutions in the field theory of meson beams and quintessence, the class of solutions due to a finite cosmological constant, the wormhole and superluminal metrics. The Kerr Newman NUT de Sitter class of exact solutions to EH uses a source-less electromagnetic field and positive vacuum energy. Finally the Godel dust solution of EH uses a pressure-less perfect fluid (dust) and a positive vacuum energy. From the point of view of ECE these are exotic, logically inconsistent and use adjustable parameters. None are true unified field theories because they are not based on the required logic of Cartan geometry.

Of these solutions the Crothers solution is the rigorously correct one, and produces a finite charge density and current density given a finite mass density. The Crothers solution also eliminates singularities, known as "Big Bang" and "black hole". ECE theory has been shown [2–9] to eliminate the wholly phenomenological concept of "dark matter" in favor of the Cartan torsion, which is an intrinsic part of geometry. The latter is the objective foundation of general relativity. The most important property of the Crothers solution is that it is rigorously correct from a geometrical point of view, and it is further discussed later in this Section. The Crothers solutions are still Riemannian solutions, without consideration of torsion, but in Eq. (23.17), the right hand side term is considered in an approximation to derive from curvature. The effect of gravitational torsion can be included in further work by changing Eq. (23.17) to:

$$\partial_\mu F^{a\mu\nu} = -A^{(0)} \left(R^a{}_\mu{}^{\mu\nu} + \omega^a_{\mu b} T^{\mu\nu b} \right) \tag{23.36}$$

where $\omega^a_{\mu b}$ is the gravitational spin connection and $T^{\mu\nu b}$ the gravitational torsion.

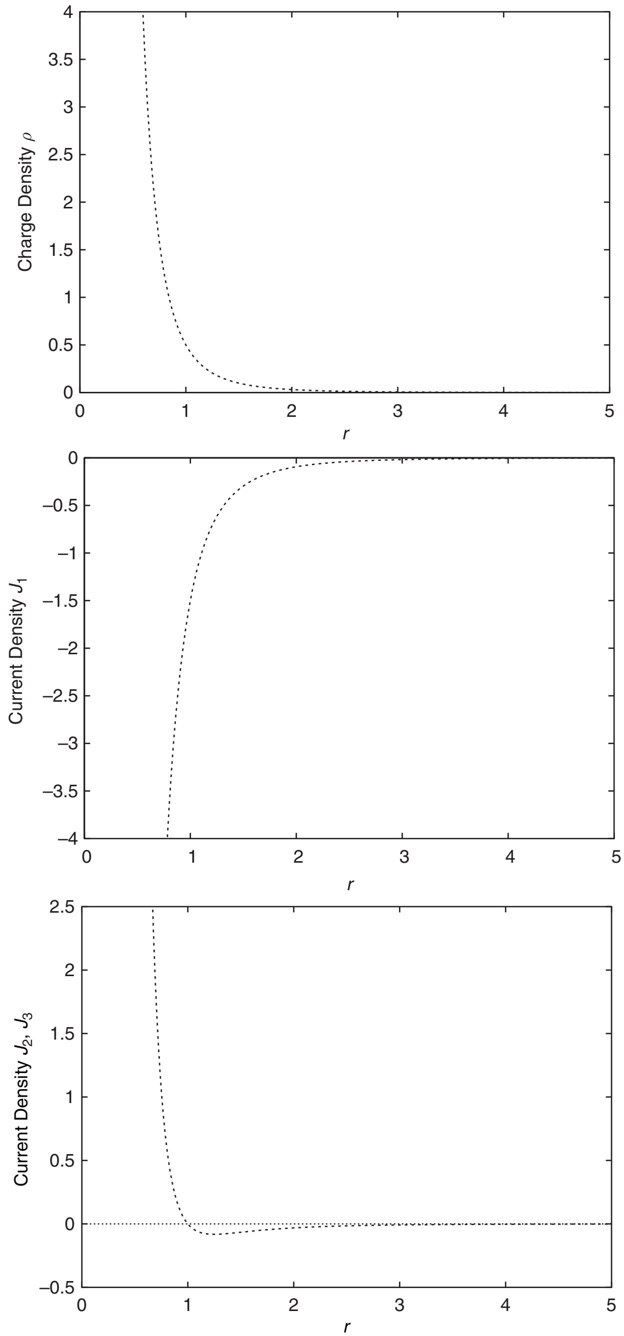

Fig. 23.1a. Spherical metric of Crothers, charge density ρ and current densities $J_r,\ J_\theta, J_\phi$ for $r_0 = 0, \alpha = 0, n = 1, A = B = 1$.

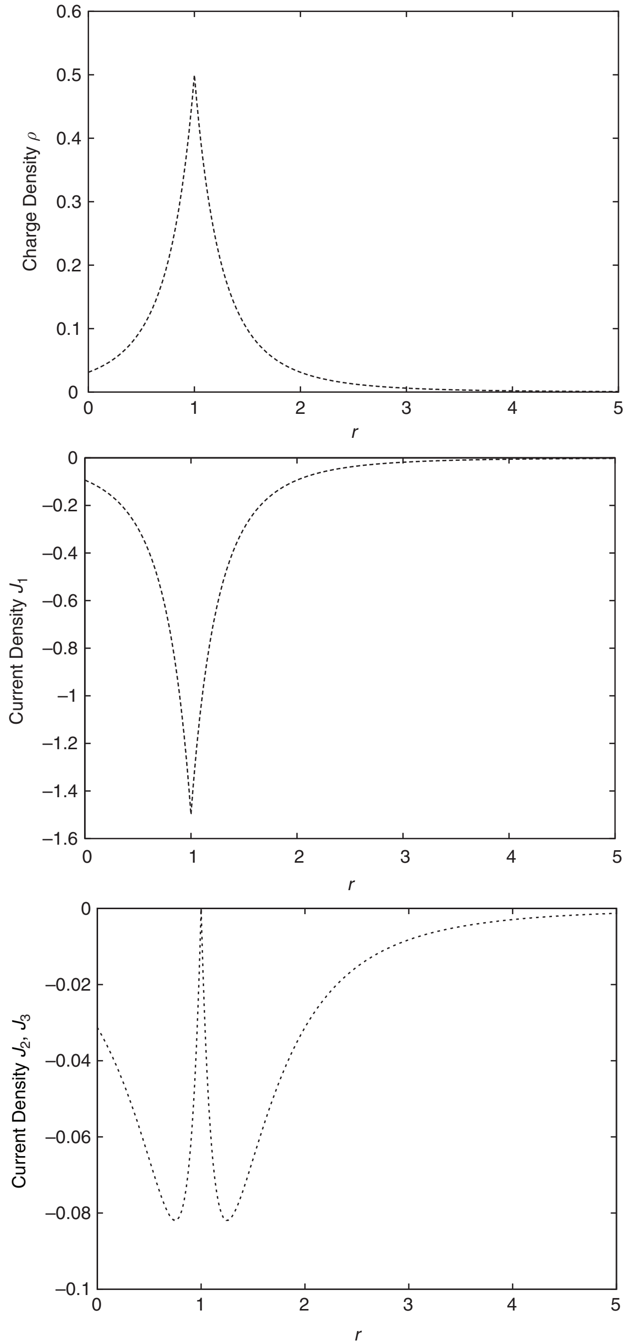

Fig. 23.1b. Spherical metric of Crothers, charge density ρ and current densities J_r, J_θ, J_ϕ for $r_0 = 1, \alpha = 1, n = 1, A = B = 1$.

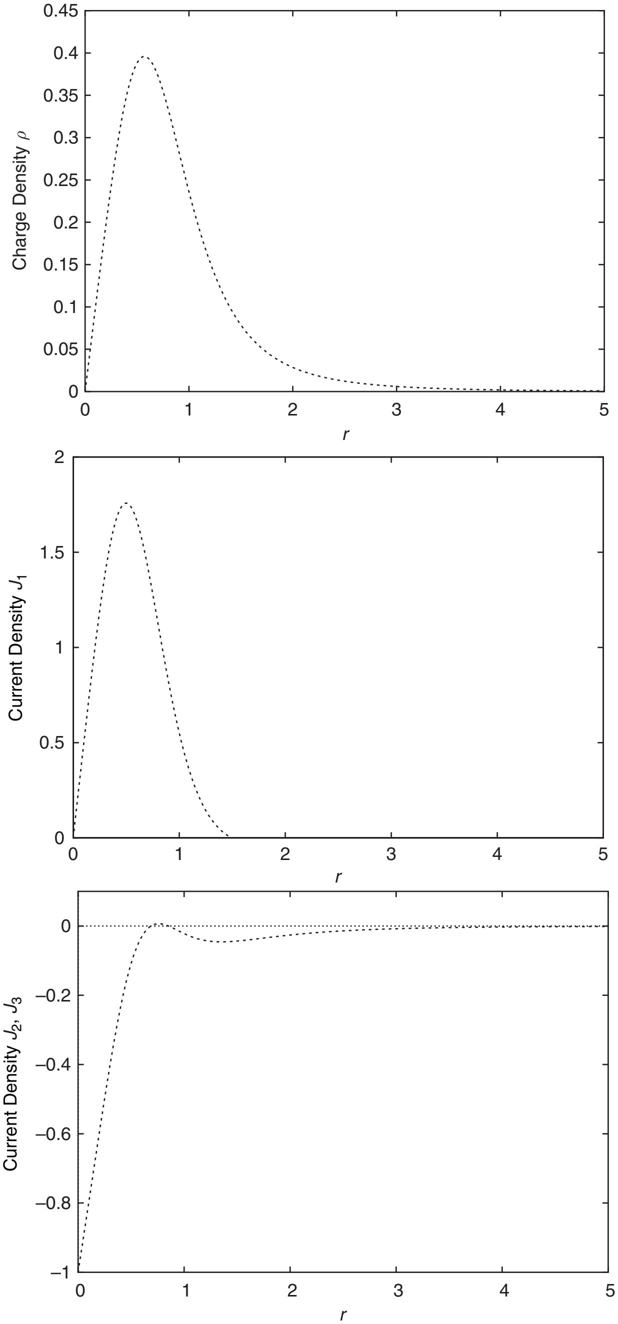

Fig. 23.1c. Spherical metric of Crothers, charge density ρ and current densities J_r, J_θ, J_ϕ for $r_0 = 0, \alpha = 1, n = 3, A = B = 1$.

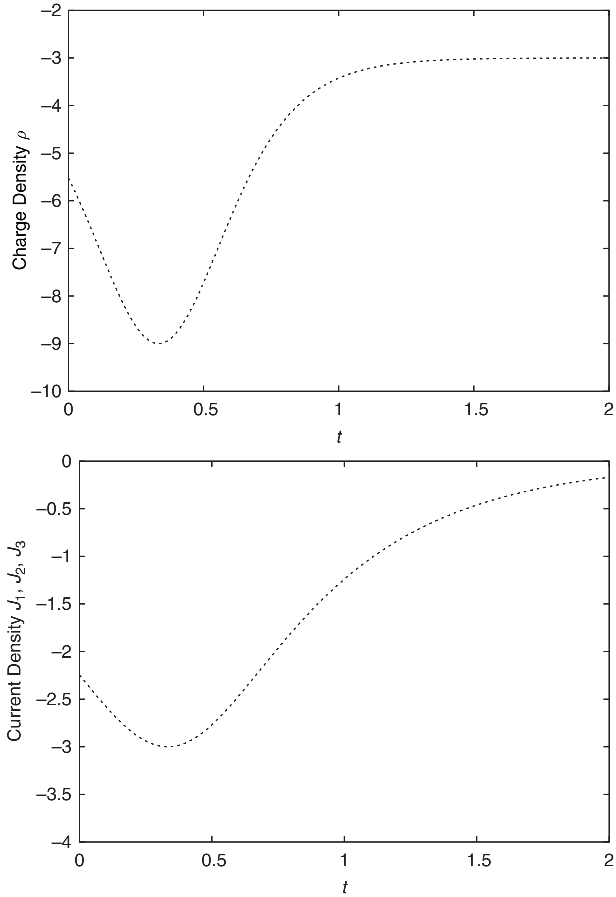

Fig. 23.2. Friedmann Dust metric, charge density ρ and current density J_x, J_y, J_z for $a = 1$.

The rigorous self-consistency of ECE theory is proven from the fact that a vacuum solution, the usually named Schwarzschild metric, results in zero charge density and current density. This proves that the ECE theory is technically correct (see Appendices) and conceptually self consistent and objective. In ECE theory there is neither a gravitational nor an electromagnetic field without mass density acting as the source of that field. Indeed, the gravitational and electromagnetic fields become unified in the same field, and also unified with the weak, strong and fermionic and other matter fields [2–9]. In ECE, field theory is unified with quantum mechanics using the tetrad postulate.

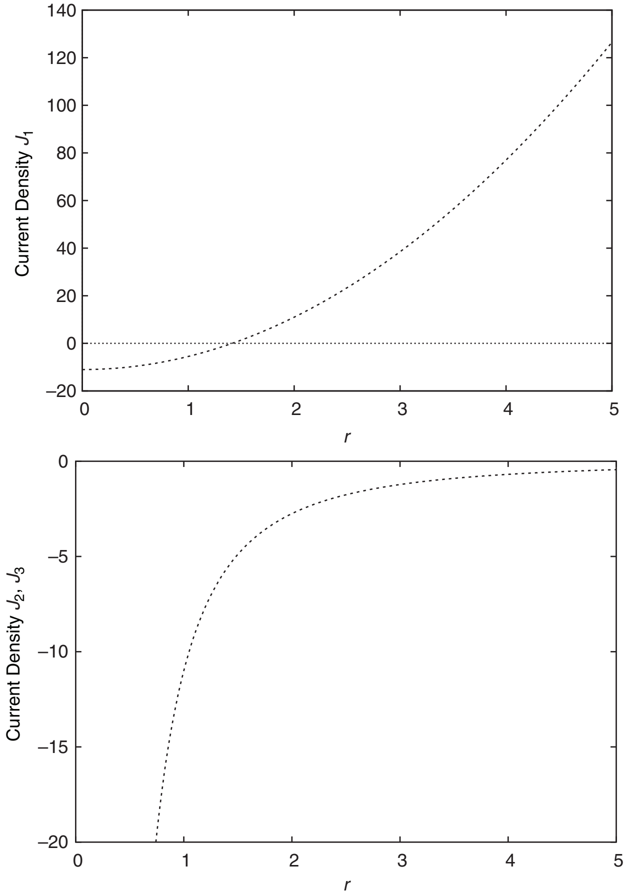

Fig. 23.3. FLRW metric, current density, r dependence of J_r and J_θ, J_ϕ for $a = t^2, t = 1, k = .5$.

It is to be noted that there are conceptual inconsistencies both in MH theory and in the class of vacuum solutions of EH, because in both cases, there is a field of force, but no source for the field. The concept of the field of force was introduced by Faraday. Maxwell considered the source to be the result of the field. The twentieth century view was that the field is produced by the source. ECE theory asserts that the field is geometry, and that the source of the gravitational field unified with the electromagnetic field is mass density.

The main results of this paper are summarized in Table 23.1a and in the figures for charge and current densities for the Coulomb and Ampère Maxwell laws for several representative metrics. Eddington deduced [16] that there is an infinite number of vacuum solutions of the Einstein Hilbert (EH) field

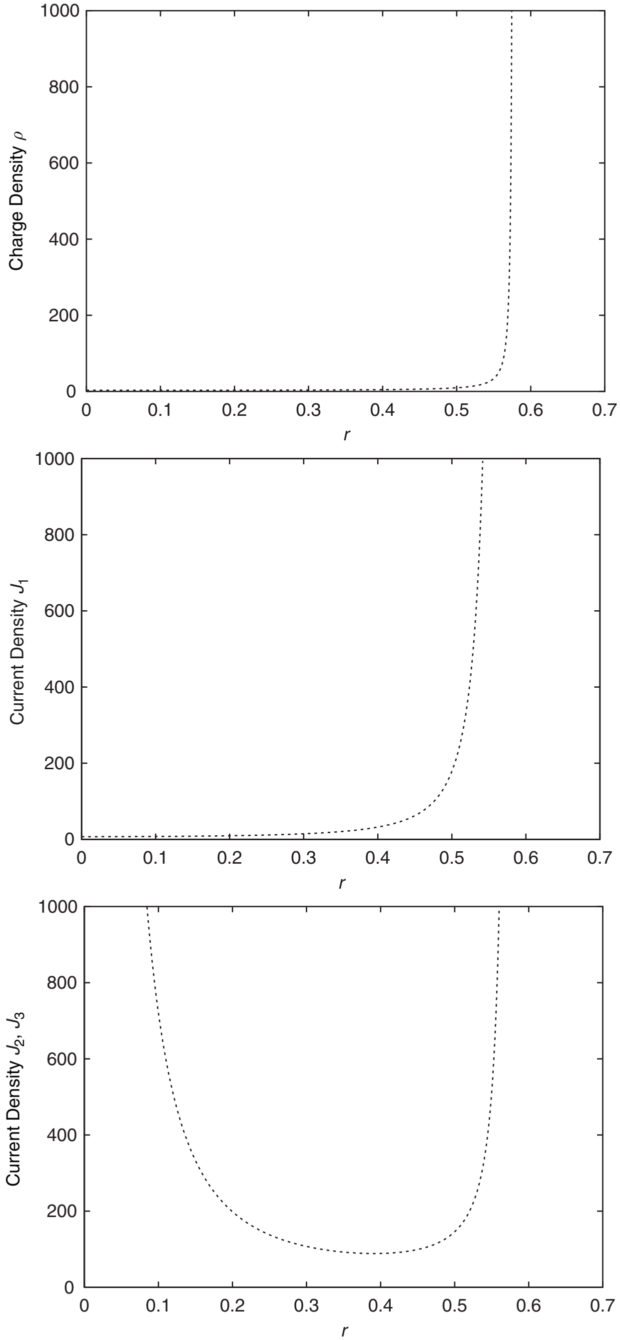

Fig. 23.4. Perfect spherical fluid, charge density ρ and current densities J_r, J_θ, J_ϕ for $a = b = 1$.

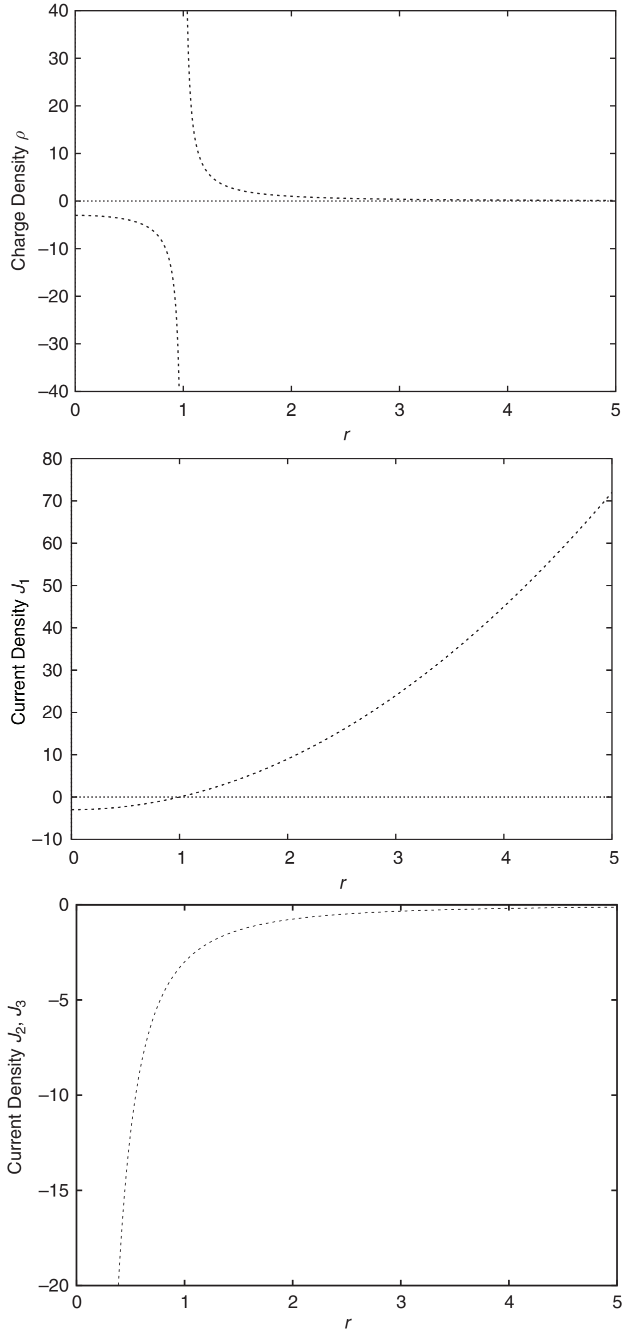

Fig. 23.5. Static De Sitter metric, charge density ρ and current densities J_r, J_θ, J_ϕ for $\alpha = 1$.

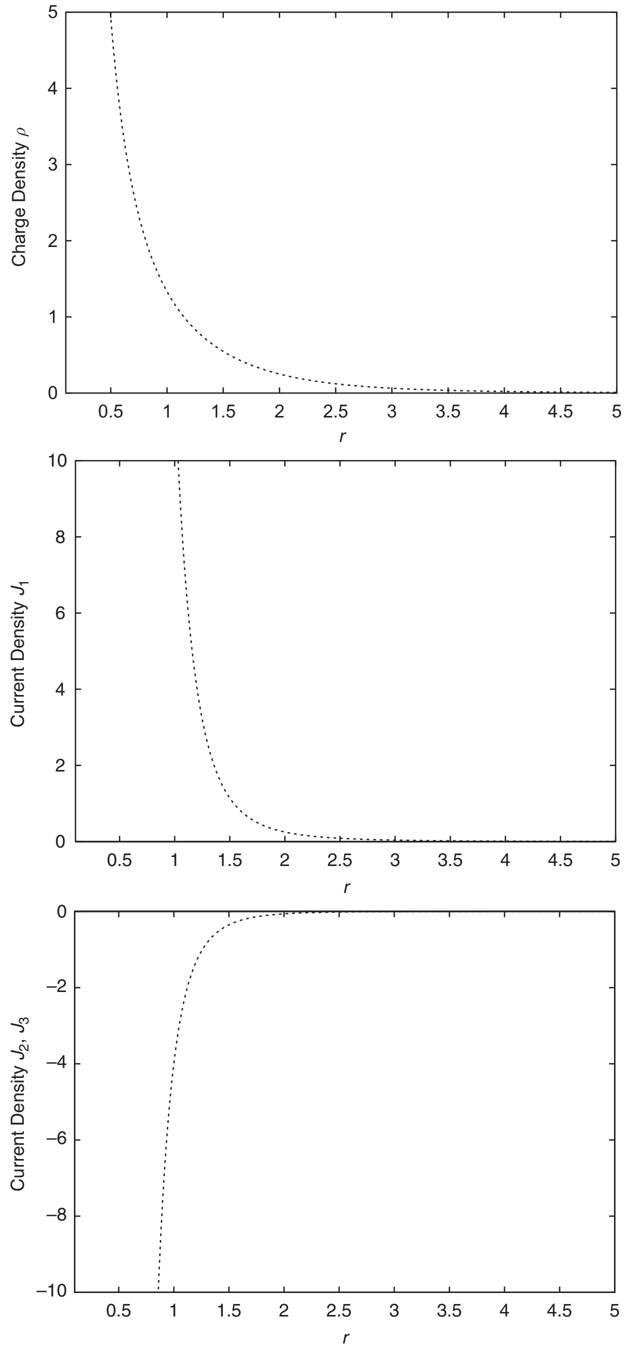

Fig. 23.6a. Reissner-Nordstrom metric, charge density ρ and current densities J_r, J_θ, J_ϕ for $M = 1$, $Q = 2$.

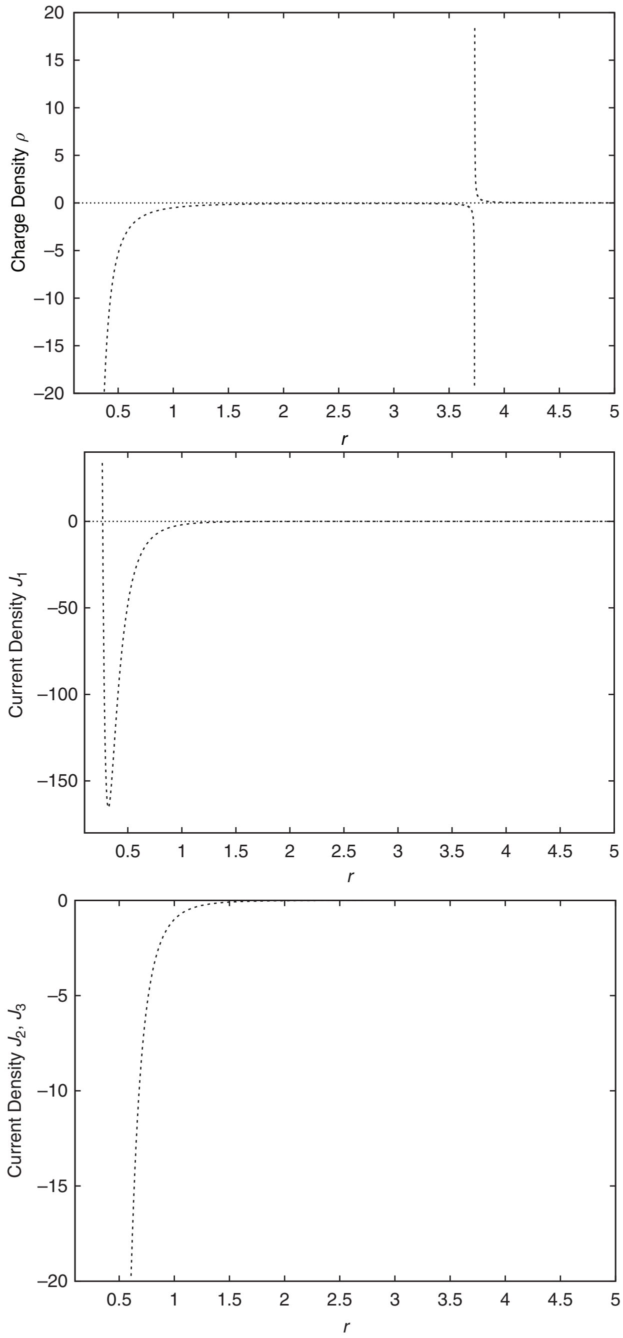

Fig. 23.6b. Reissner-Nordstrom metric, charge density ρ and current densities $J_r,\ J_\theta, J_\phi$ for $M = 2,\ Q = 1$.

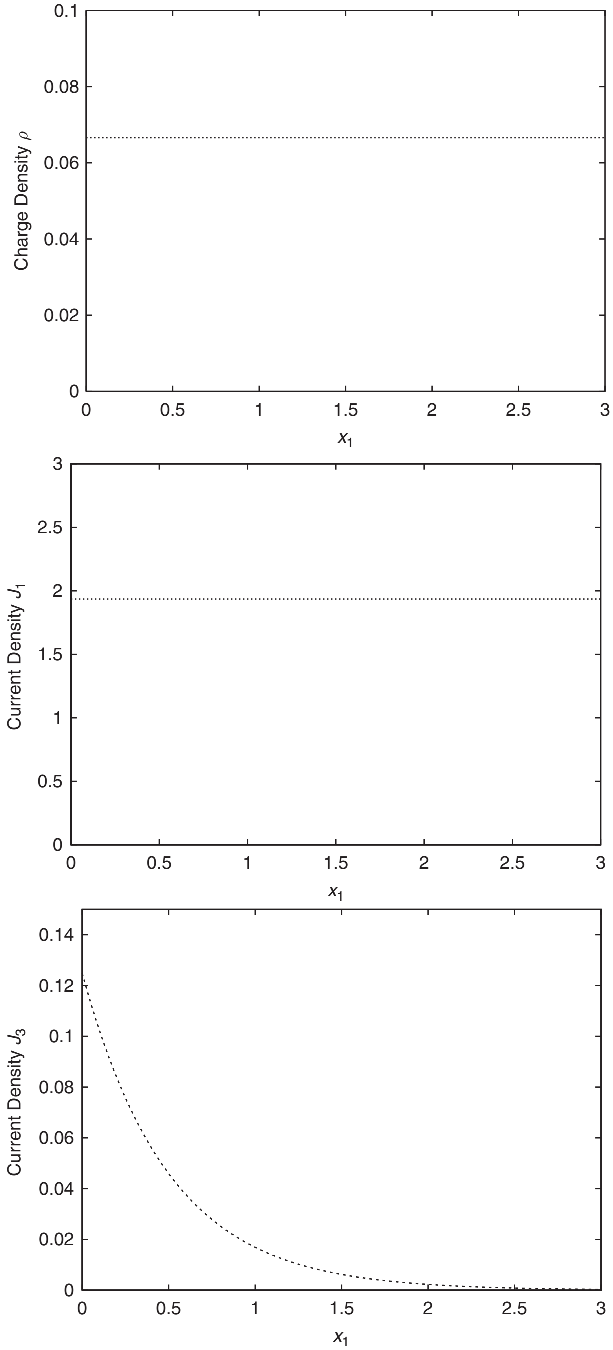

Fig. 23.7. Goedel Metric, charge density ρ and current densities J_x and J_z for $\omega = 1$.

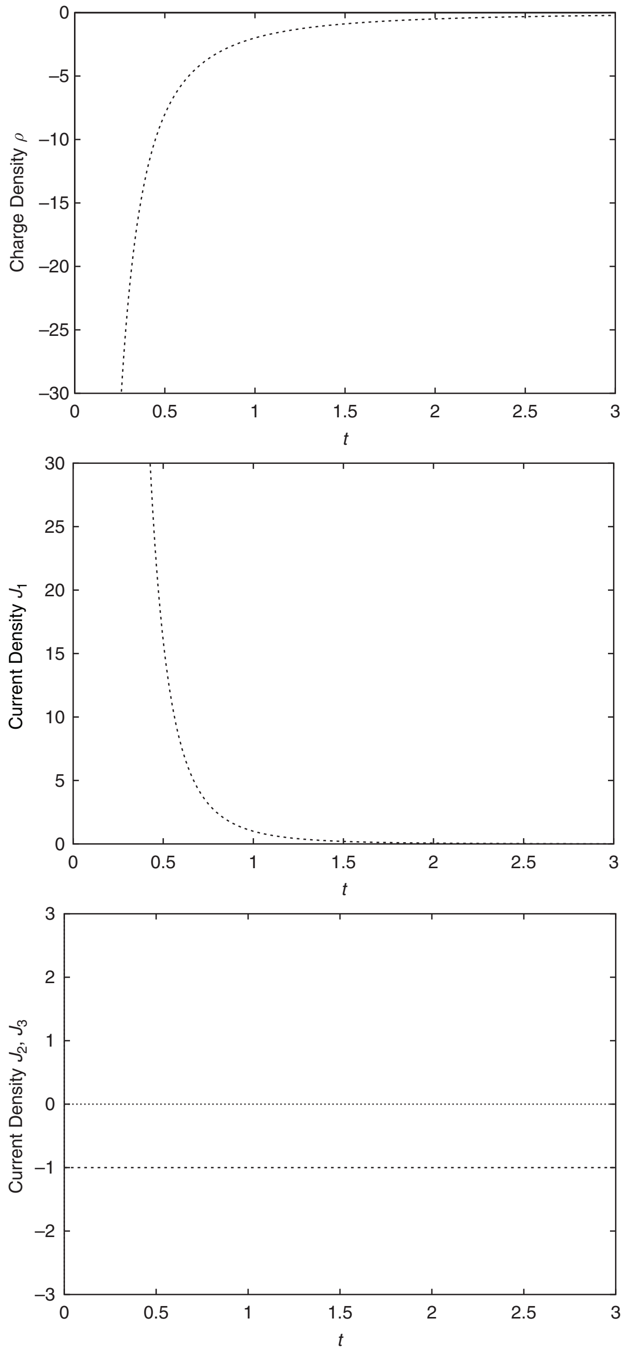

Fig. 23.8. Kasner metric, charge density ρ and current densities J_1, J_2, J_3, for $p_1 = 1$, $p_2 = -1$, $p_3 = 0$.

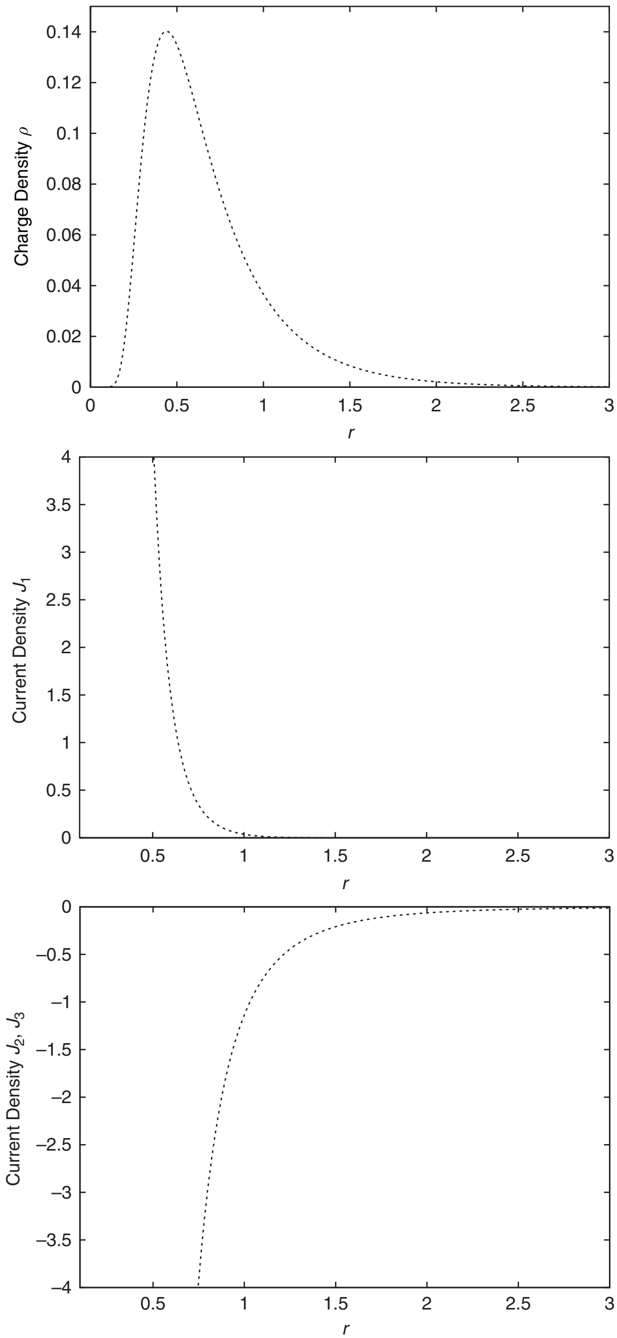

Fig. 23.9. General spherical metric, charge density ρ and current density J_r, J_θ, J_ϕ for $\alpha = 1/r$, $\beta = r$.

equation, which is therefore under determined mathematically. This fact alone shows the need for a generally covariant unified field theory which constrains severely the mathematically allowed solutions using the laws of classical electrodynamics. In a recent volume [17] many mathematical solutions are given of the EH field equation. The ECE of this paper may now be used to find which of these are meaningful in physics (i.e. physical) and which are pure mathematics with no physical meaning. In this paper the vacuum solutions are represented by a few well known types. The Minkowski or flat space-time metric does not give a finite charge or current density in a generally covariant field theory because all the Riemann elements vanish. The usually named Schwarzschild solution (actually the Hilbert solution) does not give a finite charge or current density, and its Ricci tensor elements all vanish. So this is not a valid solution of the EH field equation. It appears to be accurate in the solar system (NASA Cassini) because the weak field limit is used to calculate the light deflection. This Hilbert metric represents space-time around a static mass. The original Schwarzschild metric, as correctly attributed by Crothers [15], gives a physically valid charge and current density (Fig. (23.1c)) which go to zero as the radial coordinate goes to infinity, without nodes and singularities. At infinite separation, objects are infinitely far apart, so no interaction occurs, indicating zero charge and current densities. The general Crothers metric (Table One and Fig. (23.1a)) is a physical metric for these reasons, and is acceptable in a unified field theory.

The Godel metric is a vacuum solution that represents space-time around a spinning mass, and its charge and current densities are sketched in Fig. (23.7). As for all the metrics used in this paper, it was checked by computer that it obeys the Ricci cyclic equation:

$$R^a{}_b \wedge q^b = 0 \tag{23.37}$$

i.e.

$$R^\kappa{}_{\mu\nu\sigma} + R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} = 0 \tag{23.38}$$

in tensor notation. Therefore a spinning mass is sufficient to create charge and current densities in ECE theory. Surprisingly, it was found by computer that the simple Kerr metric for the vacuum (not shown) gave several singularities in the charge and current densities. So this metric, and the charged Kerr metric, was not considered further. The Kasner metric (Fig. (23.8)) represents an anisotropic vacuum and gives finite charge and current densities, so vacuum anisotropy is sufficient to give charge and current density in ECE theory.

Metrics that use finite canonical energy momentum density include the Friedmann dust metric and the Friedmann Lemaitre Robertson Walker (FLRW) metric. Friedmann dust (Fig. (23.2)) gives a finite charge density as r goes to infinity, and this result is considered to be unphysical for reasons stated. The FLRW metric with $a = t^2$ (Fig. (23.3)) has some more than one unphysical characteristic in that it gives radially constant charge density and a radial component of current density that increases quadratically with radial distance and becomes infinite. However, the FLRW gives the correct Coulomb Law (Table 23.1a). The metric for a perfect spherical fluid (Fig. (23.4)) is unphysical again. The charge and current density components go to zero as r goes to infinity for this metric. The static de Sitter metric (Fig. (23.5)) has an unphysical node in charge density and the radial component of current density goes to infinity. The other components of current density go to zero correctly as r goes to infinity. The Reissner Nordstrom metric does not meet the philosophical requirements of ECE theory as discussed already, and as shown in Fig. (23.6) displays an unphysical node in charge density for M = 2, Q = 1. The charge and current densities of this metric go to zero correctly as r goes to infinity. For M = 1, Q = 2 this metric gives the correct behavior as r goes to infinity. Results for the Godel metric are given in Fig. (23.7), the Kasner metric in Fig. (23.8) and the general spherical metric (general Schwarzschild) in Fig. (23.9). The Godel metric describes a homogeneous distribution of swirling dust particles, and gives a constant charge density and radial current density. This is plausible physically. The Kasner metric describes an anisotropic universe without matter, with the choice of parameters:

$$p_1 = 1,\ p_2 = -1,\ p_3 = 0 \tag{23.39}$$

the results depend only on t (Fig. (23.8)) as a consequence of this choice or definition. The charge density goes to zero as t goes to infinity and J_2, J_3 is time independent. Finally the general Schwarzschild metric (Fig. (23.9)) is illustrated for the choice of parameters:

$$\alpha = \frac{1}{r},\ \beta = r \tag{23.40}$$

so there is no intrinsic t dependence by definition. The results are physical, the charge and current densities go to zero as r goes to infinity.

This pattern is likely to be repeated for the various known exact solutions of the EH equation [17], very few metrics give the required electro-dynamical laws without unphysical flaws. So ECE gives the much needed constraint on these EH solutions. Some care must be taken in interpretation of these results, for example the unphysical characteristics in Fig. (23.1b) could be due to the fact that r is constrained to $r > \alpha$ for physical results. However the ECE method gives at least one fundamentally important finding: that spinning mass densities create electromagnetic charge and current density given the existence of the primordial voltage $cA^{(0)}$, indicating the origin of charge in elementary particles.

23.4 Schwarzschild Class of Solutions

The Schwarzschild class of solutions are developed from a generalisation of Minkowski spacetime [19], where the latter is given by

$$ds^2 = dt^2 - dr^2 - |r - r_0|^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ 0 \leq |r - r_0| < \infty, \tag{23.41}$$

although Minkowski space appears in the literature by the following line element,

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\ 0 \leq r < \infty, \tag{23.42}$$

wherein $r_0 = 0$.

The generalisation of the Minkowski line element has the form,

$$ds^2 = A\left(\sqrt{C(r)}\right)dt^2 - B\left(\sqrt{C(r)}\right)d\sqrt{C(r)}^2 - C(r)(d\theta^2 + \sin^2\theta d\varphi^2), \\ C(r) \equiv C(|r - r_0|), \tag{23.43}$$

where $A\left(\sqrt{C(r)}\right)$, $B\left(\sqrt{C(r)}\right)$ and $C(r)$ are *a priori* unknown positive-valued analytic functions that must be determined by the intrinsic geometry of the line element and associated boundary conditions, satisfying the condition $R_{\mu\nu} = 0$. The function $\sqrt{C(r)} = R_c(r)$ is the radius of curvature. Using expression (23.43) in Einstein's field equations gives the following general line element in terms one unknown analytic function,

$$ds^2 = \left(1 - \frac{\alpha}{\sqrt{C(r)}}\right)dt^2 - \left(1 - \frac{\alpha}{\sqrt{C(r)}}\right)^{-1} d\sqrt{C(r)}^2 \\ - C(r)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{23.44}$$

The admissible form of $C(r)$ that satisfies the intrinsic geometry of the line element and the required boundary conditions has been previously deduced [19], and is given by

$$\sqrt{C(r)} = R_c(r) = \left(|r - r_0|^n + \alpha^n\right)^{\frac{1}{n}}, \\ r \in \Re, \quad n \in \Re^+, \quad r \neq r_0, \tag{23.45}$$

where r_0 and n are *entirely arbitrary constants*, and α is a constant that depends upon the mass of the source of the gravitational field. Metric (23.44) is well defined on $-\infty < r < r_0 < r < \infty$, and has no singularity other than at $r = r_0$. Thus, there is no such thing as a black hole.

The metric appearing in the bulk of the literature under the name of the "Schwarzschild" solution[1], is *not* Schwarzschild's solution, and is obtained from (23.4) and (23.5) by choosing $n = 1$, $r_0 = \alpha$, $r > r_0$. It should be noted that according to (23.5) the actual radius of curvature for the usual line element is $R_c = (r - \alpha) + \alpha$, so that α drops out of the expression for the radius of curvature, but that does not mean that R_c can then go down to zero. R_c must always obey expression (23.5), which generates it. There is no possibility of the so-called "black hole". It is the standard but erroneous *assumption* that R_c can go down to zero in the usual line element that has spawned the (fallacious) concept of the black hole. Schwarzschild's true solution [21], although not well known, is obtained by choosing $n = 3$, $r_0 = 0$, $r > r_0$. Schwarzschild's actual solution is well-defined on $0 < r < \infty$, and does not admit of a black hole. Schwarzschild in fact, never made any claims in relation to what has been called a black hole, notwithstanding it being so frequently attributed to him in the literature [22].

The infinite number of metrics obtained via (23.4) and (23.5) are equivalent, and so anything proved for any one of them necessarily holds for all of them. The simplest Schwarzschild-class metric is Brillouin's solution [23–24], obtained by choosing $n = 1$, $r_0 = 0$, $r > r_0$, giving the line element,

$$ds^2 = \left(1 - \frac{\alpha}{r + \alpha}\right) dt^2 - \left(1 - \frac{\alpha}{r + \alpha}\right)^{-1} dr^2 - (r + \alpha)^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad 0 < r < \infty. \tag{23.46}$$

The requirement that a solution for Einstein's static vacuum field must admit of an infinite series of equivalent metrics was pointed put by Eddington as long ago as 1923 [25].

Although the components of the metric tensor of (23.44), based upon the supposition of the line element (23.33), are determined by the field equations, the admissible form of the *a priori unknown* $C(r)$ is *not* determined by the field equations. It is determined by the intrinsic geometry of the line element, already fixed in (23.33) by the form of the line element for Minkowski spacetime itself, and the required boundary conditions. This illustrates that satisfaction of the field equations is a necessary but insufficient condition for a model of Einstein's gravitational field. Indeed, one can substitute into (23.44) *any* analytic function for $C(r)$ without disturbing the spherical symmetry

[1] The first and correct form of this line element was in fact derived by Johannes Droste in 1916 [20].

of the line element and without violating the field equations. However, not simply any analytic function will produce a meaningful model of Einstein's gravitational field. For example, setting $C(r) = \exp(2r)$ produces a line element that is spherically symmetric and satisfies $R_{\mu\nu} = 0$, but it does not describe a model of Einstein's gravitational field. To begin with, the resulting line element is not asymptotically Minkowski, and that is sufficient to invalidate the relevant line element as a model of Einstein's gravitational field, notwithstanding its satisfaction of $R_{\mu\nu} = 0$ and its spherical symmetry.

The fundamental error in the usual analysis of spherically symmetric line elements of a Type 1 Einstein Space, has been its failure to apprehend the geometric fact that there is a distinction between the radius of curvature $R_c(r)$ and the geodesic proper radius $R_p(r)$ in a general spherically symmetric metric space such as that for Einstein's gravitational field. In Minkowski space, $R_c(r)$ and $R_p(r)$ are identical, owing to the pseudo-Euclidean[2] nature of Minkowski space. But Einstein's gravitational field is non-Euclidean (it is a pseudo-Riemannian metric manifold), and so the familiar Euclidean relations do not apply. Nonetheless, the intrinsic geometry of the line element of Einstein's gravitational field for a spherically symmetric Type 1 Einstein Space is precisely the same as the line element for Minkowski space. In both cases the geodesic proper radius is given by the integral of the square root of the component of the line element that contains the square of the differential element of the radius of curvature and the radius of curvature is the square root of the coefficient of the collected infinitesimal angular terms. It is common in the literature to find the radius of curvature referred to as an "areal" radius, or in a certain case as a "Schwarzschild" radius. However, it is in fact *the radius of curvature*, owing to its formal geometric relationship to the Gaussian curvature [26–27], and it does not determine the radial geodesic distance from the source of the gravitational field. In the case of (23.42), $R_c(r) = r$ and

$$R_p(r) = \int_0^r dr = r = R_c(r). \tag{23.47}$$

However, in the case of (23.44),

$$R_p = \int_0^{R_p} dR_p = \int_{R_c(r_0)}^{R_c(r)} \sqrt{B(R_c(r))}\, dR_c(r) = \int_{r_0}^{r} \sqrt{B(R_c(r))}\, \frac{dR_c(r)}{dr} dr, \tag{23.48}$$

where $R_c(r_0)$ is *a priori unknown* owing to the fact that $R_c(r)$ is *a priori unknown*. One cannot simply assume that because $0 \le r < \infty$ in (23.42) that it must follow that in (23.44) $0 \le R_c(r) < \infty$. In other words, one

[2] Actually, pseudo-Efcleethean, after the geometry of Efcleethees.

cannot simply *assume* that $\sqrt{C(r_0)} = R_c(r_0) = 0$. In (23.44) and (23.45) the quantity r is just a parameter, and the radius of curvature and the proper radius are not the same in general. Furthermore, according to (23.44) and (23.45), $R_c(r_0) = \alpha$ and $R_p(r_0) = 0$, $\forall\ r_0$ [19–26].

For the sake of completeness, a similar analysis extends the foregoing results to encompass the Reissner-Nordstrom, Kerr and Kerr-Newman configurations. The generalised line element, in Boyer Lindquist coordinates, is given by [28],

$$ds^2 = \frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 - \frac{\sin^2\theta}{\rho^2}\left[\left(R_c^2 + a^2\right)d\varphi - a dt\right]^2 - \frac{\rho^2}{\Delta}dR_c^2 - \rho^2 d\theta^2, \tag{23.49}$$

$$R_c = R_c(r) = \left(\left|r - r_0\right|^n + \beta^n\right)^{\frac{1}{n}}, \quad \beta = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - (q^2 + a\cos^2\theta)}, \tag{23.50}$$

$$a = \frac{2L}{\alpha}, \quad \rho^2 = R_c^2 + a^2\cos^2\theta, \quad a^2 + q^2 < \frac{\alpha^2}{4}, \quad \Delta = R_c^2 - \alpha R_c + q^2 + a^2, \tag{23.51}$$

$$r \in \Re, \quad n \in \Re^+, \tag{23.52}$$

wherein L is the angular momentum, q is the charge, and the constants r_0 and n are entirely arbitrary. In can be seen that when the charge $q = 0$, the Kerr configuration class is recovered; when the angular momentum is zero, the Reissner-Nordstrom configuration class is recovered; and when the charge and the angular momentum are both zero, the Schwarzschild class of solutions is recovered. In no case is a black hole possible. In all cases an infinite number of equivalent metrics is obtained.

It must be emphasized that the Schwarzschild class of solutions, and also the Reissner-Nordstrom, Kerr and Kerr-Newman line elements, all describe the source of the gravitational field in terms of a *centre of mass*, and so the lone singularity that occurs in these line elements has *no physical significance*.

A full description of Einstein's gravitational field for $R_{\mu\nu} = 0$ requires *two* line elements – one for the voluminous interior of the source of the field and one for the region outside the source, the latter being a centre of mass description, such as one from the Schwarzschild class of solutions. This is illustrated further by the class of solutions for the idealised case of a sphere of homogeneous incompressible fluid.

23.5 The Homogeneous Incompressible Sphere of Fluid

Schwarzschild obtained, in 1916, a solution for a sphere of homogeneous incompressible fluid [29]. His solution has been generalised for an infinite class of equivalent metrics [30]. These solutions demonstrate that there is an upper bound and a lower bound on the size of a sphere of homogeneous incompressible fluid that can exist.

The generalised Schwarzschild line element for this configuration is [30],

$$
\begin{gathered}
ds^2 = \left[\frac{3\cos|\chi_a - \chi_0| - \cos|\chi - \chi_0|}{2}\right]^2 dt^2 - \frac{3}{\kappa\rho_0}d\chi^2 - \frac{3\sin^2|\chi - \chi_0|}{\kappa\rho_0}(d\theta^2 + \sin^2\theta d\varphi^2), \\
\sin|\chi - \chi_0| = \sqrt{\frac{\kappa\rho_0}{3}}\eta^{\frac{1}{3}}, \quad \eta = |r - r_0| + \rho, \quad \kappa = 8\pi k^2, \\
\rho = \left(\frac{\kappa\rho_0}{3}\right)^{-\frac{3}{2}} \left\{\frac{3}{2}\sin^3|\chi_a - \chi_0| - \frac{9}{4}\cos|\chi_a - \chi_0|\left[|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right]\right\}, \\
r \in \Re, \quad \chi \in \Re, \\
0 \le |\chi - \chi_0| \le |\chi_a - \chi_0| < \frac{\pi}{2},
\end{gathered}
\tag{23.53}
$$

where ρ_0 is the constant density of the sphere of fluid, the subscript a denotes values at the surface of the fluid sphere, k^2 is Gauss' gravitational constant, χ_0 denotes the arbitrary location of the centre of spherical symmetry of the sphere in the gravitational field, and r_0 the arbitrary parametric location of the centre of spherical symmetry. Schwarzschild's solution is recoverd by choosing $\chi_0 = 0$, $r_0 = 0$, $\chi \ge 0$ and $r \ge 0$. The foregoing line element is non-singular.

Outside the sphere of fluid, where the sphere is described in terms of its *centre of mass*, the Schwarzschild class of line elements for $R_{\mu\nu} = 0$ is affected by the distribution of mass of the sphere of fluid, and becomes

$$
\begin{gathered}
ds^2 = \left(1 - \frac{\alpha}{R_c}\right)dt^2 - \left(1 - \frac{\alpha}{R_c}\right)^{-1} dR_c^2 - R_c^2(d\theta^2 + \sin^2 d\varphi^2), \\
R_c = \left(|r - r_0|^n + \epsilon^n\right)^{\frac{1}{n}}, \quad \alpha = \sqrt{\frac{3}{\kappa\rho_0}}\sin^3|\chi_a - \chi_0|, \\
\epsilon = \sqrt{\frac{3}{\kappa\rho_0}}\left\{\frac{3}{2}\sin^3|\chi_a - \chi_0| - \frac{9}{4}\cos|\chi_a - \chi_0|\left[|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right]\right\}^{\frac{1}{3}}, \\
r \in \Re, \quad 0 < |\chi_a - \chi_0| < \frac{\pi}{2}, \quad 0 < |r_a - r_0| < \infty,
\end{gathered}
\tag{23.54}
$$

where n, χ_0 and r_0 are entirely arbitrary constants. Schwarzschild's original solution for the region outside the sphere of fluid is recovered by choosing $n = 3$, $r_0 = 0$, $\chi_0 = 0$, $r > 0$ and $\chi_a > 0$.

It is clear from this solution that the constant α appearing in the Schwarzschild class of solutions is not the Newtonian mass. Only in a very weak field, such as that of the Sun is α approximately the Newtonian mass, but that mass is not assigned by comparison to a far field Newtonian potential, as is usually done. It is determined by calculation using quantities associated with the line element for the interior of the source of the gravitational field in all cases, be they weak or strong fields. In the former case the result differs little from the Newtonain value, but in strong fields the difference becomes significant. Furthermore, there are two masses to consider: the passive mass (substantial mass), as determined by the line element for the interior of the source of the gravitational field, and the active mass (gravitational mass) as determined for the line element for the region exterior to the source of the field but which is still obtained from an expression determined by quantities for the field inside the source of the gravitational field. These masses are not the same - the passive mass is greater than the active mass.

The passive mass of the sphere of fluid is determined by the line element for the interior of the sphere, by multiplying the constant density of the sphere into the volume V of the sphere, and is given by

$$\begin{aligned} M = \rho_0 V = \rho_0 \left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}} \int_{\chi_0}^{\chi_a} \sin^2 |\chi - \chi_0| \frac{(\chi - \chi_0)}{|\chi - \chi_0|} d\chi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi \\ = 2\pi\rho_0 \left(\frac{3}{\kappa\rho_0}\right)^{\frac{3}{2}} \left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right). \end{aligned} \tag{23.55}$$

The active mass of the sphere is given by $2m = \frac{\alpha}{k^2}$, i.e.

$$m = \frac{\alpha}{2k^2} = \frac{1}{2k^2}\sqrt{\frac{3}{\kappa\rho_0}} \sin^3 |\chi_a - \chi_0|. \tag{23.56}$$

The ratio of the active to passive mass is,

$$\frac{m}{M} = \frac{2\sin^3 |\chi_a - \chi_0|}{3\left(|\chi_a - \chi_0| - \frac{1}{2}\sin 2|\chi_a - \chi_0|\right)}. \tag{23.57}$$

The escape velocity for the sphere of fluid is given by $v_a = \sin|\chi_a - \chi_0|$. Thus, as the escape velocity increases, the ratio $\frac{m}{M}$ decreases owing to the increase in the mass concentration.

In addition, the proper radius of the sphere of fluid can only be determined from the line elements for its interior [30]. The line elements for the region outside the source of the field can say nothing about the proper radius of the source of the field. This is not surprising, since the line element for the region beyond the surface of the sphere of fluid describes the sphere in terms of its *centre of mass*, and as such treats the source as a point-mass, which has no extension.

23.6 Cosmological Models

A similar fundamental situation arises in the case of an Einstein cosmology. In the case of the FLRW model, for example, there is a line element containing an *a priori unknown* analytic function $\exp(g(t))$. This line element satisfies the field equations, but that does not of itself mean that it yields a valid cosmological model. The form of $\exp(g(t))$ must be determined by the intrinsic geometry of the line element and the boundary conditions. It is *not* determined by the field equations. One must demonstrate that there exists some $\exp(g(t))$ for the FLRW line element before any meaning can be given to it as an Einstein cosmological model. Therefore, any analysis that proceeds by utilising the a priori unknown function $\exp(g(t))$ may well be invalid since it has not been determined beforehand if $\exp(g(t))$ admits of a suitable form for an Einstein cosmological model. Now it has been shown [31] that $\exp(g(t))$ has has only one form meeting the required boundary conditions. In fact, the intrinsic geometry of the FLRW line element implies, with the necessary boundary conditions, an infinite and unbounded Universe. From where does this infinity come? Precisely from $\exp(g(t))$, so that $\exp(g(t))$ is infinite for all values of t. In other words, the FLRW line element modelling an Einstein cosmology is actually independent of time. Therefore, any analysis that proceeds by treating $\exp(g(t))$ in the FLRW line element as finite at any given time, insofar as it is alleged to model an Einstein cosmology, must fail. The Standard Cosmological Model (Big Bang) has failed to correctly consider the intrinsic geometry of the line element and the boundary conditions on $\exp(g(t))$, and so it is invalid. It has merely been *assumed* in the Standard Cosmological Model that $\exp(g(t))$ can be well-defined, never proving that $\exp(g(t))$ has an admissible well-defined form.

The FLRW line element is based upon the assertion that it is possible to express the spherically symmetric line element most generally in co-moving coordinates as [32]

$$ds^2 = e^{\nu} dt^2 + 2adrdt - e^{\lambda} dr^2 - e^{\mu}(r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \tag{23.58}$$

wherein ν, λ and μ are functions of the variables r and t. Then, by a series of transformations and use of the field equations, the FLRW line element is obtained:

$$ds^2 = dt^2 - \frac{e^{g(t)}}{\left[1 + \frac{k}{4}r^2\right]^2}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right), \tag{23.59}$$

where k is a constant. Note that the field equations have *not* determined a form for $g(t)$. This must be determined from the intrinsic geometry of the line element and relevant boundary conditions. A question to be answered therefore is whether or not the intrinsic geometry and boundary conditions admit of a form for $g(t)$ that relates to an Einstein cosmological model. Furthermore, the range on the parameter r must also be determined from the intrinsic geometry of the line element and the boundary conditions. One cannot merely *assume* that in (23.7), $0 \leq r < \infty$. Indeed, the assumption is also demonstrably false.

Since a geometry is entirely determined by the *form* of its line element [32], everything must be determined from it. One cannot, as is usually done, merely foist assumptions upon it. The intrinsic geometry of the line element and the consequent geometrical relations between the components of the metric tensor and associated boundary conditions determine all.

In (23.7) the quantity r is not a radial geodesic distance. It is not even a radius of curvature on (23.7). It is merely a parameter for the radius of curvature and the proper radius, both of which are well-defined by the *form* of the line element (describing a spherically symmetric metric manifold). The radius of curvature, R_c , for (23.7), is

$$R_c = e^{\frac{1}{2}g(t)}\frac{r}{1 + \frac{k}{4}r^2}. \tag{23.60}$$

The proper radius is

$$R_p = e^{\frac{1}{2}g(t)}\int \frac{dr}{1 + \frac{k}{4}r^2} = \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}}\left(\arctan\frac{\sqrt{k}}{2}r + n\pi\right), \quad n = 0, 1, 2, ... \tag{23.61}$$

Since $R_p \geq 0$ by definition, $R_p = 0$ is satisfied when $r = 0 = n$. So $r = 0$ is the lower bound on r. The upper bound on r must also be ascertained from the line element and boundary conditions.

It is noted that the spatial component of (23.8) has a maximum of $\frac{1}{\sqrt{k}}$ at any time t, when $r = \frac{2}{\sqrt{k}}$. Thus, as $r \to \infty$, the spatial component of R_c runs from 0 (at $r = 0$) to the maximum $\frac{1}{\sqrt{k}}$ (at $r = \frac{2}{\sqrt{k}}$), then back to 0, since

$$\lim_{r\to\infty}\frac{r}{1 + \frac{k}{4}r^2} = 0. \tag{23.62}$$

Transform (23.7) by setting

$$R = R(r) = \frac{r}{1 + \frac{k}{4}r^2}, \tag{23.63}$$

which carries (23.7) into

$$ds^2 = dt^2 - e^{g(t)} \left[\frac{dR^2}{1 - kR^2} + R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right]. \tag{23.64}$$

The quantity R appearing in (23.11) is not a radial geodesic distance. It is only a component of the radius of curvature in that it relates to the Gaussian curvature $G = \frac{1}{e^{g(t)}R^2}$. The radius of curvature of (23.11) is

$$R_c = \frac{1}{\sqrt{G}} = e^{\frac{1}{2}g(t)}R, \tag{23.65}$$

and the proper radius of Einstein's universe is, by (23.11),

$$R_p = e^{\frac{1}{2}g(t)} \int \frac{dR}{1 - kR^2} = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left(\arcsin \sqrt{k}R + 2m\pi \right), \quad m = 0, 1, 2, ... \tag{23.66}$$

Now according to (23.10), the minimum value of R is $R(r = 0) = 0$. Also, according to (23.10), the maximum value R is $R(r = \frac{2}{\sqrt{k}}) = \frac{1}{\sqrt{k}}$. $R = \frac{1}{\sqrt{k}}$ makes (23.11) singular, although (23.7) is not singular at $r = \frac{2}{\sqrt{k}}$. Since by (23.10), $r \to \infty \Rightarrow R(r) \to 0$, then if $0 \leq r < \infty$ on (23.7), it follows that the proper radius of Einstein's universe is, according to (23.10),

$$R_p = e^{\frac{1}{2}g(t)} \int_0^0 \frac{dR}{1 - kR^2} \equiv 0. \tag{23.67}$$

Therefore, $0 \leq r < \infty$ on (23.7) is false. Furthermore, since the proper radius of Einstein's universe cannot be zero and cannot depend upon a set of coordinates (it must be an invariant), expressions (23.9) and (23.13) must agree. Similarly, the radius of curvature of Einstein's universe must be an invariant (independent of a set of coordinates), so expressions (23.8) and (23.12) must also agree, in which case $0 \leq R < \frac{1}{\sqrt{k}}$ and $0 \leq r < \frac{2}{\sqrt{k}}$. Then by (23.9), the proper radius of Einstein's universe is

$$R_p = \lim_{\alpha \to \frac{2}{\sqrt{k}}} e^{\frac{1}{2}g(t)} \int_0^\alpha \frac{dr}{1 + \frac{k}{4}r^2} = \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left[\left(\frac{\pi}{4} + n\pi \right) - m\pi \right], \quad n, m = 0, 1, 2, ... \quad n \geq m. \tag{23.68}$$

Setting $p = n - m$ gives for the proper radius,

$$R_p = \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left(\frac{\pi}{4} + p\pi\right), \quad p = 0, 1, 2, ... \tag{23.69}$$

Now by (23.13), the proper radius of Einstein's univese is

$$R_p = \lim_{\alpha \to \frac{1}{\sqrt{k}}} e^{\frac{1}{2}g(t)} \int_0^\alpha \frac{dR}{\sqrt{1 - kR^2}} = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left[\left(\frac{\pi}{2} + 2n\pi\right) - m\pi\right], \quad n, m = 0, 1, 2, ... \tag{23.70}$$

$$2n \geq m. \tag{23.71}$$

Setting $q = 2n - m$ gives the proper radius of Einstein's universe as,

$$R_p = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left(\frac{\pi}{2} + q\pi\right), \quad q = 0, 1, 2, ... \tag{23.72}$$

Expressons (23.16) and (23.17) must be equal for all values of p and q. This can only occur if $g(t)$ is infinite for all values of t. Thus, the proper radius of Einstein's universe is infinite, and hence, the radius of curvature of Einstein's universe is also infinite. In addition, it follows from the line elements, that the volume and the area of Einstein's universe are infinite for all time t. Thus, Einstein's universe is infinite and unbounded and independent of time. Therefore, the Standard Cosmological Model (Big Bang) is inconsistent with General Relativity and is therefore invalid.

The standard static cosmological models suffer from the same fundamental defects, and are therefore invalid. The line element for Einstein's cylindrical model is,

$$ds^2 = dt^2 - \left[1 - \left(\lambda - 8\pi P_0\right) R_c^2\right]^{-1} dR_c^2 - R_c^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{23.73}$$

This has no Lorentz signature solution for $\frac{1}{\sqrt{\lambda - 8\pi P_0}} < R_c(r) < \infty$ [33]. For $1 - \left(\lambda - 8\pi P_0\right) R_c^2 > 0$ and $R_c = R_c(r) \geq 0$,

$$0 \leq R_c < \frac{1}{\sqrt{\lambda - 8\pi P_0}}. \tag{23.74}$$

The proper radius is

$$R_p = \lim_{\alpha \to \frac{1}{\sqrt{\lambda - 8\pi P_0}}} \int_0^{\alpha} \frac{dR_c}{\sqrt{1 - (\lambda - 8\pi P_0)\, R_c^2}} = \frac{(1+4n)\,\pi}{2\sqrt{\lambda - 8\pi P_0}}, \quad n = 0, 1, 2, ... \tag{23.75}$$

which is arbitrarily large.

The spherical model of de Sitter is given by the line element

$$ds^2 = \left(1 - \frac{\lambda + 8\pi\rho_{00}}{3} R_c^2\right) dt^2 - \left(1 - \frac{\lambda + 8\pi\rho_{00}}{3} R_c^2\right)^{-1} dR_c^2 - R_c^2 (d\theta^2 + \sin^2\theta d\varphi^2), \tag{23.76}$$

where ρ_{00} is the macroscopic density of the Universe. This line element has no Lorentz signature solution on $\sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}} < R_c < \infty$ [33], so $0 \leq R_c < \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}}$. The proper radius is

$$R_p = \lim_{\alpha \to \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}}} \int_0^{\alpha} \frac{dR_c}{R_c \sqrt{\lambda + 8\pi\rho_{00}}} = \sqrt{\frac{3}{\lambda + 8\pi\rho_{00}}} \frac{(1+4n)\,\pi}{2}, \quad n = 0, 1, 2, ... \tag{23.77}$$

which is arbitrarily large.

It is also worth noting that it has recently been shown that the likely source of the Cosmic Microwave Background (CMB) is not the Cosmos but the oceans of the Earth [15–23], and therefore the CMB has nothing to do with the Standard Cosmological Model (Big Bang). It is anticipated that the PLANCK satellite, soon to be launched to the 2nd Lagrange Point, will verify the oceans, the Earth Microwave Background (EMB), as the source of the CMB. The PLANCK satellite is equipped with absolute measuring instruments whereas the WMAP satellite has only differential instruments and so cannot take an absolute measurement, which simply means that the interpretation of its data (and that of COBE) as a verification of the Big Bang source of the CMB is invalid. Indeed, the WMAP data appears to have no relevance for cosmology at all.

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A

Appendix 1: Hodge Dual Transformation

The general Hodge dual of a tensor is defined 11 as:

$$\widetilde{V}_{\mu_1\ldots\mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1\ldots\nu_p}_{\mu_1\ldots\mu_{n-p}} V_{\nu_1\ldots\nu_p} \tag{A.1}$$

where:

$$\epsilon_{\mu_1\mu_2\ldots\mu_n} = |g|^{\frac{1}{2}}\epsilon_{\mu_1\mu_2\ldots\mu_n} \tag{A.2}$$

is the totally anti-symmetric tensor, defined as the square root of the modulus of the determinant of the metric multiplied by the Levi-Civita symbol:

$$\bar{\epsilon}_{\mu_1\mu_2\ldots\mu_n} = \left\{ \begin{array}{rl} 1 & \text{for even permutation} \\ -1 & \text{for odd permutation} \\ 0 & \text{otherwise} \end{array} \right\}. \tag{A.3}$$

Using the metric compatibility condition [11]:

$$D_\mu g_{\nu p} = 0 \tag{A.4}$$

it is seen that:

$$D_\mu |g|^{\frac{1}{2}} = \partial_\mu |g|^{\frac{1}{2}} = 0 \tag{A.5}$$

because the determinant of the metric is made up of individual elements of the metric tensor. The covariant derivative of each element vanishes by Eq. (A.4), so we obtain Eq. (A.5). The pre-multiplier $|g|^{\frac{1}{2}}$ is a scalar, and we use the fact that the covariant derivative of a scalar is the same as its four-derivative [11]:

$$D_\mu V = \partial_\mu V. \tag{A.6}$$

The homogeneous field equation (23.4) in tensor notation is:

$$\begin{aligned}&\partial_\mu F^a_{\nu\rho} + \partial_\rho F^a_{\mu\nu} + \partial_\nu F_{\rho\mu} = \\ &\quad - A^{(0)} \left(R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} + \omega^a_{\mu b} T^b_{\nu\rho} + \omega^a_{\rho b} T^b_{\mu\nu} + \omega^a_{\nu b} T^b_{\rho\mu}\right)\end{aligned} \tag{A.7}$$

and this is equivalent [2–18, 12] to:

$$\partial_\mu \widetilde{F}^{\alpha\mu\nu} = \mu_0 \tilde{j}^{\nu a} := -A^{(0)} \left(\widetilde{R}^a_{\mu^{\mu\nu}} + \omega^a_{\mu b} \widetilde{T}^{b\mu\nu}\right) \tag{A.8}$$

The Hodge dual of a two-form in four-dimensional space-time is another two-form. For example:

$$\widetilde{F}^{a\mu\nu} = \frac{1}{2}|g|^{\frac{1}{2}} \bar{\epsilon}^{\mu\nu\rho\sigma} F^a_{\rho\sigma}, \tag{A.9}$$

$$\widetilde{T}^{a\mu\nu} = \frac{1}{2}|g|^{\frac{1}{2}} \bar{\epsilon}^{\mu\nu\rho\sigma} T^a{}_{\rho\sigma}\,, \tag{A.10}$$

$$\widetilde{R}^{a\mu\nu}{}_b = \frac{1}{2}|g|^{\frac{1}{2}} \bar{\epsilon}^{\mu\nu\rho\sigma} R^a{}_{b\rho\sigma}. \tag{A.11}$$

The Bianchi identity:

$$d \wedge T^a_{\mu\nu} + \omega^a{}_b \wedge T^b_{\mu\nu} := -q \wedge R^a{}_{b\mu\nu} \tag{A.12}$$

is an identity between two-forms. So it remains true for:

$$d \wedge \widetilde{F}^a_{\mu\nu} = -A^{(0)} \left(\widetilde{R}^a{}_{b\mu\nu} + \omega^a{}_b \wedge \widetilde{T}^a_{\mu\nu}\right) \tag{A.13}$$

because $\widetilde{F}_{\mu\nu}$, $\widetilde{R}_{\mu\nu}$, and $\widetilde{T}_{\mu\nu}$ are two-forms, antisymmetric in their last two indices. In other words if we write down the sum:

$$\partial_\mu \widetilde{F}^a_{\nu\rho} + \partial_\rho \widetilde{F}^a_{\mu\nu} + \partial_\nu \widetilde{F}^a_{\rho\mu} := d \wedge \widetilde{F}^a \tag{A.14}$$

it is identically equal to the sum:

$$\begin{aligned}&- A^{(0)} \left(\widetilde{R}^a_{\mu\nu\rho} + \widetilde{R}^a_{\rho\mu\nu} + \widetilde{R}^a_{\nu\rho\mu} + \omega^a_{\mu b} \widetilde{T}^b_{\nu\rho} + \omega^a_{\rho b} \widetilde{T}^b_{\mu\nu} + \omega^a_{\nu b} \widetilde{T}^b_{\rho\mu}\right) \\ &:= -A^{(0)} \left(q^b \wedge \widetilde{R}^a{}_b + \omega^a{}_b \wedge \widetilde{T}^b\right)\end{aligned} \tag{A.15}$$

So the inhomogeneous field equation is:

$$d \wedge \tilde{F}^a = \mu_0 J^a = -A^{(0)} \left(q^b \wedge \widetilde{R}^a{}_b + \omega^a{}_b \wedge \widetilde{T}^b \right) \tag{A.16}$$

which is equivalent to:

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu} \tag{A.17}$$

as given in the text.

B

Appendix 2: Equivalence of Indices in the Field Equations

The homogeneous and inhomogeneous field equations can be written in equivalent ways, and the equivalence is proven in this Appendix. The first method of writing the homogeneous field equation is the sum:

$$\partial_\mu F^a_{\nu\rho} + \partial_\rho F^a_{\mu\nu} + \partial_\nu F^a_{\rho\mu} = \mu_0 \left(j^a_{\mu\nu\rho} + j^a_{\rho\mu\nu} + j^a_{\nu\rho\mu} \right) \tag{B.1}$$

where the charge current density three-forms are defined by:

$$\begin{aligned} j^a_{\mu\nu\rho} &+ j^a_{\rho\mu\nu} + j^a_{\nu\rho\mu} \\ &:= -\frac{A^{(0)}}{\mu_0} \left(R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} + \omega^a_{\mu b} T^b_{\nu\rho} + \omega^a_{\rho b} T^b_{\mu\nu} + \omega^a_{\nu b} T^b_{\rho\mu} \right). \end{aligned} \tag{B.2}$$

Considering individual tensor elements such as those defined by

$$\begin{aligned} \partial_0 \tilde{F}^{a01} + \partial_2 \tilde{F}^{a21} + \partial_3 \tilde{F}^{a31} &= \frac{1}{2} |g| \bar{\epsilon}^{\mu 1 \rho\sigma} \partial_\mu F^a_{\rho\sigma} \\ = \frac{1}{2} |g|^{\frac{1}{2}} \left(\bar{\epsilon}^{01\rho 0} \partial_0 F^a_{\rho\sigma} + \bar{\epsilon}^{21\rho\sigma} \partial_2 F^a_{\rho\sigma} + \bar{\epsilon}^{31\rho\sigma} \partial_3 F^a_{\rho\sigma} \right) & \\ &= |g|^{\frac{1}{2}} \left(\partial_0 F^a{}_{23} + \partial_2 F^a{}_{30} + \partial_3 F^a{}_{02} \right) \end{aligned} \tag{B.3}$$

which is a special case of the general result:

$$\partial_\mu \widetilde{F}^{a\mu\nu} = |g|^{\frac{1}{2}} \left(\partial_\mu F^a_{\nu\rho} + \partial^a_\rho F_{\mu\nu} + \partial_\nu F^a_{\rho\mu} \right). \tag{B.4}$$

Now consider the following current term for $\sigma = 1$ to obtain:

$$\tilde{j}^{a\sigma} = \frac{1}{6} |g|^{\frac{1}{2}} \bar{\epsilon}^{\mu\nu\rho\sigma} j^a_{\mu\nu\rho}, (\sigma = 1) \tag{B.5}$$

$$\bar{j}^{a1} = \frac{1}{3}|g|^{\frac{1}{2}} \left(j^a{}_{023} + j^a{}_{302} + j^a{}_{230} \right). \tag{B.6}$$

Similarly, the other two current terms

$$\tilde{j}^{a\sigma} = \frac{1}{6}|g|^{\frac{1}{2}} \bar{\epsilon}^{\rho\mu\nu\sigma} j^a_{\rho\mu\nu} \tag{B.7}$$

and

$$\tilde{j}^{a\sigma} = \frac{1}{6}|g|^{\frac{1}{2}} \bar{\epsilon}^{\nu\rho\mu\sigma} j^a_{\nu\rho\mu} \tag{B.8}$$

give Eq. (B.6) two more times. So the right hand side of Eq. (B.1) for $\nu = 1$ is:

$$\tilde{j}^{a1} = |g|^{\frac{1}{2}} \left(j^a{}_{023} + j^a{}_{302} + j^a{}_{230} \right). \tag{B.9}$$

Finally use Eq. (A.5) to find that:

$$\partial_\mu \left(|g|^{\frac{1}{2}} F^a_{\nu\rho} \right) = |g|^{\frac{1}{2}} \partial_\mu F^a_{\nu\rho} \tag{B.10}$$

and so derive Eq. (23.8) from Eq. (B.1), Q.E.D. Note that the pre-multiplier $|g|^{1/2}$ cancels out on either side of Eq. (23.8).

Similarly it can be shown that the following expression of the inhomogeneous field equation:

$$\begin{aligned} &\partial_\mu \widetilde{F}^a_{\nu\rho} + \partial_\rho \widetilde{F}^a_{\mu\nu} + \partial_\nu \widetilde{F}^a_{\rho\mu} \\ &\quad = -A^{(0)} \left(\widetilde{R}^a{}_{\mu\nu\rho} + \widetilde{R}^a{}_{\rho\mu\nu} + \widetilde{R}^a{}_{\nu\rho\mu} + \omega^a{}_{\mu b} \widetilde{T}_{\nu\rho} + \omega^a_{\mu b} \widetilde{T}_{\mu\nu} + \omega^a_{\nu b} \widetilde{T}_{\rho\mu} \right) \end{aligned} \tag{B.11}$$

is equivalent to:

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu} \tag{B.12}$$

as used in the text.

As a familiar example of Appendices 1 and 2 consider the Maxwell Heaviside (MH) equations in free space. The homogeneous MH equation in differential form notation is

$$d \wedge F = 0 \tag{B.13}$$

which is either:

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \tag{B.14}$$

or

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \tag{B.15}$$

in tensor notation. The inhomogeneous MH equation in differential form notation is:

$$d \wedge \tilde{F} = 0 \tag{B.16}$$

which is either:

$$\partial_\mu F^{\mu\nu} = 0 \tag{B.17}$$

or

$$\partial_\mu \tilde{F}_{\nu\rho} + \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} = 0 \tag{B.18}$$

in tensor notation. The individual Hodge dual tensors are defined by:

$$\tilde{F}^{\nu\rho} = \frac{1}{2}\epsilon^{\nu\rho\mu\sigma} F_{\mu\sigma} \quad \text{etc.} \tag{B.19}$$

and indices are lowered as follows:

$$\tilde{F}_{\nu\rho} = g_{\nu\rho} g_{\rho\kappa} \tilde{F}^{\rho\kappa} \quad \text{etc.} \tag{B.20}$$

where $g_{\mu\nu}$ is the Minkowski metric in this case. The equivalent ECE equations in free space have the same properties exactly except of the addition of the index a to every tensor in the equations. Finally, the homogeneous ECE equation in form notation is:

$$d \wedge F^a = \mu_0 j^a \tag{B.21}$$

which is

$$\partial_\mu \tilde{F}^{\mu\nu a} = \mu_0 \tilde{j}^{\nu a} \tag{B.22}$$

in tensor notation. The inhomogeneous ECE equation in form notation is:

$$d \wedge \tilde{F}^a = \mu_0 J^a \tag{B.23}$$

which is

$$\partial_\mu \tilde{F}^{\mu\nu a} = \mu_0 J^{\nu a} \tag{B.24}$$

in tensor notation. The individual Hodge duals are:

$$\tilde{F}^{\nu\rho a} = \frac{1}{2}|g|^{\frac{1}{2}}\bar{\epsilon}^{\nu\rho\mu\sigma}F^a_{\mu\sigma} \quad \text{etc.} \tag{B.25}$$

and indices are lowered with the metric of the base manifold:

$$\tilde{F}^a_{\nu\rho} = g_{\nu\rho}g_{\rho\kappa}\tilde{F}^{\nu\rho a} \quad \text{etc.} \tag{B.26}$$

C

Appendix 3: Reduction to Vector Notation

In this appendix the tensorial form of the inhomogeneous ECE equation is reduced to the vector form, giving the Coulomb and Ampère Maxwell laws in generally covariant unified field theory. Begin with the inhomogeneous field equation:

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu} = -\frac{A^{(0)}}{\mu_0}\left(R^a{}_\mu \mu\nu + \omega^a{}_{\mu b} T^{b\mu\nu}\right). \tag{C.1}$$

In the Einstein Hilbert limit:

$$T^{b\mu 0} = 0 \tag{C.2}$$

so the equation becomes:

$$\partial_\mu F^{a\mu\nu} = -\frac{A^{(o)}}{\mu_0} R^{a\mu\nu}{}_\mu. \tag{C.3}$$

The indices in the Riemann tensor elements are raised using the metric of the base manifold as follows:

$$R^a{}_\mu{}^{\sigma\rho} = g^{\sigma\nu} g^{\rho\kappa} R^a{}_{\mu\nu\kappa}. \tag{C.4}$$

The Coulomb Law is obtained for:

$$\nu = 0 \tag{C.5}$$

and is:

$$\partial_\mu F^{a\mu 0} = -\frac{A^{(0)}}{\mu_0}\left(R^a{}_1{}^{10} + R^a{}_2{}^{20} + R^a{}_3{}^{30}\right) \tag{C.6}$$

where summation over repeated μ indices has been carried out. The vector form of Eq. (C.6) is:

$$(\boldsymbol{\nabla} \cdot \boldsymbol{E})^a = -\phi \left(R^a{}_1{}^{10} + R^a{}_2{}^{20} + R^a{}_3{}^{30} \right). \tag{C.7}$$

The only possible value of a (see also Appendix D) for the Coulomb Law is:

$$a = 0 \tag{C.8}$$

so we obtain the generally covariant Coulomb Law:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = (\boldsymbol{\nabla} \cdot \boldsymbol{E})^0 = -\phi \left(R^0{}_1{}^{10} + R^0{}_2{}^{20} + R^0{}_3{}^{30} \right). \tag{C.9}$$

Both sides are scalar valued quantities, so the time-like, or scalar, index $a = 0$ is used. Here ϕ is the scalar potential, having the units of volts. The units of $\boldsymbol{E}$ are volt/m and those of the R elements are inverse meters squared, so units are consistent.

The generally covariant Ampère Maxwell law is obtained with:

$$\nu = 1, 2, 3. \tag{C.10}$$

When:

$$\nu = 1 \tag{C.11}$$

Eq. (C.3) becomes:

$$\partial_0 F^{a01} + \partial_2 F^{a21} + \partial_3 F^{a31} = -\frac{A^{(0)}}{\mu_0} R^a{}_\mu{}^{\mu 1}. \tag{C.12}$$

The vector form of this equation is:

$$(\boldsymbol{\nabla} \times \boldsymbol{B})_1^a = \frac{1}{c^2} \frac{\partial E_1^a}{\partial t} + \frac{\mu_0}{c} J_1^a. \tag{C.13}$$

Here, the 1 subscript denotes a component in a particular coordinate system. For example in the spherical polar system:

$$1 = r \tag{C.14}$$

or in the Cartesian system:

$$1 = X. \tag{C.15}$$

So Eq. (C.13) is the r or X component of the Ampère Maxwell Law. If we adopt the spherical polar system for the Riemann elements (see Appendix 5) the value of a in Eq. (C.13) must also be 1. If the complex circular basis [2–9] is chosen then:

$$a = (1), (2), (3). \tag{C.16}$$

However, if the complex circular basis is chosen, then the relevant Riemann elements are:

$$R^{(1)}{}_{\mu}{}^{\mu 1}, R^{(2)}{}_{\mu}{}^{\mu 2}, R^{(3)}{}_{\mu}{}^{\mu 3} \tag{C.17}$$

in which one index is complex circular, and the other three are spherical polar. It is possible to use either system, or any other system of coordinates for a. Therefore the generally covariant Ampère Maxwell Law is:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{\mu_0}{c}\boldsymbol{J} \tag{C.18}$$

where the charge current density is defined as:

$$\boldsymbol{J} = J_1^1 \boldsymbol{e}_r + J_2^2 \boldsymbol{e}_\theta + J_3^3 \boldsymbol{e}_\phi \tag{C.19}$$

with the scalar valued components:

$$J_1^1 = -\frac{A^{(0)}}{\mu_0}\left(R^1{}_0{}^{01} + R^1{}_2{}^{21} + R^1{}_3{}^{31}\right), \tag{C.20}$$

$$J_2^2 = -\frac{A^{(0)}}{\mu_0}\left(R^2{}_0{}^{02} + R^2{}_1{}^{12} + R^2{}_3{}^{32}\right), \tag{C.21}$$

$$J_3^3 = -\frac{A^{(0)}}{\mu_0}\left(R^3{}_0{}^{03} + R^3{}_1{}^{13} + R^3{}_2{}^{23}\right), \tag{C.22}$$

A particular metric may finally be used to calculate these Riemann components exactly, and example is given in detail in Appendix 5.

The main result is that in the presence of space-time curvature, the electrodynamical properties of light are changed, in addition to the well known effects of Einstein Hilbert theory there are polarization changes in light deflected by gravitation. These are due to the charge current density $\boldsymbol{J}$, which does not exist in the free space limit of Maxwell Heaviside theory. So these are predictions of ECE theory that are known already to be corroborated qualitatively [2–9], because of observations of polarization changes in light deflected by a white dwarf for example.

D

Appendix 4: The Meaning of the a Index

It is first noted that the ECE field equations originate in the Bianchi identity:

$$D \wedge F^a := R^a{}_b \wedge A^b \tag{D.1}$$

where the a and b indices denote those of a tangent space-time at point P in a base manifold in differential geometry. Thus:

$$D \wedge F^a = D_\mu F^a_{\nu\sigma} + D_\sigma F^a_{\mu\nu} + D_\nu F^a_{\sigma\mu} \tag{D.2}$$

where the Greek indices of the base manifold have been restored. In generally covariant unified field theory the electromagnetic field tensor is therefore a vector-valued two-form, i.e. an anti-symmetric tensor for each a. The field two-form is defined as:

$$F^a_{\mu\nu} = q^a_\kappa F^\kappa_{\mu\nu} \tag{D.3}$$

where $F^\kappa_{\mu\nu}$ is a tensor in the base manifold with three indices. It is seen that:

$$D \wedge F^\kappa := R^\kappa{}_b \wedge A^b \tag{D.4}$$

using the tetrad postulate:

$$D_\mu q^a_\kappa = 0. \tag{D.5}$$

Therefore Eq. (D.1) can be written in the base manifold:

$$D_\mu F^\kappa_{\nu\sigma} + D_\sigma F^\kappa_{\mu\nu} + D_\nu F^\kappa_{\sigma\mu} := A^{(0)} \left(R^\kappa{}_{\mu\nu\sigma} + R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} \right). \tag{D.6}$$

In general relativity and unified field theory the base manifold is four dimensional space-time in which curvature and torsion are both present in general. So the electromagnetic field in this base manifold is a rank three tensor, not a rank two tensor as in special relativity and Minkowski space-time. In the latter type of space-time there is no curvature and no torsion, so Minkowski space-time is flat space-time.

For example, consider the electric field components:

$$F^{\kappa}_{\nu\sigma} = F^{\kappa}{}_{10}\,, F^{\kappa}{}_{20}\,, F^{\kappa}{}_{30}\,. \tag{D.7}$$

In the complex circular basis:

$$\kappa = (1), (2), (3) \tag{D.8}$$

and we recover the three vector components $E_X^{(1)}$, $E_Y^{(1)}$, and $E_Z^{(1)}$. The first two denote complex conjugate plane waves:

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}\,(\boldsymbol{i} - i\boldsymbol{j})\, e^{i(\omega t - \kappa z)}. \tag{D.9}$$

So the meaning of κ superimposed on $\nu\sigma$ is that one coordinate system is superimposed on another. When one coordinate system is imposed on the same coordinate system the only possibilities are:

$$F^{\kappa}_{\nu\sigma} = F^{1}_{10}, F^{2}_{20}, F^{3}_{30} = E^{1}_{1}, E^{2}_{2}, E^{3}_{3} \tag{D.10}$$

as used in Appendix 4.

E

Appendix 5: Christoffel Symbols and Riemann Elements

The non-vanishing Christoffel symbols and Riemann elements of each line element used in this paper were computed using a program written by Horst Eckardt based on Maxima [4], after first hand checking the program for correctness. For the spherically symmetric line element:

$$ds^2 = -e^{2\alpha}dt^2c^2 = e^{2\beta}dr^2 + r^2d\Omega^2 \tag{E.1}$$

$$\begin{gathered} g_{\infty} = -e^{2\alpha}, g_{11} = e^{2\beta}, g_{22} = r^2, \\ g_{33} = r^2\sin^2\theta, \end{gathered} \tag{E.2}$$

it was checked by hand calculation and by computer that the Christoffel symbols and Riemann elements are as given by Carroll [11] as follows:

$$\begin{aligned} &\Gamma^0{}_{00} = \partial_0\alpha, \Gamma^0{}_{01} = \partial_1\alpha, \Gamma^0{}_{11} = e^{2(\beta-\alpha)}\partial_0\beta, \\ &\Gamma^1{}_{00} = e^{2(\alpha-\beta)}\partial_1\alpha, \Gamma^1{}_{01} = \partial_0\beta, \Gamma^1{}_{11} = \partial_1\beta, \\ &\Gamma^2{}_{12} = \frac{1}{r}, \Gamma^1{}_{22} = -re^{-2\beta}, \Gamma^3{}_{13} = \frac{1}{r}, \\ &\Gamma^1{}_{33} = -re^{-2\beta}\sin^2\theta, \Gamma^2{}_{33} = -\sin\theta\cos\theta, \Gamma^3{}_{23} = \frac{\cos\theta}{\sin\theta}, \end{aligned} \tag{E.3}$$

$$
\begin{aligned}
R^0{}_{101} &= e^{2(\beta-\alpha)}\left(\partial_0^2\beta + (\partial_0\beta)^2 - \partial_0\alpha\partial_0\beta\right) \\
&\quad + \partial_1\alpha\partial_1\beta - \partial_1(\partial_1\alpha) - (\partial_1\alpha)^2 \\
R^0{}_{202} &= -re^{2\beta}\partial_1\alpha, \\
R^0{}_{303} &= R^0_{202}\sin^2\theta, \\
R^1{}_{212} &= re^{-2\beta}\partial_1\beta, \\
R^1{}_{313} &= re^{-2\beta}\partial_1\beta\sin^2\theta = R^1_{212}\sin^2\theta, \\
R^2{}_{323} &= (1-e^{2\beta})\sin^2\theta, \\
(0,1,2,3) :&= (t,r,\theta,\phi)
\end{aligned}
\tag{E.4}
$$

The inverse metric elements are related to the metric elements as follows:

$$
\begin{aligned}
g^{00} &= g_{00}^{-1} = -e^{-2\alpha}, \\
g^{11} &= g_{11}^{-1} = e^{-2\beta}, \\
g^{22} &= g_{22}^{-1} = \frac{1}{r^2} \\
g^{33} &= g_{33}^{-1} = \frac{1}{(r^2\sin^2\theta)}
\end{aligned}
\tag{E.5}
$$

in the spherical polar coordinate system $(r,\ \theta,\ \phi)$. The Christoffel symbol in Riemann geometry is:

$$
\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)
\tag{E.6}
$$

where summation is implied over repeated indices in the covariant - contravariant system.

The non-vanishing Riemann elements are calculated from the Christoffel symbols using the definition of the Riemann tensor:

$$
R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}
\tag{E.7}
$$

where summation is again implied over repeated indices.

In order to calculate the Coulomb law and Ampère Maxwell laws, indices must be raised with the metrics:

$$
\begin{aligned}
R^0{}_1{}^{10} &= -R^0{}_1{}^{01} = -g^{11}g^{00}R^0{}_{101} = R^0{}_{101}, \\
R^0{}_2{}^{20} &= -R^0{}_2{}^{20} = -g^{22}g^{00}R^0{}_{202}, \\
R^0{}_3{}^{30} &= -R^0{}_3{}^{03} = -g^{33}g^{00}R^0{}_{303},
\end{aligned}
\tag{E.8}
$$

and this procedure was adhered to for each line element.

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